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Article

Neutrosophic Triplet G-Module

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Abstract: In this study, the neutrosophic triplet G-module is introduced and the properties of neutrosophic triplet G-module are studied. Furthermore, reducible, irreducible, and completely reducible neutrosophic triplet G-modules are defined, and relationships of these structures with each other are examined. Also, it is shown that the neutrosophic triplet G-module is different from the G-module.

Keywords: neutrosophic triplet G-module; neutrosophic triplet group; neutrosophic triplet vector space

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1. Introduction

Neutrosophy is a branch of philosophy, firstly introduced by Smarandache in 1980. Neutrosophy [1] is based on neutrosophic logic, probability, and set. Neutrosophic logic is a generalized form of many logics such as fuzzy logic, which was introduced by Zadeh [2], and intuitionistic fuzzy logic, which was introduced by Atanassov [3]. Furthermore, Bucolo et al. [4] studied complex dynamics through fuzzy chains; Chen [5] introduced MAGDM based on intuitionistic 2–Tuple linguistic information, and Chen [6] obtain some q–Rung Ortopair fuzzy aggregation operators and their MAGDM. Fuzzy set has function of membership; intuitionistic fuzzy set has function of membership and function of non-membership. Thus; they do not explain the indeterminancy states. However, the neutrosophic set has a function of membership, a function of indeterminacy, and a function of non-membership. Also, many researchers have studied the concept of neutrosophic theory in [7–12]. Recently, Olgun et al. [13] studied the neutrosophic module; Şahin et al. [14] introduced Neutrosophic soft lattices; Şahin et al. [15] studied the soft normed ring; Şahin et al. [16] introduced the centroid single-valued neutrosophic triangular number and its applications; Şahin et al. [17] introduced the centroid points of transformed single-valued neutrosophic number and its applications; Ji et al. [18] studied multi-valued neutrosophic environments and their applications. Also, Smarandache et al. [19] studied neutrosophic triplet (NT) theory and [20,21] neutrosophic triplet groups. A NT has a form <x, neut(x), anti(x)>, in which neut(x) is neutral of "x" and anti(x) is opposite of "x". Furthermore, neut(x) is different from the classical unitary element. Also, the neutrosophic triplet group is different from the classical group. Recently, Smarandache et al. [22] studied the NT field and [23] the NT ring; Şahin et al. [24] introduced the NT metric space, the NT vector space, and the NT normed space; Şahin et al. [25] introduced the NT inner product.

The concept of G-module [26] was introduced by Curties. G-modules are algebraic structures constructed on groups and vector spaces. The concept of group representation was introduced by Frobenious in the last two decades of the 19th century. The representation theory is an important algebraic structure that makes the elements, which are abstract concepts, more evident. Many important results could be proved only for representations over algebraically closed fields. The module theoretic approach is better suited to deal with deeper results in representation theory. Moreover, the module theoretic approach adds more elegance to the theory. In particular, the G-module structure has been extensively used for the study of representations of finite groups. Also, the representation theory of groups describes all the ways in which group G may be embedded in any linear group GL (V). The G-module also holds an important place in the representation theory of groups. Recently some researchers have been dealing with the G-module. For example, Fernandez [27] studied fuzzy G-modules. Sinho and Dewangan [28] studied isomorphism theory for fuzzy submodules of G-modules. Şahin et al. [29] studied soft G-modules. Sharma and Chopra [30] studied the injectivity of intuitionistic fuzzy G-modules.

