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Article

$\mathcal{N}$-Hyper Sets

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Abstract: To deal with the uncertainties, fuzzy set theory can be considered as one of the mathematical tools by Zadeh. As a mathematical tool to deal with negative information, Jun et al. introduced a new function, which is called a negative-valued function, and constructed $\mathcal{N}$-structures in 2009. Since then, $\mathcal{N}$-structures are applied to algebraic structures and soft sets, etc. Using the $\mathcal{N}$-structures, the notions of (extended) $\mathcal{N}$-hyper sets, $\mathcal{N}$-substructures of type 1, 2, 3 and 4 are introduced, and several related properties are investigated in this research paper.

Keywords: $\mathcal{N}$-structure; (extended) $\mathcal{N}$-hyper set; $\mathcal{N}$-substructure of types $1,2,3,4$

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1. Introduction

Most mathematical tools for computing, formal modeling and reasoning are crisp, deterministic and precise in many characters. However, several problems in economics, environment, engineering, social science, medical science, etc. do not always involve crisp data in real life. Consequently, we cannot successfully use the classical method because of various types of uncertainties presented in the problem. To deal with the uncertainties, fuzzy set theory can be considered as one of the mathematical tools (see [1]). A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0,1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. Thus far, most of the generalization of the crisp set has been conducted on the unit interval $[0,1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0,1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tools. To attain such object, Jun et al. [2] introduced a new function, which is called negative-valued function, and constructed $\mathcal{N}$-structures. Since then, $\mathcal{N}$-structures are applied to rings (see [3]), BCH-algebras (see [4]), (ordered) semigroups (see [5–8]). The combination of soft sets and $\mathcal{N}$-structures is dealt with in [9,10] and [11]. The purpose of this paper is to introduce the notions of (extended) $\mathcal{N}$-hyper sets, $\mathcal{N}$-substructures of type 1, 2, 3 and 4, and to investigate several related properties. In our consecutive research in future, we will try to study several applications based on $\mathcal{N}$-structures, for example, another type of algebra, soft and rough set theory, decision-making problems, etc. In particular, we will study complex dynamics through $\mathcal{N}$-structures based on the paper [12].

2. Preliminaries

Denote by $\mathcal{F}(X,[-1,0])$ the collection of all functions from a set X to $[-1,0]$. We say that an element of $\mathcal{F}(X,[-1,0])$ is a *negative-valued function* from X to $[-1,0]$ (briefly, $\mathcal{N}$*-function* on X). By an $\mathcal{N}$*-structure*, we mean an ordered pair (X,ρ) of X and an $\mathcal{N}$-function ρ on X (see [2]).

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\}, & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\}, & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Given a subset A of $[-1,0]$, we define

$$\ell(A) = \bigvee\{a \mid a \in A\} - \bigwedge\{a \mid a \in A\}.$$

3. (Extended) $\mathcal{N}$-Hyper Sets

Definition 1. *Let X be an initial universe set. By an $\mathcal{N}$-hyper set over X, we mean a mapping $\mu : X \to \mathcal{P}^*([-1,0])$, where $\mathcal{P}^*([-1,0])$ is the collection of all nonempty subsets of $[-1,0]$.*

In an $\mathcal{N}$-hyper set $\mu : X \to \mathcal{P}^*([-1,0])$ over X, we consider two $\mathcal{N}$-structures $(X,\mu_\wedge)$, $(X,\mu_\vee)$ and a fuzzy structure (X,μ_ℓ) in which

$$\mu_\wedge : X \to [-1,0],\ \ x \mapsto \bigwedge\{\mu(x)\}, \tag{1}$$

$$\mu_\vee : X \to [-1,0],\ \ x \mapsto \bigvee\{\mu(x)\}, \tag{2}$$

$$\mu_\ell : X \to [0,1],\ \ x \mapsto \ell(\mu(x)). \tag{3}$$

It is clear that $\mu_\ell(x) = \mu_\vee(x) - \mu_\wedge(x)$ for all $x \in X$.

Example 1. *Let $X = \{a,b,c,d\}$ and define an $\mathcal{N}$-hyper set $\mu : X \to \mathcal{P}^*([-1,0])$ over X by Table 1.*

Table 1. $\mathcal{N}$-hyper set.

