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On the Semigroup Whose Elements Are Subgraphs of a Complete Graph

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Abstract: Let K_n be a complete graph on n vertices. Denote by SK_n the set of all subgraphs of K_n. For each $G, H \in SK_n$, the ring sum of G and H is a graph whose vertex set is $V(G) \cup V(H)$ and whose edges are that of either G or H, but not of both. Then SK_n is a semigroup under the ring sum. In this paper, we study Green's relations on SK_n and characterize ideals, minimal ideals, maximal ideals, and principal ideals of SK_n. Moreover, maximal subsemigroups and a class of maximal congruences are investigated. Furthermore, we prescribe the natural order on SK_n and consider minimal elements, maximal elements and covering elements of SK_n under this order.

Keywords: complete graph; Green's relations; ideal; natural order; maximal subsemigroup; maximal congruence

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1. Introduction and Preliminaries

One of several ways to study the algebraic structures in mathematics is to consider the relations between graph theory and semigroup theory known as algebraic graph theory. It is a branch of mathematics concerning the study of graphs in connection to semigroup theory in which algebraic methods are applied to problems about graphs. Cayley graphs of semigroups are special graphs such that many authors have widely studied, see for examples [1–3]. Studying on characterization of those graphs is a way of considering the relations between graphs and semigroups in the sense that such graphs are constructed from semigroups. On the other hand, the construction of a semigroup from a given graph is also interesting to study. However, there are no authors considering the properties of semigroups which are constructed from graphs. Certain special types of connected graphs are also interesting to study and look for some applications, especially a complete graph which is a graph in which every two distinct vertices are adjacent. Some authors considered a complete graph for applying its structure to complete their research, see for examples [4,5]. Furthermore, an algebraic formation on connected graphs has been studied in the sense of defining some binary operations among a pair of such graphs. Several authors investigated some properties of families of graphs together which graph operations, see for examples [6–8].

In this paper, we consequently construct a new semigroup from a complete graph and study some valuable properties of such a semigroup. We need to consider that all sets mentioned in this paper are assumed to be finite sets. Some basic preliminaries, useful notations, and valuable mathematical terminologies needed in what follows are prescribed. Note that a graph G is an order pair $(V(G), E(G))$ of a nonempty vertex set $V(G)$ and an edge set $E(G)$. For all undefined notions and notations, we refer the reader to [9,10].

Now, we give the description of the semigroup focused in this paper. Let K_n be a complete graph on n vertices and SK_n the set of all subgraphs of K_n. For each $G, H \in SK_n$, the *ring sum* $G \oplus H$ of G and H is a graph whose vertex set $V(G \oplus H) = V(G) \cup V(H)$ and whose edges are that of either G or H, but not of both, that is, $E(G \oplus H) = [E(G) \cup E(H)] \backslash [E(G) \cap E(H)]$. It is easy to verify that $(SK_n, \oplus)$ is a commutative semigroup. For convenience, we write GH instead of $G \oplus H$.

Throughout this paper, we shall denote the set of n vertices of K_n by the set $X_n = \{v_1, v_2, \ldots, v_n\}$. For each a nonempty subset A of X_n, denote by ϕ_A a graph with a vertex set $V(\phi_A) = A$ and $E(\phi_A) = \emptyset$ which is called an *empty graph*. For convenience, if $A = \{v\}$, then we write ϕ_v instead of $\phi_{\{v\}}$. We obviously obtain that $G^2 = \phi_{V(G)}$ and $G\phi_A = G$ for every $G \in SK_n$ and $A \subseteq V(G)$.

In this section, we consider the regularity and Green's relations on the semigroup SK_n. Moreover, we also determine the rank of SK_n.

Proposition 1. *SK_n is a regular semigroup.*

Proof. Let G be an element of SK_n. We will show that $G^3 = G$. It is obvious that $V(G^3) = V(G)$. Consider

$$\begin{aligned} E(G^3) &= E(G^2G) \\ &= [E(G^2) \cup E(G)] \backslash [E(G^2) \cap E(G)] \\ &= [\emptyset \cup E(G)] \backslash [\emptyset \cap E(G)] \\ &= E(G), \end{aligned}$$

we conclude that $G^3 = G$ which implies that G is regular. □

Moreover, we observe that the set of all empty graphs in SK_n forms the set of all idempotents of SK_n, denoted by $\mathcal{E}(SK_n)$, that is,

$$\mathcal{E}(SK_n) = \{\phi_A \in SK_n : \emptyset \neq A \subseteq X_n\}.$$

Clearly, $|\mathcal{E}(SK_n)| = 2^n - 1$. Furthermore, since SK_n is regular and its idempotents commute, it follows from Theorem 5.1.1 [9] (p. 145) that SK_n is an inverse semigroup.

Next, we will describe Green's relations on SK_n.