In this paper, we study neutrosophic triplet G-Modules in order to obtain a new algebraic constructed on neutrosophic triplet groups and neutrosophic triplet vector spaces. Also we define the reducible neutrosophic triplet G-module, the irreducible neutrosophic triplet G-module, and the completely reducible neutrosophic triplet G-module. In this study, in Section 2, we give some preliminary results for neutrosophic triplet sets, neutrosophic triplet groups, the neutrosophic triplet field, the neutrosophic triplet vector space, and G-modules. In Section 3, we define the neutrosophic triplet G-module, and we introduce some properties of a neutrosophic triplet G-module. We show that the neutrosophic triplet G-module is different from the G-module, and we show that if certain conditions are met, every neutrosophic triplet vector space or neutrosophic triplet group can be a neutrosophic triplet G-module at the same time. Also, we introduce the neutrosophic triplet G-module homomorphism and the direct sum of neutrosophic triplet vector space. In Section 4, we define the reducible neutrosophic triplet G-module, the irreducible neutrosophic triplet G-module, and the completely reducible neutrosophic triplet G-module, and we give some properties and theorems for them. Furthermore, we examine the relationships of these structures with each other, and we give some properties and theorems. In Section 5, we give some conclusions.

2. Preliminaries

Definition 1. *Let N be a set together with a binary operation *. Then, N is called a neutrosophic triplet set if for any $a \in N$ there exists a neutral of "a" called neut(a) that is different from the classical algebraic unitary element and an opposite of "a" called anti(a) with neut(a) and anti(a) belonging to N, such that [21]:*

$a * neut(a) = neut(a) * a = a$,

and

$a * anti(a) = anti(a) * a = neut(a)$.

Definition 2. *Let (N,*) be a neutrosophic triplet set. Then, N is called a neutrosophic triplet group if the following conditions are satisfied [21].*

(1) *If (N,*) is well-defined, i.e., for any $a, b \in N$, one has $a*b \in N$.*
(2) *If (N,*) is associative, i.e., $(a*b)*c = a*(b*c)$ for all $a, b, c \in N$.*

Theorem 1. *Let (N,*) be a commutative neutrosophic triplet group with respect to * and $a, b \in N$, in which a and b are both cancellable [21],*

(i) $neut(a)*neut(b) = neut(a*b)$.
(ii) $anti(a)*anti(b) = anti(a*b)$.

Definition 3. *Let (NTF,*, #) be a neutrosophic triplet set together with two binary operations * and #. Then (NTF,*, #) is called neutrosophic triplet field if the following conditions hold [22].*

1. *(NTF,*) is a commutative neutrosophic triplet group with respect to *.*
2. *(NTF, #) is a neutrosophic triplet group with respect to #.*
3. $a\#(b*c) = (a\#b)*(a\#c)$ *and* $(b*c)\#a = (b\#a)*(c\#a)$ *for all* $a, b, c \in NTF$.

Theorem 2. *Let (N,*) be a neutrosophic triplet group with respect to *. For (left or right) cancellable $a \in N$, one has the following [24]:*

(i) *neut(neut(a)) = neut(a);*
(ii) *anti(neut(a)) = neut(a);*
(iii) *anti(anti(a)) = a;*
(iv) *neut(anti(a)) = neut(a).*

Definition 4. *Let (NTF, $*_1$, $\#_1$) be a neutrosophic triplet field, and let (NTV, $*_2$, $\#_2$) be a neutrosophic triplet set together with binary operations " $*_2$" and "$\#_2$". Then (NTV, $*_2$, $\#_2$) is called a neutrosophic triplet vector space if the following conditions hold. For all $u, v \in NTV$, and for all $k \in NTF$, such that $u*_2v \in NTV$ and $u \#_2k \in NTV$ [24];*

(1) $(u*_2v) *_2t = u*_2 (v*_2t)$; $u, v, t \in NTV$;
(2) $u*_2v = v*_2u$; $u, v \in NTV$;
(3) $(v*_2u) \#_2k = (v\#_2k) *_2(u\#_2k)$; $k \in NTF$ and $u, v \in NTV$;
(4) $(k*_1t) \#_2u = (k\#_2v) *_1(u\#_2v)$; $k, t \in NTF$ and $u \in NTV$;
(5) $(k\#_1t) \#_2u = k\#_1(t\#_2u)$; $k, t \in NTF$ and $u \in NTV$;
(6) *There exists any $k \in NTF$ such that $u\#_2neut(k) = neut(k) \#_2u = u$; $u \in NTV$.*

Definition 5. *Let G be a finite group. A vector space M over a field K is called a G-module if for every $g \in G$ and $m \epsilon M$ there exists a product (called the action of G on M) $m.g \in M$ satisfying the following axioms [26]:*

(i) $m.1_G = m, \forall\, m \epsilon M$ *(1_G being the identity element in G);*
(ii) $m.(g.h) = (m.g).h, \forall\, m \epsilon M; g, h \epsilon G$;
(iii) $(k_1m_1 + k_2m_2).g = k_1(m_1.g)+ k_2(m_2.g)$; $k_1, k_2 \epsilon K$; $m_1, m_2 \epsilon M$, *and* $g \epsilon G$.