X	***a***	***b***	***c***	***d***
μ	$[-0.5,0]$	$(-0.6,-0.3)$	$[-0.4,-0.2)$	$(-1,-0.8]$

Then, μ generates two $\mathcal{N}$-structures $(X,\mu_\wedge)$ and $(X,\mu_\vee)$, and a fuzzy structure (X,μ_ℓ) as Table 2.

Table 2. $\mathcal{N}$-structures $(X,\mu_\wedge)$, $(X,\mu_\vee)$ and (X,μ_ℓ).

X	***a***	***b***	***c***	***d***
$\mu_\wedge$	-0.5	-0.6	-0.4	-1
$\mu_\vee$	0	-0.3	-0.2	-0.8
μ_ℓ	0.5	0.3	0.2	0.2

Definition 2. *Given an $\mathcal{N}$-structure (X,φ) over X, define a map*

$$\varphi^e : \mathcal{P}^*(X) \to \mathcal{P}^*([-1,0]),\ \ A \mapsto \{\varphi(a) \mid a \in A\}, \tag{4}$$

where $\mathcal{P}^*(X)$ is the set of all nonempty subsets of X. We call φ^e the extended $\mathcal{N}$-hyper set over X.

Example 2. *Let $X = \{a, b, c, d\}$ be an initial universe set and let (X, φ) be an $\mathcal{N}$-structure over X given by Table 3.*

Table 3. $\mathcal{N}$-structure (X, φ).

X	a	b	c	d
φ	-0.5	-0.3	-0.4	-0.8

Then, the extended $\mathcal{N}$-hyper set φ^e over X is described as Table 4.

Table 4. The extended $\mathcal{N}$-hyper set φ^e over X.

$A \in \mathcal{P}^*(X)$	$\varphi^e(A)$	$A \in \mathcal{P}^*(X)$	$\varphi^e(A)$
$\{a\}$	$\{-0.5\}$	$\{b\}$	$\{-0.3\}$
$\{c\}$	$\{-0.4\}$	$\{d\}$	$\{-0.8\}$
$\{a, b\}$	$\{-0.5, -0.3\}$	$\{a, c\}$	$\{-0.5, -0.4\}$
$\{a, d\}$	$\{-0.5, -0.8\}$	$\{a, b, c\}$	$\{-0.5, -0.4, -0.3\}$
$\{a, b, d\}$	$\{-0.5, -0.3, -0.8\}$	$\{a, b, c, d\}$	$\{-0.5, -0.4, -0.3, -0.8\}$

Definition 3. *Let X be an initial universe set with a binary operation $*$. An $\mathcal{N}$-structure (X, φ) over X is called.*

- *an $\mathcal{N}$-substructure of $(X, *)$ with type 1 (briefly, $\mathcal{N}_1$-substructure of $(X, *)$) if it satisfies:*

$$(\forall x, y \in X) \left(\varphi(x * y) \leq \bigvee \{\varphi(x), \varphi(y)\} \right), \tag{5}$$

- *an $\mathcal{N}$-substructure of $(X, *)$ with type 2 (briefly, $\mathcal{N}_2$-substructure of $(X, *)$) if it satisfies:*

$$(\forall x, y \in X) \left(\varphi(x * y) \geq \bigwedge \{\varphi(x), \varphi(y)\} \right), \tag{6}$$

- *an $\mathcal{N}$-substructure of $(X, *)$ with type 3 (briefly, $\mathcal{N}_3$-substructure of $(X, *)$) if it satisfies:*

$$(\forall x, y \in X) \left(\varphi(x * y) \geq \bigvee \{\varphi(x), \varphi(y)\} \right), \tag{7}$$

- *an $\mathcal{N}$-substructure of $(X, *)$ with type 4 (briefly, $\mathcal{N}_4$-substructure of $(X, *)$) if it satisfies:*

$$(\forall x, y \in X) \left(\varphi(x * y) \leq \bigwedge \{\varphi(x), \varphi(y)\} \right). \tag{8}$$

It is clear that every $\mathcal{N}_4$-substructure of $(X, *)$ is an $\mathcal{N}_1$-substructure of $(X, *)$, and every $\mathcal{N}_3$-substructure of $(X, *)$ is an $\mathcal{N}_2$-substructure of $(X, *)$.