Proposition 2. *Let $G, H \in SK_n$. Then $G = KH$ for some $K \in SK_n$ if and only if $V(H) \subseteq V(G)$. Consequently, $G\mathcal{L}H$ if and only if $V(G) = V(H)$.*

Proof. Assume that $G = KH$ for some $K \in SK_n$. Then $V(G) = V(KH) = V(K) \cup V(H)$. Hence $V(H) \subseteq V(G)$. Conversely, assume that $V(H) \subseteq V(G)$. Define K to be the graph with the vertex set $V(K) = V(G)$ and the edge set $E(K) = [E(G) \backslash E(H)] \cup [E(H) \backslash E(G)]$. We will show that $G = KH$. It is easy to see that $V(KH) = V(K) \cup V(H) = V(G) \cup V(H) = V(G)$ and

$$\begin{aligned} E(KH) &= [E(K) \cup E(H)] \backslash [E(K) \cap E(H)] \\ &= [E(G) \cup E(H)] \backslash [E(H) \backslash E(G)] \\ &= E(G). \end{aligned}$$

Therefore, $G = KH$. □

Furthermore, we can directly obtain that $\mathcal{L} = \mathcal{R} = \mathcal{H} = \mathcal{D} = \mathcal{J}$ since SK_n is commutative.

Given a nonempty subset A of a semigroup S, denote by $\langle A \rangle$ the subsemigroup of S generated by A. The *rank* of S, denoted by rank(S), is the minimum cardinality of a generating set for a semigroup S.

In order to consider the rank of SK_n, we shall denote by $H[e]$ an induced subgraph of H induced by e where $e \in E(H)$. Let T_{ij} denote a graph in SK_n with $V(T_{ij}) = \{v_i, v_j\}$ where $i \neq j$ and $E(T_{ij}) = \{v_i v_j\}$.

Theorem 1. *rank(SK_n) =* $\binom{n}{2} + n$.

Proof. Let $\mathcal{M} = \{T_{ij} : i \neq j\}$ and $\mathcal{N} = \{\phi_v : v \in X_n\}$. We claim that every element of SK_n can be generated by some elements of $\mathcal{M} \cup \mathcal{N}$. Let $T \in SK_n$. If T contains isolated vertices, then those isolated vertices can be generated by corresponding elements in $\mathcal{N}$. So we now consider in the case where T has no isolated vertices. It is not difficult to verify that the set of all subgraphs $T[e]$ where $e \in E(T)$ is a subset of $\mathcal{M}$ and $T = \bigoplus_{e \in E(T)} T[e]$. This means that T can be written as a product of some elements of $\mathcal{M}$ under the operation $\oplus$. Hence $\mathcal{M} \cup \mathcal{N}$ is a generating set of SK_n which implies that

$$\text{rank}(SK_n) \leq |\mathcal{M} \cup \mathcal{N}| = \binom{n}{2} + n.$$

Moreover, we can easily observe that both of $H \in \mathcal{N}$ and $G \in \mathcal{M}$ cannot be written as a product of other elements in SK_n. Therefore, all elements in $\mathcal{M} \cup \mathcal{N}$ must belong to every generating set of SK_n. Consequently,

$$\text{rank}(SK_n) = \binom{n}{2} + n. \quad \square$$

2. Ideals of SK_n

This section presents the characterizations of ideals, minimal ideals, maximal ideals, and principal ideals of SK_n.

Let $\mathcal{C}$ be a nonempty subset of a power set $P(X_n)$. The set $\mathcal{C}$ is said to be an *upper set* of $P(X_n)$ if $\mathcal{C}$ satisfies the condition that if $A \in \mathcal{C}$ and $A \subseteq B$ for some $B \in P(X_n)$, then $B \in \mathcal{C}$. Note that if $A \in \mathcal{C}$, then $A \cup B \in \mathcal{C}$ for all $B \in P(X_n)$.

Now, we present the characterization of ideals of SK_n as follows.

Theorem 2. *The ideals of SK_n are precisely the sets*

$$I_{\mathcal{C}} = \{G \in SK_n : V(G) \in \mathcal{C}\}$$

where $\mathcal{C}$ is an upper set of $P(X_n)$.

Proof. Let $\mathcal{C}$ be an upper set of $P(X_n)$. We will show that $I_{\mathcal{C}}$ is an ideal of SK_n. Since $\mathcal{C} \neq \emptyset$, we get $I_{\mathcal{C}} \neq \emptyset$. Let $G \in I_{\mathcal{C}}$ and $H \in SK_n$. Then $V(G) \in \mathcal{C}$ and hence $V(GH) = V(G) \cup V(H) \in \mathcal{C}$ by the previous note which implies that $GH \in I_{\mathcal{C}}$.

Conversely, let I be any ideal of SK_n and let $\mathcal{A} = \{V(G) : G \in I\}$. Then $\mathcal{A}$ is a nonempty subset of $P(X_n)$. We will prove that $\mathcal{A}$ is an upper set of $P(X_n)$. Let $V(G) \in \mathcal{A}$ and $A \in P(X_n)$ in which $V(G) \subseteq A$. We get that $\phi_A G \in I$ since I is an ideal of SK_n. Thus $A = A \cup V(G) = V(\phi_A G) \in \mathcal{A}$. Hence $\mathcal{A}$ is an upper set of $P(X_n)$. Therefore, $I_{\mathcal{A}} = \{G \in SK_n : V(G) \in \mathcal{A}\} = \{G \in SK_n : G \in I\} = I$. This certainly completes the proof of our assertion. $\square$

In what follows, we define a subset of SK_n,

$$S(r) = \{G \in SK_n : |V(G)| = r\} \text{ where } 1 \leq r \leq n,$$

which plays an essential role for characterizing minimal ideals and maximal ideals of SK_n.