Definition 6. *Let M be a G-module. A vector subspace N of M is a G-submodule if N is also a G-module under the same action of G [26].*

Definition 7. *Let M and M^* be G-modules. A mapping ϕ [26]: $M \to M^*$ is a G-module homomorphism if*

(i) $\phi(k_1.m_1 + k_2.m_2) = k_1.\, \phi(m_1) + k_2.\phi(m_2)$;
(ii) $\phi(m.g) = \phi(m).g$; $k_1, k_2 \epsilon K$; $m, m_1, m_2 \epsilon M$; $g \epsilon G$.

Further, if ϕ is 1-1, then ϕ is an isomorphism. The G-modules *M* and *M** are said to be isomorphic if there exists an isomorphism ϕ of *M* onto *M**. Then we write $M \cong M^*$.

Definition 8. *Let M be a nonzero G-module. Then, M is irreducible if the only G-submodules of M are M and {0}. Otherwise, M is reducible [26].*

Definition 9. *Let $M_1, M_2, M_3, \ldots, M_n$ be vector spaces over a field K [31]. Then, the set $\{m_1 + m_2 + \ldots + m_n; m_i \epsilon M_i\}$ becomes a vector space over K under the operations*

$$(m_1+m_2 + \ldots + m_n) + (m_1{}' + m_2{}' + \ldots + m_n{}') = (m_1 + m_1{}') + (m_2 + m_2{}') + \ldots + (m_n + m_n{}') \text{ and}$$

$$\alpha(m_1+m_2 + \ldots + m_n) = \alpha m_1 + \alpha m_2 + \ldots + \alpha m_n;\ \alpha \epsilon K, m_n{}' \epsilon M_i$$

It is the called direct sum of the vector spaces $M_1, M_2, M_3, \ldots, M_n$ and is denoted by ${}^{n}_{i=1}\oplus M_i$.

Remark 1. *The direct sum $M = {}^{n}_{i=1}\oplus M_i$ of vector spaces M_i has the following properties [31]:*

(i) *Each element $m \epsilon M$ has a unique expression as the sum of elements of M_i.*
(ii) *The vector subspaces $M_1, M_2, M_3, \ldots, M_n$ of M are independent.*
(iii) *For each $1 \le i \le n$, $M_j \cap (M_1 + M_2 + \ldots + M_{j-1} + M_{j+1} + \ldots + M_n) = \{0\}$.*

Definition 10. *A nonzero G-module M is completely reducible if for every G-submodule N of M there exists a G-submodule N* of M such that M = N ⊕ N* [26].*

Proposition 1. *A G-submodule of a completely reducible G-module is completely reducible [26].*

3. Neutrosophic Triplet G-Module

Definition 11. *Let (G, *) be a neutrosophic triplet group, (NTV,$*_1$, $\#_1$) be a neutrosophic triplet vector space on a neutrosophic triplet field (NTF,$*_2$, $\#_2$), and g*m ϵ NTV for g ϵ G, m ϵ NTV. If the following conditions are satisfied, then (NTV,$*_1$, $\#_1$) is called neutrosophic triplet G-module.*

(a) *There exists g ∈ G such that m*neut(g) = neut(g)*m = m for every m ∈ NTV;*
(b) $m{*_1}(g{*_1}h) = (m{*_1}g)\,{*_1}h$, ∀ m ∈ NTV; g, h ∈ G;
(c) $(k_1\#_1m_1 *_1 k_2\, \#_1m_2)*g = k_1\#_1\,(m_1*g)*_1k_2\#_1\,(m_2*g)$, $\forall k_1, k_2 \epsilon$ NTF; $m_1, m_2\ \epsilon$ NTV; g ϵ G.