Example 3. *Let X be the set of all integers and let $*$ be a binary operation on X defined by*

$$(\forall x, y \in X) \, (x * y = -(|x| + |y|)).$$

(1) Define an $\mathcal{N}$-structure (X, φ) over X by

$$\varphi : X \to [-1, 0], \;\; x \mapsto -1 + \tfrac{1}{e^{|x|}}.$$

Then, $\varphi(0) = 0$, $\lim\limits_{|x| \to \infty} \varphi(x) = -1$ and

$$\varphi(x * y) = -1 + \tfrac{1}{e^{|x|+|y|}} \leq \bigwedge \left\{-1 + \tfrac{1}{e^{|x|}},\ -1 + \tfrac{1}{e^{|y|}}\right\} = \bigwedge\{\varphi(x), \varphi(y)\}$$

for all $x, y \in X$. *Therefore,* (X, φ) *is an* $\mathcal{N}_4$*-substructure of* $(X, *)$*, and hence it is also an* $\mathcal{N}_1$*-substructure of* $(X, *)$*.*

(2) Let (X, φ) *be an* $\mathcal{N}$*-structure over* X *in which* φ *is given by*

$$\varphi : X \to [-1, 0],\ \ x \mapsto \tfrac{-1}{1+|x|}.$$

Then,

$$\begin{aligned} \varphi(x * y) &= \varphi(-(|x| + |y|)) = \tfrac{-1}{1+|-(|x|+|y|)|} \\ &= \tfrac{-1}{1+|x|+|y|} \geq \bigvee \left\{\tfrac{-1}{1+|x|},\ \tfrac{-1}{1+|y|}\right\} \\ &= \bigvee\{\varphi(x),\ \varphi(y)\} \end{aligned}$$

for all $x, y \in X$. *Therefore,* (X, φ) *is an* $\mathcal{N}_3$*-substructure of* $(X, *)$*, and hence it is also an* $\mathcal{N}_2$*-substructure of* $(X, *)$*.*

For any initial universe set X with binary operations, let $\mathcal{H}(X)$ denote the set of all $(X, *)$ where $*$ is a binary operation on X, that is,

$$\mathcal{H}(X) := \{(X, *) \mid * \text{ is a binary operation on } X\}.$$

We consider the following subsets of $\mathcal{H}(X)$:

$$\begin{aligned} \mathcal{N}_1(\varphi) &:= \{(X, *) \in \mathcal{H}(X) \mid \varphi \text{ is an } \mathcal{N}_1\text{-substructure of } (X, *)\}, \\ \mathcal{N}_2(\varphi) &:= \{(X, *) \in \mathcal{H}(X) \mid \varphi \text{ is an } \mathcal{N}_2\text{-substructure of } (X, *)\}, \\ \mathcal{N}_3(\varphi) &:= \{(X, *) \in \mathcal{H}(X) \mid \varphi \text{ is an } \mathcal{N}_3\text{-substructure of } (X, *)\}, \\ \mathcal{N}_4(\varphi) &:= \{(X, *) \in \mathcal{H}(X) \mid \varphi \text{ is an } \mathcal{N}_4\text{-substructure of } (X, *)\}. \end{aligned}$$

Theorem 1. *Given an* $\mathcal{N}$*-structure* (X, φ) *over an initial universe set* X*, if* $(X, *) \in \mathcal{N}_1(\varphi)$*, then* $(\mathcal{P}^*(X), *) \in \mathcal{N}_1\left(\varphi^e_\wedge\right)$*.*

Proof. If $(X, *) \in \mathcal{N}_1(\varphi)$, then φ is an $\mathcal{N}_1$-substructure of $(X, *)$, that is, Equation (5) is valid. Let $A, B \in \mathcal{P}^*(X)$. Then,

$$\varphi^e_\wedge(A * B) = \bigwedge\{\varphi(a * b) \mid a \in A,\ b \in B\}. \tag{9}$$

Note that

$$(\forall \varepsilon > 0)(\exists a_0 \in X) \left(\varphi(a_0) < \bigwedge\{\varphi(a) \mid a \in A\} + \varepsilon\right)$$

and

$$(\forall \varepsilon > 0)(\exists b_0 \in X) \left(\varphi(b_0) < \bigwedge\{\varphi(b) \mid b \in B\} + \varepsilon\right).$$