The following lemma shows some facts about ideals of SK_n which are useful for proving the next theorem.

Lemma 1. *Let I be an ideal of SK_n. If $S(1) \subseteq I$, then $I = SK_n$.*

Proof. We assume that $S(1) \subseteq I$. Let $T \in SK_n$. For each $v \in V(T)$, we obtain that $T = T\phi_v$. Since $\phi_v \in S(1) \subseteq I$ and I is an ideal of SK_n, we have $T \in I$ which certainly implies that $I = SK_n$, as required. □

An ideal M of a semigroup S is said to be *minimal* if every ideal I of S contained in M coincides with M. Further, M is said to be *maximal* if every proper ideal of S containing M coincides with M.

The following results describe the characterizations of minimal ideals and maximal ideals of SK_n, respectively.

Theorem 3. *$S(n)$ is the unique minimal ideal of SK_n.*

Proof. We first show that $S(n)$ is an ideal of SK_n. It is easy to investigate that $\{X_n\}$ is an upper set. By Theorem 2, we obtain that $I_{\{X_n\}}$ is an ideal of SK_n.

Consider
$$\begin{aligned} I_{\{X_n\}} &= \{G \in SK_n : V(G) \in \{X_n\}\} \\ &= \{G \in SK_n : V(G) = X_n\} \\ &= \{G \in SK_n : |V(G)| = n\} \\ &= S(n), \end{aligned}$$

we can conclude that $S(n)$ is an ideal of SK_n. Next, let I be an ideal of SK_n such that $I \subseteq S(n)$. Let $H \in S(n)$. Thus $V(H) = V(G)$ and $H = H\phi_{V(G)} \in I$ for any $G \in I$ which implies that $S(n) \subseteq I$. Therefore, $S(n) = I$, that is, $S(n)$ is a minimal ideal of SK_n.

We now let J be a minimal ideal of SK_n and $G \in J$. Hence $\phi_{V(G)} \in J$ and $\phi_{X_n} = \phi_{X_n}\phi_{V(G)} \in J$. Let $K \in S(n)$. Then $K = K\phi_{X_n} \in J$, we obtain that $S(n) \subseteq J$. It follows that $S(n) = J$ by the minimality of J. □

Theorem 4. *Maximal ideals of SK_n are precisely the sets $SK_n \backslash \{\phi_v\}$ where $v \in X_n$.*

Proof. We first prove that $SK_n \backslash \{\phi_v\}$ is an ideal of SK_n. Let T denote the set $SK_n \backslash \{\phi_v\}$ where $v \in X_n$. Let $G \in T$ and $H \in SK_n$. Suppose to the contrary that $GH = \phi_v$. Then $V(G) \cup V(H) = \{v\}$. Hence $V(G) = \{v\}$ which implies that $G = \phi_v$, a contradiction. Therefore, $GH \in T$, this means that T is an ideal of SK_n. It is uncomplicated to investigate that T is maximal.

Next, let I be a maximal ideal of SK_n. If $S(1) \subseteq I$, then $I = SK_n$ by Lemma 1 which contradicts to the maximality of I. Hence $S(1) \nsubseteq I$, that is, there exists $\phi_v \notin I$ for some $v \in X_n$ which implies that $I \subseteq SK_n \backslash \{\phi_v\}$. Since I is maximal in SK_n, we can conclude that $I = SK_n \backslash \{\phi_v\}$. □

Let S be a semigroup and $a \in S$. The smallest ideal of S containing a is $S^1 a S^1$ where S^1 is the monoid obtained from S by adjoining an identity 1 if necessary. We shall call it the *principal ideal of S generated by a*. The following theorem provides the necessary and sufficient conditions for ideals of SK_n to be principal.

Theorem 5. *$I_{\mathcal{C}}$ is a principal ideal of SK_n if and only if there exists unique $A \in \mathcal{C}$ such that $|A| = k$ where $k = \min\{|C| : C \in \mathcal{C}\}$ and $A \subseteq X$ for all $X \in \mathcal{C}$.*

Proof. Assume that $I_{\mathcal{C}}$ is a principal ideal of SK_n. Then $I_{\mathcal{C}} = (SK_n^1)G(SK_n^1)$ for some $G \in SK_n$. Let $A \in \mathcal{C}$ be such that $|A| = k$. Suppose that there exists $B \in \mathcal{C}$ such that $|B| = k$. Hence