Corollary 1. *Neutrosophic G-modules are generally different from the classical G-modules, since there is a single unit element in classical G-module. However, the neutral element neut(g) in neutrosophic triplet G-module is different from the classical one. Also, neutrosophic triplet G-modules are different from fuzzy G-modules, intuitionistic fuzzy G-modules, and soft G-modules, since neutrosophic triplet set is a generalized form of fuzzy set, intuitionistic fuzzy set, and soft set.*

Example 1. *Let X be a nonempty set and let P(X) be set of all subset of X. From Definition 4, (P(X), ∪, ∩) is a neutrosophic triplet vector space on the (P(X), ∪, ∩) neutrosophic triplet field, in which the neutrosophic triplets with respect to ∪; neut(A) = A and anti(A) = B, such that A, B ∈ P(X); A ⊆ B; and the neutrosophic triplets with respect to ∩; neut(A) = A and anti(A) = B, such that A, B ∈ P(X); B ⊇ A. Furthermore, (P(X), ∪) is a neutrosophic triplet group with respect to ∪, in which neut(A) = A and anti(A) = B such that A, B ⊂ P(X); A ⊆ B. We show that (P(X), ∪, ∩) satisfies condition of neutrosophic triplet G-module. From Definition 11:*

(a) *It is clear that if A = B, there exists any A ∈ P(X) for every B ∈ P(X), such that B ∪ neut(A) = neut(A) ∪ B = A.*
(b) *It is clear that A ∪ (B ∪ C) = (A ∪ B) ∪ C, ∀ A ∈ P(X); B, C ∈ P(X).*
(c) *It is clear that*

$((A_1 \cap B_1) \cup (A_2 \cap B_2)) \cup C = (A_1 \cap B_1) \cup C)) \cup (A_2 \cap B_2) \cup C)), \forall A_1, A_2\ \epsilon\ P(X); B_1, B_2\ \epsilon\ P(X); C\ \epsilon\ P(X).$ *Thus, (P(X), ∪, ∩) is a neutrosophic triplet G-module.*

Corollary 2. *If G = NTV, * = $*_1$, then each (NTV,$*_1$, $\#_1$) neutrosophic triplet vector space is a neutrosophic triplet G-module at the same time. Thus, if G = NTV and * = $*_1$, then every neutrosophic triplet vector space or neutrosophic triplet group can be a neutrosophic triplet G-module at the same time. It is not provided by classical G-module.*

Proof of Corollary 1. *If G = NTV, * = $*_1$;*

(a) *There exists a g ϵ NTV such that m*neut(g) = neut(g)*m = m, ∀m ϵ NTV;*
(b) *It is clear that m*(g*h) = (m*g)*h, as (NTV,*) is a neutrosophic triplet group; ∀ m, g, h ∈ NTV;*
(c) *It is clear that* $(k_1\#_1m_1 *_1 k_2\, \#_1m_2)*g = k_1\#_1(m_1*g)*_1k_2\#_1\,(m_2*g)$, *since (NTV,$*_1$, $\#_1$) is a neutrosophic triplet vector space;* $\forall\, g, k_1, k_2\ \epsilon$ NTF; $m_1, m_2\ \epsilon$ NTV.

Definition 12. *Let (NTV,$*_1$, $\#_1$) be a neutrosophic triplet G-module. A neutrosophic triplet subvector space (N, $*_1$, $\#_1$) of (NTV,$*_1$, $\#_1$) is a neutrosophic triplet G-submodule if (N,$*_1$, $\#_1$) is also a neutrosophic triplet G-module.*

Example 2. *From Example 1; for N ⊆ X, (P(N), ∪, ∩) is a neutrosophic triplet subvector space of (P(X), ∪, ∩). Also, (P(N), ∪, ∩) is a neutrosophic triplet G-module. Thus, (P(N), ∪, ∩) is a neutrosophic triplet G-submodule of (P(X), ∪, ∩).*

Example 3. *Let (NTV,$*_1$, $\#_1$) be a neutrosophic triplet G-module. N = {neut(x)} $\in$ NTV is a neutrosophic triplet subvector space of (NTV,$*_1$, $\#_1$). Also, N = {neut(x) = x} $\in$ NTV is a neutrosophic triplet G-submodule of (NTV,$*_1$, $\#_1$).*

Definition 13. *Let (NTV,$*_1$, $\#_1$) and (NTV^*,$*_3$, $\#_3$) be neutrosophic triplet G-modules on neutrosophic triplet field (NTF,$*_2$, $\#_2$) and (G, *) be a neutrosophic triplet group. A mapping ϕ: NTV$\rightarrow NTV^*$ is a neutrosophic triplet G-module homomorphism if*

(i) $\phi(neut(m)) = neut(\phi(m))$
(ii) $\phi(anti(m)) = anti(\phi(m))$
(iii) $\phi((k_1\#_1 m_1) *_1 (k_2\#_1 m_2)) = (k_1\#_3\phi(m_1))*_3(k_2\#_3\phi(m_2))$
(iv) $\phi(m*g) = \phi(m)*g$; $\forall\, k_1, k_2 \in NTF$; $m, m_1, m_2 \in M$; $g \in G$.

Further, if ϕ is 1-1, then ϕ is an isomorphism. The neutrosophic triplet G-modules (NTV,$*_1$, $\#_1$) and (NTV^*,$*_3$, $\#_3$) are said to be isomorphic if there exists an isomorphism ϕ: $NTV \rightarrow NTV^*$. Then, we write $NTV \cong NTV^*$.

Example 4. *From Example 1, (P(X), $\cup$, $\cap$) is neutrosophic triplet vector space on neutrosophic triplet field (P(X), $\cup$, $\cap$). Furthermore, (P(X), $\cup$, $\cap$) is a neutrosophic triplet G-module. We give a mapping ϕ: P(X) $\rightarrow$ P(X), such that ϕ(A) = neut(A). Now, we show that ϕ is a neutrosophic triplet G-module homomorphism.*

(i) $\phi(neut(A)) = neut(neut(A)) = neut(\phi(A))$
(ii) $\phi(anti(A)) = neut(anti(A))$; *from Theorem 2,* $neut(anti(A)) = neut(A)$.

$anti(\phi(A)) = anti(neut(A))$; *from Theorem 2,* $anti(neut(A)) = neut(A)$. *Then* $\phi(anti(A)) = anti(\phi(A))$.
(iii) $\phi((A_1 \cap B_1) \cup (A_2 \cap B_2)) = neut(A_1 \cap B_1) \cup (A_2 \cap B_2))$; *from Theorem 1, as* $neut(a)*neut(b) = neut(a*b)$;

$neut(A_1 \cap B_1) \cup (A_2 \cap B_2)) = neut(A_1 \cap B_1) \cup neut(A_2 \cap B_2) =$

$((neut(A_1) \cap neut(B_1)) \cup ((neut(A_2) \cap neut(B_2))$. *From Example 1, as* $neut(A) = A$,

$((neut(A_1) \cap neut(B_1)) \cup ((neut(A_2) \cap neut(B_2)) = (A_1 \cap neut(B_1)) \cup (A_2 \cap neut(B_2)) =$

$(A_1 \cap neut(B_1)) \cup (A_2 \cap neut(B_2)) = (A_1 \cap \phi(B_1)) \cup (A_2 \cap \phi(B_2))$.
(iv) $\phi(A*B) = neut(A*B)$; *from Theorem 1, as* $neut(a)*neut(b) = neut(a*b)$, $neut(A*B) = neut(A)*neut(B)$. *From Example 1, as* $neut(A) = A$, $neut(A)*neut(B) = A*neut(B) = A*\phi(B)$.