$\phi_A, \phi_B \in I_\mathcal{C} = (SK_n^1)G(SK_n^1)$, that is, $\phi_A = G_1GG_2$ and $\phi_B = H_1GH_2$ for some $G_1, G_2, H_1, H_2 \in SK_n^1$. Thus $A = V(G_1) \cup V(G) \cup V(G_2)$ and $B = V(H_1) \cup V(G) \cup V(H_2)$ which implies that $V(G) \subseteq A$ and $V(G) \subseteq B$. Since $G \in I_\mathcal{C}$ and A, B are elements of $\mathcal{C}$ having the minimum cardinality k, we obtain that $A = V(G) = B$. We next let $X \in \mathcal{C}$. Hence $\phi_X \in I_\mathcal{C}$. So there exist $G_3, G_4 \in SK_n^1$ in which $\phi_X = G_3GG_4$. Then $X = V(G_3) \cup V(G) \cup V(G_4)$ which implies that $A = V(G) \subseteq X$, as desired.

Conversely, assume that the conditions hold. Since $A \in \mathcal{C}$, we have $\phi_A \in I_\mathcal{C}$ which leads to the fact that $(SK_n^1)\phi_A(SK_n^1) \subseteq I_\mathcal{C}$. Now, let $G \in I_\mathcal{C}$. Then $V(G) \in \mathcal{C}$ and $A \subseteq V(G)$ by our assumption. Therefore, $G = G\phi_A\phi_A \in (SK_n^1)\phi_A(SK_n^1)$ which follows that $I_\mathcal{C} = (SK_n^1)\phi_A(SK_n^1)$. Consequently, $I_\mathcal{C}$ is a principal ideal of SK_n generated by ϕ_A which completes the proof of our assertion. □

Remark 1. *Let A be a nonempty subset of X_n. Define $\mathcal{A}$ to be the family of all supersets of A. By Theorem 5, we can conclude that $I_\mathcal{A}$ is a principal ideal of SK_n. It is not difficult to verify that if $A \neq B$, then $I_\mathcal{A} \neq I_\mathcal{B}$. Therefore, the number of principal ideals of SK_n equals the number of nonempty subsets of X_n which equals $2^n - 1$, certainly.*

Example 1. *This example illustrates the ideal, minimal ideal, maximal ideal, and principal ideal of SK_3. All elements of SK_3 are shown in Figure 1 where each block is an $\mathcal{L}$-class of SK_3. In addition, we observe that*

$$
\begin{aligned}
S(1) &= \{G_1, G_2, G_3\}, \\
S(2) &= \{G_4, G_5, G_6, G_7, G_8, G_9\} \text{ and} \\
S(3) &= \{G_{10}, G_{11}, G_{12}, G_{13}, G_{14}, G_{15}, G_{16}, G_{17}\}.
\end{aligned}
$$

- *$SK_3 \backslash \{G_2, G_3\}$ is an ideal of SK_3.*
- *$S(3)$ is the unique minimal ideal of SK_3.*
- *$SK_3 \backslash \{G_1\}, SK_3 \backslash \{G_2\}$ and $SK_3 \backslash \{G_3\}$ are all maximal ideals of SK_3.*
- *$\{G_4, G_5, G_{10}, G_{11}, \ldots, G_{17}\}$ is a principal ideal of SK_3 generated by G_4.*

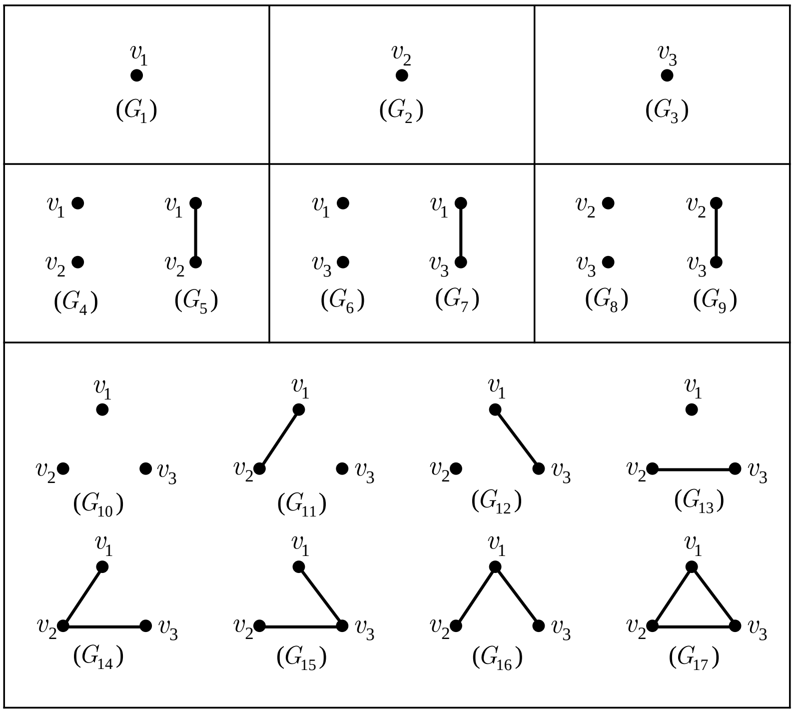

Figure 1. All elements of SK_3.