4. Reducible, Irreducible, and Completely Reducible Neutrosophic Triplet G-Modules

Definition 14. *Let (NTV,$*_1$, $\#_1$) be neutrosophic triplet G-modules on neutrosophic triplet field (NTF,$*_2$, $\#_2$). Then, (NTV,$*_1$, $\#_1$) is irreducible neutrosophic triplet G-modules if the only neutrosophic triplet G-submodules of (NTV,$*_1$, $\#_1$) are (NTV,$*_1$, $\#_1$) and {neut(x) = x}, x $\in$ NTV. Otherwise, (NTV,$*_1$, $\#_1$) is reducible neutrosophic triplet G-module.*

Example 5. *From Example 2, for N = {1,2} $\subseteq$ {1,2,3} = X, (P(N), $\cup$, $\cap$) is a neutrosophic triplet subvector space of (P(X), $\cup$, $\cap$). Also, (P(N), $\cup$, $\cap$) is a neutrosophic triplet G-module. Thus, (P(N), $\cup$, $\cap$) is a neutrosophic triplet G-submodule of (P(X), $\cup$, $\cap$). Also, from Definition 14, (P(X), $\cup$, $\cap$) is a reducible neutrosophic triplet G-module.*

Example 6. *Let X = G = {1, 2} and P(X) be power set of X. Then, (P(X), *, ∩) is a neutrosophic triplet vector space on the (P(X), *, ∩) neutrosophic triplet field and (G, *) is a neutrosophic triplet group, in which*

$$A^*B = \begin{cases} B\backslash A, s(A) < s(B)\Lambda B \supset A \wedge A' = B \\ A\backslash B, s(B) < s(A)\Lambda A \supset B \wedge B' = A \\ (A\backslash B)', s(A) > s(B) \wedge A \supset B \wedge B\prime \neq A \\ (B\backslash A)', s(B) > s(A) \wedge B \supset A \wedge A\prime \neq B \\ X, s(A) = s(B) \wedge A \neq B \\ \varnothing, A = B \end{cases}$$

Here, s(A) means the cardinal of A, and A′ means the complement of A.
*The neutrosophic triplets with respect to *;*
neut(∅) = ∅, anti(∅) = ∅; neut({1}) = {1, 2}, anti({1}) = {2}; neut({2}) = {1, 2}, anti({2}) = {1}; neut({1, 2}) = ∅, anti({1,2}) = {1, 2};
The neutrosophic triplets with respect to ∩;
neut(A) = A and anti(A) = B, in which B ⊃ ⋂ A.
*Also, (P(X), *, ∩) is a neutrosophic triplet G-module. Here, only neutrosophic triplet G-submodules of (P(X), *, ∩) are (P(X), *, ∩) and {neut(∅) = ∅}. Thus, (P(X),*, ∩) is a irreducible neutrosophic triplet G-module.*

Definition 15. *Let* $(NTV_1, *_1, \#_1)$, $(NTV_2, *_1, \#_1)$, ..., $(NTV_n, *_1, \#_1)$ *be neutrosophic triplet vector spaces on* $(NTF, *_2, \#_2)$. *Then, the set* $\{m_1 + m_2 + \ldots + m_n;\ m_i \in NTV_i\}$ *becomes a neutrosophic triplet vector space on* $(NTF, *_2, \#_2)$, *such that*

$$(m_1 *_1 m_2 *_1 \ldots *_1 m_n)\ *_1 (m_1{}' *_1 m_2{}' *_1 \ldots *_1 m_n{}') = (m_1 *_1 m_1{}') *_1 (m_2\ *_1 m_2{}')\ *_1 \ldots *_1 (m_n *_1 m_n{}')\ and$$

$$\alpha\#_1(m_1 *_1 m_2 *_1 \ldots *_1 m_n) = (\alpha\#_1 m_1) *_1 \alpha\#_1 m_2) *_1 \ldots *_1 (\alpha\#_1 m_n);\ \alpha \in NTf,\ m_n{}' \in NTV_i.$$

It is called the direct sum of the neutrosophic triplet vector spaces $NTV_1, NTV_2, NTV_3, \ldots, NTV_n$ and is denoted by ${}^{n}_{i=1}\oplus NTV_i$.

Remark 2. *The direct sum* $NTV = {}^{n}_{i=1}\oplus NTV_i$ *of neutrosophic triplet vector spaces* NTV_i *has the following properties.*

(i) *Each element m* $\in$ *NTV has a unique expression as the sum of elements of* NTV_i.
(ii) *For each* $1 \le i \le n$, $NTV_j \cap (NTV_1 + NTV_2 + \ldots + NTV_{j-1} + NTV_{j+1} + \ldots + NTV_n)$ *= {x: neut(x) = x}.*

Definition 16. *Let* $(NTV, *_1, \#_1)$ *be neutrosophic triplet G-modules on neutrosophic triplet field* $(NTF, *_2, \#_2)$, *such that NTV* $\neq$ *{neut(x) = x}. Then,* $(NTV, *_1, \#_1)$ *is a completely reducible neutrosophic triplet G-module if for every neutrosophic triplet G-submodule* $(N_1, *_1, \#_1)$ *of* $(NTV, *_1, \#_1)$ *there exists a neutrosophic triplet G-submodule* $(N_2, *_1, \#_1)$ *of* $(NTV, *_1, \#_1)$, *such that* $NTV = N_1 \oplus N_2$.

Example 7. *From Example 5, for N = {1, 2}, (P(N), ∪, ∩) is a neutrosophic triplet vector space on (P(N), ∪, ∩) and a neutrosophic triplet G-module. Also, the neutrosophic triplet G-submodules of (P(N), ∪, ∩) are (P(N), ∪, ∩), (P(M), ∪, ∩), (P(K), ∪, ∩), and (P(L), ∪, ∩). Here, M = {1}, K = {2}, and T = {∅}, in which P(M)⊕P(K) = P(N), P(K)⊕P(M) = P(N), P(N)⊕P(T) = P(N), and P(T)⊕P(N) = P(N). Thus, (P(N), ∪, ∩) is a completely reducible neutrosophic triplet G-module.*

Theorem 3. *A neutrosophic triplet G-submodule of a completely reducible neutrosophic triplet G-module is completely neutrosophic triplet G-module.*

Proof of Theorem 1. *Let* $(NTV, *_1, \#_1)$ *is a completely reducible neutrosophic triplet G-module on neutrosophic triplet field* $(NTF, *_2, \#_2)$. *Assume that* $(N, *_1, \#_1)$ *is a neutrosophic triplet G-submodule of* $(NTV, *_1, \#_1)$ *and* $(M, *_1, \#_1)$ *is a neutrosophic triplet G-submodule of* $(N, *_1, \#_1)$. *Then,* $(M, *_1, \#_1)$ *is a neutrosophic triplet*

G-submodule of (NTV, $*_1$, $\#_1$). There exists a neutrosophic triplet G-submodule (T, $*_1$, $\#_1$), such that NTV = M$\oplus$T, since (NTV, $*_1$, $\#_1$) is a completely reducible neutrosophic triplet G-module. Then, we take N' = T $\cap$ N. From Remark 2,

$$N' \cap M \subset M \cap T = \{x : \text{neut}(x) = x\} \tag{1}$$

Then, we take y $\in$ N. If y $\in$ N, y $\in$ NTV and y = m $*_1$ t, in which m $\in$ M; t $\in$ T. Therefore, we obtain t $\in$ N. Thus,

$$tN' = T \cap N \text{ and } y = m*_1\, tN' \oplus M \tag{2}$$

From (i) and (ii), we obtain N = $N'\oplus$M. Thus, (N, $*_1$, $\#_1$) is completely reducible neutrosophic triplet G-module.