3. Maximal Subsemigroups and a Class of Maximal Congruences of SK_n

This section presents the results of maximal subsemigroups and maximal congruences of SK_n.

A nonempty proper subset M of a semigroup S is called a *maximal subsemigroup* if M is a subsemigroup of S and any proper subsemigroup of S containing M must be M.

Theorem 6. *A maximal subsemigroup of SK_n is one of the following forms.*

(i) $SK_n \backslash \{\phi_v\}$ *for some* $v \in X_n$;
(ii) $SK_n \backslash \{T_{ij}\}$ *for some* $i \neq j$.

Proof. We have known that $SK_n \backslash \{\phi_v\}$ is a subsemigroup of SK_n for all $v \in X_n$ by Theorem 4. So we need to prove that $SK_n \backslash \{T_{ij}\}$ is a subsemigroup of SK_n for any distinct $i, j \in \{1, 2, \ldots, n\}$. Let $i \neq j$ and $G, H \in SK_n \backslash \{T_{ij}\}$. Suppose that $GH = T_{ij}$. Thus $V(G) \cup V(H) = \{v_i, v_j\}$ and $E(GH) = \{v_i v_j\}$, that is, $V(G)$ and $V(H)$ are subsets of $\{v_i, v_j\}$. If both $E(G)$ and $E(H)$ contain $\{v_i v_j\}$, then $E(GH) \neq \{v_i v_j\}$ which is impossible. Then there exists only one of them containing $\{v_i v_j\}$. Without loss of generality, suppose that $E(G)$ contains $\{v_i v_j\}$. Since $V(G) \subseteq \{v_i, v_j\}$, we have $G = T_{ij}$ which is a contradiction. Consequently, $SK_n \backslash \{T_{ij}\}$ is a subsemigroup of SK_n. It is easy to see that $SK_n \backslash \{\phi_v\}$ and $SK_n \backslash \{T_{ij}\}$ are maximal.

Let S be a maximal subsemigroup of SK_n. We consider the following two cases.

Case 1: $\{\phi_v : v \in X_n\} \subseteq S$. Then there exists $T_{ij} \in SK_n \backslash S$, otherwise $\{T_{ij} : i \neq j\} \subseteq S$ which implies that $S = SK_n$ since $\{\phi_v : v \in X_n\} \cup \{T_{ij} : i \neq j\}$ is a generating set of SK_n, a contradiction. So $S \subseteq SK_n \backslash \{T_{ij}\}$ and by the maximality of S, we get $S = SK_n \backslash \{T_{ij}\}$, that is, S is of the form (ii).

Case 2: $\{\phi_v : v \in X_n\} \nsubseteq S$. Then $\phi_v \notin S$ for some $v \in X_n$. Thus $S \subseteq SK_n \backslash \{\phi_v\}$ and hence $S = SK_n \backslash \{\phi_v\}$ since S is a maximal subsemigroup of SK_n. Therefore, S is of the form (i). $\square$

Let ρ be a congruence on a semigroup S. We call ρ a *maximal congruence* if δ is a congruence on S with $\rho \subsetneq \delta \subseteq S \times S$ implies $\delta = S \times S$.

Theorem 7. *Let $v \in X_n$. Then $\rho = [(SK_n \backslash \{\phi_v\}) \times (SK_n \backslash \{\phi_v\})] \cup \{(\phi_v, \phi_v)\}$ is a maximal congruence on SK_n.*

Proof. It is clear that ρ is an equivalence relation on SK_n. Let $G, H, K \in SK_n$ be such that $(H, K) \in \rho$. Then $(H, K) \in (SK_n \backslash \{\phi_v\}) \times (SK_n \backslash \{\phi_v\})$ or $H = \phi_v = K$. If $(H, K) \in (SK_n \backslash \{\phi_v\}) \times (SK_n \backslash \{\phi_v\})$, then $GH, GK \in SK_n \backslash \{\phi_v\}$ since $SK_n \backslash \{\phi_v\}$ is an ideal of SK_n. Thus $(GH, GK) \in (SK_n \backslash \{\phi_v\}) \times (SK_n \backslash \{\phi_v\}) \subseteq \rho$. If $H = \phi_v = K$, then $GH = G\phi_v = GK$. Thus $(GH, GK) \in \rho$ which implies that ρ is a congruence on SK_n.

Next, we show that ρ is a maximal congruence on SK_n. Assume that δ is a congruence on SK_n such that $\rho \subsetneq \delta \subseteq SK_n \times SK_n$. Then there exists $(\phi_v, K) \in \delta$ where $K \in SK_n \backslash \{\phi_v\}$. Let $(G, H) \in SK_n \times SK_n$. If $G, H \in SK_n \backslash \{\phi_v\}$, then $(G, H) \in \rho \subseteq \delta$. If $G = \phi_v = H$, then $(G, H) \in \rho \subseteq \delta$. If $G = \phi_v$ and $H \in SK_n \backslash \{\phi_v\}$, then $(H, K) \in (SK_n \backslash \{\phi_v\}) \times (SK_n \backslash \{\phi_v\}) \subseteq \rho \subseteq \delta$. From $(\phi_v, K), (K, H) \in \delta$, we obtain by the transitivity of δ that $(G, H) = (\phi_v, H) \in \delta$. Thus $\delta = SK_n \times SK_n$, as required. $\square$

4. Natural Order on SK_n

In this section, we prescribe the natural order on SK_n and investigate minimal elements and maximal elements of SK_n with respect to this order. Furthermore, we consider lower covers and upper covers of elements that are not minimal and maximal, respectively. We also give the necessary and sufficient conditions for the existence of an infimum and a supremum of a nonempty subset of SK_n.

On an inverse semigroup S, the *natural order* is defined in a natural way. For given $a, b \in S$, we define $a \leq b$ if there exists an idempotent $e \in S$ such that $a = be$. The following theorem characterizes the natural order on SK_n.

Theorem 8. *Let $G, H \in SK_n$. Then $G \leq H$ if and only if $V(H) \subseteq V(G)$ and $E(H) = E(G)$.*

Proof. Assume that $G \leq H$. Then there exists $K \in \mathcal{E}(SK_n)$ such that $G = HK$. Thus $V(H) \subseteq V(H) \cup V(K) = V(G)$ and

$$\begin{aligned} E(G) &= [E(H) \cup E(K)] \backslash [E(H) \cap E(K)] \\ &= [E(H) \cup \emptyset] \backslash [E(H) \cap \emptyset] \\ &= E(H). \end{aligned}$$

Conversely, assume that $V(H) \subseteq V(G)$ and $E(H) = E(G)$. Therefore, $\phi_{V(G)} \in \mathcal{E}(SK_n)$. It is obvious that $V(G) = V(H) \cup V(\phi_{V(G)}) = V(H\phi_{V(G)})$ since $V(H) \subseteq V(G)$. Since $E(G) = E(H) = [E(H) \cup E(\phi_{V(G)})] \backslash [E(H) \cap E(\phi_{V(G)})] = E(H\phi_{V(G)})$, we obtain that $G = H\phi_{V(G)}$. Consequently, $G \leq H$. □

In particular, we write $G < H$ for $G \leq H$ but $G \neq H$, that is,

$$G < H \text{ if and only if } V(H) \subsetneq V(G) \text{ and } E(H) = E(G).$$

Remark 2. *In fact, the order relation $\leq$ is compatible with the multiplication on a semigroup S which can be seen in [9] (p.152).*

Let $(P, \leq)$ be a partially ordered set. An element a of P is called *minimal* if for each $p \in P$, $p \leq a$ implies $p = a$. For the definition of a *maximal element*, we can define in an analogous manner of a minimal element.

We now present the characterizations of minimal elements and maximal elements of SK_n, respectively.

Theorem 9. *Let $G \in SK_n$. Then G is minimal if and only if $V(G) = X_n$.*

Proof. Assume that G is minimal. We have known that $V(G) \subseteq X_n$. Suppose that there exists $v \in X_n \backslash V(G)$. Let H be a graph such that $V(H) = V(G) \cup \{v\}$ and $E(H) = E(G)$. Thus $V(G) \subsetneq V(H)$ and then $H < G$ which contradicts to the minimality of G. Hence $V(G) = X_n$.

Conversely, assume that $V(G) = X_n$. Let $H \in SK_n$ be such that $H \leq G$. By Theorem 8, we have $X_n = V(G) \subseteq V(H)$ and $E(G) = E(H)$. Hence $V(H) = X_n = V(G)$ which leads to $H = G$. Therefore, G is minimal. □

Theorem 10. *Let $G \in SK_n$. Then G is maximal if and only if either $G = \phi_v$ for some $v \in X_n$ or G contains no isolated vertices.*

Proof. Assume that G is maximal. Suppose that $G \neq \phi_v$ for all $v \in X_n$ and G contains an isolated vertex, say v, that is, $\deg(v) = 0$. Let H be a graph such that $V(H) = V(G) \backslash \{v\}$ and $E(H) = E(G)$. Then $V(H) \subsetneq V(G)$ which implies that $G < H$ by Theorem 8. This contradicts the maximality of G.

Conversely, it is easy to verify that G is maximal when $G = \phi_v$ for all $v \in X_n$. Now, we assume that G contains no isolated vertices. Let $H \in SK_n$ be such that $G \leq H$. Then $V(H) \subseteq V(G)$ and $E(H) = E(G)$. Let $v \in V(G)$. Since $\deg(v) > 0$, there exists $u \in V(G)$ such that $vu \in E(G) = E(H)$, and thus $v \in V(H)$. Hence $V(H) = V(G)$ which leads to $H = G$. Therefore, G is maximal in SK_n. □

Let $(P, \leq)$ be a partially ordered set. A *lower cover* of $p \in P$ is an element l of P such that $l < p$ and there is no $l' \in P$ in which $l < l' < p$. An *upper cover* of $p \in P$ is an element $u \in P$ such that $p < u$ and there is no $u' \in P$ in which $p < u' < u$.