Theorem 4. *Let (NTV, $*_1$, $\#_1$) be a completely reducible neutrosophic triplet G-module on neutrosophic triplet field (NTF, $*_2$, $\#_2$). Then, there exists a irreducible neutrosophic triplet G-submodule of (NTV, $*_1$, $\#_1$).*

Proof of Theorem 2. *Let (NTV, $*_1$, $\#_1$) be a completely reducible neutrosophic triplet G-module and (N, $*_1$, $\#_1$) be a neutrosophic triplet G-submodule of (NTV, $*_1$, $\#_1$). We take y $\neq$ neut(y) $\in$ N, and we take collection sets of neutrosophic triplet G-submodules of (N, $*_1$, $\#_1$) such that do not contain element y. This set is not empty, because there is {x: x = neut(x)} neutrosophic triplet G-submodule of (N, $*_1$, $\#_1$). From Zorn's Lemma, the collection has maximal element (M, $*_1$, $\#_1$). From Theorem 3, (N, $*_1$, $\#_1$) is a completely reducible neutrosophic triplet G-module, and there exists a (N_1, $*_1$, $\#_1$) neutrosophic triplet G-submodule, such that N = M$\oplus N_1$. We show that (N_1*_1, $\#_1$) is a irreducible neutrosophic triplet G–submodule. Assume that (N_1, $*_1$, $\#_1$) is a reducible neutrosophic triplet G–submodule. Then, there exists (K_1, $*_1$, $\#_1$) and (K_2, $*_1$, $\#_1$) neutrosophic triplet G-submodules of (N_1, $*_1$, $\#_1$), such that y $\in$ N_1, N_2, and from Theorem 3, N_1 = $K_1 \oplus K_2$, in which, as N = M$\oplus N_1$, N = M$\oplus K_1 \oplus K_2$. From Remark 2, ($M*_1K_1$) $\cap$ K_2 = {neut(x) = x} or ($M*_1K_2$) $\cap$ K_1 = neut(x) = x}. Then, y $\notin$ ($M*_1K_1$) $\cap$ K_2 or y $\notin$ ($M*_1K_2$) $\cap$ K_1. Hence, y $\notin$ ($M*_1K_1$) or y $\notin$ ($M*_1K_2$). This is a contraction. Thus, (N_1*_1, $\#_1$) is an irreducible neutrosophic triplet G-submodule.*

Theorem 5. *Let (NTV, $*_1$, $\#_1$) be a completely reducible neutrosophic triplet G-module. Then, (NTV, $*_1$, $\#_1$) is a direct sum of irreducible neutrosophic triplet G-modules of (NTV, $*_1$, $\#_1$).*

Proof of Theorem 3. *From Theorem 3, (N_i,$*_1$, $\#_1$) (i = 1, 2, ..., n), neutrosophic triplet G-submodules of (NTV, $*_1$, $\#_1$) are completely reducible neutrosophic triplet G-modules, such that NTV = $N_{i-k} \oplus N_k$ (k = 1, 2, ..., $i-1$). From Theorem 4, there exists (M_i, $*_1$, $\#_1$) irreducible neutrosophic triplet G-submodules of (N_i, $*_1$, $\#_1$). Also, from Theorem 3, (M_i, $*_1$, $\#_1$) are completely reducible neutrosophic triplet G-modules, such that N_i = $N_{i-k} \oplus N_k$ (k = 1, 2, ..., $i-1$). If these steps are followed, we obtained (NTV, $*_1$, $\#_1$), which is a direct sum of irreducible neutrosophic triplet G-modules of (NTV, $*_1$, $\#_1$).*

5. Conclusions

In this paper; we studied the neutrosophic triplet G-module. Furthermore, we showed that neutrosophic triplet G-module is different from the classical G-module. Also, we introduced the reducible neutrosophic triplet G-module, the irreducible neutrosophic triplet G-module, and the completely reducible neutrosophic triplet G-module. The neutrosophic triplet G-module has new properties compared to the classical G-module. By using the neutrosophic triplet G-module, a theory of representation of neutrosophic triplet groups can be defined. Thus, the usage areas of the neutrosophic triplet structures will be expanded.

Author Contributions: Florentin Smarandache defined and studied neutrosophic triplet G-module, Abdullah Kargın defined and studied reducible neutrosophic triplet G-module, irreducible neutrosophic triplet G-module, and the completely reducible neutrosophic triplet G-module. Mehmet Şahin provided the examples and organized the paper.

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