The following lemma describes the existence of lower covers and upper covers of elements in SK_n.

Lemma 2. *Let $G \in SK_n$. Then the following statements hold:*

(i) *if G is not minimal, then G has a lower cover;*
(ii) *if G is not maximal, then G has an upper cover.*

Proof. (i) Let $G \in SK_n$ be not minimal. Thus $X_n \backslash V(G) \neq \emptyset$ by Theorem 9. Let $v \in X_n \backslash V(G)$. Define H to be a graph with a vertex set $V(H) = V(G) \cup \{v\}$ and an edge set $E(H) = E(G)$. Thus $V(G) \subsetneq V(H)$ which implies that $H < G$. Suppose that there exists $K \in SK_n$ in which $H < K < G$. We obtain that $V(G) \subsetneq V(K) \subsetneq V(H) = V(G) \cup \{v\}$, which is impossible. Consequently, H is a lower cover of G.

(ii) Assume that G is not maximal. Then $|V(G)| > 1$ and G must contain an isolated vertex, say v, by Theorem 10. Define H to be a graph with $V(H) = V(G) \backslash \{v\}$ and $E(H) = E(G)$. Then $V(H) \subsetneq V(G)$, that is, $G < H$. Suppose that there exists $K \in SK_n$ such that $G < K < H$. Hence $V(G) \backslash \{v\} = V(H) \subsetneq V(K) \subsetneq V(G)$, which is impossible. Therefore, H is an upper cover of G. $\square$

Theorem 11. *Let $G \in SK_n$ be such that G is not minimal. Then $H \in SK_n$ is a lower cover of G if and only if $V(H) = V(G) \cup \{v\}$ for some $v \in X_n \backslash V(G)$ and $E(H) = E(G)$. Consequently, the number of lower covers of G equals $|X_n \backslash V(G)|$.*

Proof. By Lemma 2, G has a lower cover. Assume that H is a lower cover of G. Then $H < G$ which implies that $V(G) \subsetneq V(H)$ and $E(H) = E(G)$. We will show that $|V(H) \backslash V(G)| = 1$. Suppose to the contrary that there exist two different vertices $v_1, v_2 \in V(H) \backslash V(G)$. Define K to be a graph with $V(K) = V(G) \cup \{v_1\}$ and $E(K) = E(G)$. Then $V(G) \subsetneq V(K)$ which implies that $K < G$. Since $V(K) = V(G) \cup \{v_1\} \subsetneq V(G) \cup \{v_1, v_2\} \subseteq V(H)$ and $E(K) = E(H)$, we have $H < K$ which is a contradiction. Hence $|V(H) \backslash V(G)| = 1$. Therefore, $V(H) = V(G) \cup \{v\}$ for some $v \in X_n \backslash V(G)$.

Conversely, assume that the conditions hold. By the same proof as given in Lemma 2(i), we obtain that H is a lower cover of G. $\square$

Now, we define the notation which is useful for proving the following theorem. Let G be a graph and v any vertex of G. Then $G - \{v\}$ denotes the subgraph of G by deleting the vertex v and all edges of G which are incident with v.

Theorem 12. *Let $G \in SK_n$ be such that G is not maximal. Then $H \in SK_n$ is an upper cover of G if and only if $H = G - \{v\}$ where v is an isolated vertex of G. Consequently, the number of upper covers of G equals the number of isolated vertices in G.*

Proof. By Lemma 2, G has an upper cover. Let $H \in SK_n$ be an upper cover of G. Then $G < H$, that is, $V(H) \subsetneq V(G)$ and $E(H) = E(G)$. It follows that $V(G) \backslash V(H) \neq \emptyset$ and every element in $V(G) \backslash V(H)$ is an isolated vertex of G. Suppose that $|V(G) \backslash V(H)| \geq 2$. Let $v_1, v_2 \in V(G) \backslash V(H)$ be different. Define K to be the graph such that $V(K) = V(H) \cup \{v_1\}$ and $E(K) = E(G) = E(H)$. Then $V(H) \subsetneq V(K) \subsetneq V(G)$ since $v_2 \in V(G) \backslash V(K)$. Hence $G < K < H$ which is a contradiction since H is an upper cover of G. Thus $|V(G) \backslash V(H)| = 1$, that is, $H = G - \{v\}$ where v is an isolated vertex.

Conversely, let $H = G - \{v\}$ where v is an isolated vertex. By the same proof as given in Lemma 2(ii), we obtain that H is an upper cover of G. $\square$

If X is a nonempty subset of a partially ordered set $(P, \leq)$, an element a of P is said to be an *infimum* of X if a satisfies the following conditions:

(i) $a \leq x$ for every $x \in X$;
(ii) for each $p \in P$ in which $p \leq x$ for all $x \in X$, if $a \leq p$, then $a = p$.

Similarly, we leave it to the reader to provide the analogous definition of a *supremum* of X.

Theorem 13. *Let $\mathcal{A}$ be a nonempty subset of SK_n. Then $\mathcal{A}$ has an infimum if and only if $E(G) = E(H)$ for all $G, H \in \mathcal{A}$.*

Proof. Assume that $\mathcal{A}$ has an infimum, say N. Hence $N \le G$ for all $G \in \mathcal{A}$ which implies that $E(N) = E(G)$ for all $G \in \mathcal{A}$. Thus $E(G) = E(H)$ for all $G, H \in \mathcal{A}$.

Conversely, assume that $E(G) = E(H)$ for all $G, H \in \mathcal{A}$. Define N to be a graph with a vertex set $V(N) = \bigcup\limits_{G \in \mathcal{A}} V(G)$ and an edge set $E(N) = E(T)$ for some $T \in \mathcal{A}$, that is, $E(N) = E(G)$ for all $G \in \mathcal{A}$. It is clear that $V(G) \subseteq V(N)$ for all $G \in \mathcal{A}$ which implies that $N \le G$ for all $G \in \mathcal{A}$. Let $K \in SK_n$ be such that $K \le G$ for all $G \in \mathcal{A}$ and $N \le K$, that is, $V(G) \subseteq V(K) \subseteq V(N)$ and $E(N) = E(G) = E(K)$ for all $G \in \mathcal{A}$. Hence $V(N) = \bigcup\limits_{G \in \mathcal{A}} V(G) \subseteq V(K)$, it follows that $K = N$. Consequently, N is an infimum of $\mathcal{A}$ which completes the proof of our assertion. □

Theorem 14. *Let $\mathcal{A}$ be a nonempty subset of SK_n. Then $\mathcal{A}$ has a supremum if and only if $\bigcap\limits_{G \in \mathcal{A}} V(G) \neq \emptyset$ and $E(G) = E(H)$ for all $G, H \in \mathcal{A}$.*

Proof. Assume that $\mathcal{A}$ has a supremum, say M. Then $G \le M$ for all $G \in \mathcal{A}$, that is, $V(M) \subseteq V(G)$ and $E(M) = E(G)$ for all $G \in \mathcal{A}$. Hence $V(M) \subseteq \bigcap\limits_{G \in \mathcal{A}} V(G)$ which implies that $\bigcap\limits_{G \in \mathcal{A}} V(G) \neq \emptyset$ and $E(G) = E(H)$ for all $G, H \in \mathcal{A}$.

Conversely, assume that the statements hold. Let M be a graph such that $V(M) = \bigcap\limits_{G \in \mathcal{A}} V(G)$ and $E(M) = E(H)$ for some $H \in \mathcal{A}$. We will show that M is a supremum. It is clear that $V(M) \subseteq V(G)$ and $E(M) = E(G)$ for all $G \in \mathcal{A}$. Thus $G \le M$ for all $G \in \mathcal{A}$. Next, let $K \in SK_n$ be such that $G \le K$ for all $G \in \mathcal{A}$ and $K \le M$. Thus $V(M) \subseteq V(K) \subseteq V(G)$ and $E(K) = E(G) = E(M)$ for all $G \in \mathcal{A}$. It follows that $V(K) \subseteq \bigcap\limits_{G \in \mathcal{A}} V(G) = V(M)$ and then $M = K$. This means that M is a supremum of $\mathcal{A}$, as required. □

Example 2. *The Figure 2 shows the Hasse diagram of SK_3 in which elements of SK_3 are illustrated in Example 1.*

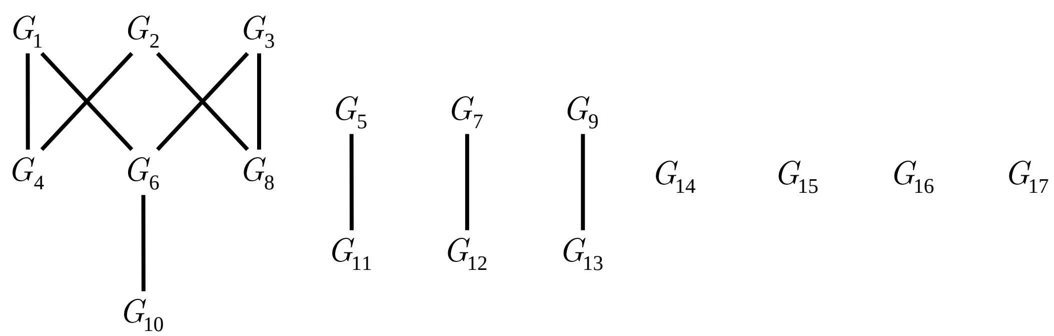

Figure 2. The Hasse diagram of SK_3.

5. Conclusions

In summary, we have found that Green's relations on an inverse semigroup SK_n coincide. All ideals, maximal ideals, and principal ideals have been characterized. In particular, the minimal ideal of SK_n is unique. Moreover, we have investigated maximal subsemigroups and maximal congruences on SK_n. Furthermore, the natural order on SK_n has been defined for considering the characterizations of minimal and maximal elements in SK_n. The necessary and sufficient conditions for a nonempty subset of SK_n to have an infimum and a supremum have been provided with certainty.

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