It follows that

$$\begin{aligned}
&\bigwedge\{\varphi(a*b) \mid a \in A,\ b \in B\} \leq \varphi(a_0 * b_0) \leq \bigvee\{\varphi(a_0), \varphi(b_0)\} \\
&\leq \bigvee\left\{\bigwedge\{\varphi(a) \mid a \in A\} + \varepsilon,\ \bigwedge\{\varphi(b) \mid b \in B\} + \varepsilon\right\} \\
&= \bigvee\{\varphi^e_\wedge(A) + \varepsilon,\ \varphi^e_\wedge(A) + \varepsilon\} \\
&= \bigvee\{\varphi^e_\wedge(A),\ \varphi^e_\wedge(A)\} + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, it follows that

$$\varphi^e_\wedge(A * B) \leq \bigvee\{\varphi^e_\wedge(A),\ \varphi^e_\wedge(A)\}.$$

Therefore, $(\mathcal{P}^*(X), *) \in \mathcal{N}_1(\varphi^e_\wedge)$. □

Theorem 2. *Given an $\mathcal{N}$-structure (X, φ) over an initial universe set X, if $(X, *) \in \mathcal{N}_2(\varphi)$, then $(\mathcal{P}^*(X), *) \in \mathcal{N}_2(\varphi^e_\vee)$.*

Proof. If $(X, *) \in \mathcal{N}_2(\varphi)$, then φ is an $\mathcal{N}_2$-substructure of $(X, *)$, that is, Equation (6) is valid. Let $A, B \in \mathcal{P}^*(X)$. Then,

$$\varphi^e_\vee(A * B) = \bigvee\{\varphi(a*b) \mid a \in A,\ b \in B\}. \tag{10}$$

Let ε be any positive number. Then, there exist $a_0, b_0 \in X$ such that

$$\begin{aligned}
&\left(\varphi(a_0) > \bigvee\{\varphi(a) \mid a \in A\} - \varepsilon\right), \\
&\left(\varphi(b_0) > \bigvee\{\varphi(b) \mid b \in B\} - \varepsilon\right),
\end{aligned}$$

respectively. It follows that

$$\begin{aligned}
&\bigvee\{\varphi(a*b) \mid a \in A,\ b \in B\} \geq \varphi(a_0 * b_0) \geq \bigwedge\{\varphi(a_0),\ \varphi(b_0)\} \\
&\geq \bigwedge\left\{\bigvee\{\varphi(a) \mid a \in A\} - \varepsilon,\ \bigvee\{\varphi(b) \mid b \in B\} - \varepsilon\right\} \\
&= \bigwedge\{\varphi^e_\vee(A) - \varepsilon,\ \varphi^e_\vee(B) - \varepsilon\} \\
&= \bigwedge\{\varphi^e_\vee(A),\ \varphi^e_\vee(B)\} - \varepsilon,
\end{aligned}$$

which shows that $\varphi^e_\vee(A * B) \geq \bigwedge\{\varphi^e_\vee(A),\ \varphi^e_\vee(B)\}$. Therefore, $(\mathcal{P}^*(X), *) \in \mathcal{N}_2(\varphi^e_\vee)$. □

Definition 4. *Given $\mathcal{N}$-hyper sets μ and λ over an initial universe set X, we define hyper-union ($\tilde{\cup}$), hyper-intersection ($\tilde{\cap}$), hyper complement ($'$) and hyper difference ($\setminus$) as follows:*

$$\begin{aligned}
&\mu \,\tilde{\cup}\, \lambda : X \to \mathcal{P}^*([-1, 0]),\ \ x \mapsto \mu(x) \cup \lambda(x), \\
&\mu \,\tilde{\cap}\, \lambda : X \to \mathcal{P}^*([-1, 0]),\ \ x \mapsto \mu(x) \cap \lambda(x), \\
&\mu \setminus \lambda : X \to \mathcal{P}^*([-1, 0]),\ \ x \mapsto \mu(x) \setminus \lambda(x), \\
&\mu' : X \to \mathcal{P}^*([-1, 0]),\ \ x \mapsto [-1, 0] \setminus \{t \in [-1, 0] \mid t \notin \mu(x)\}.
\end{aligned}$$

Proposition 1. *If μ and λ are $\mathcal{N}$-hyper sets over an initial universe set X, then*

$$(\forall x \in X)\left((\mu \,\tilde{\cup}\, \lambda)_\ell(x) \geq \bigvee\{\mu_\ell(x), \lambda_\ell(x)\}\right), \tag{11}$$

and

$$(\forall x \in X)\left((\mu \tilde{\cap} \lambda)_{\ell}(x) \leq \bigwedge\{\mu_{\ell}(x), \lambda_{\ell}(x)\}\right). \tag{12}$$

Proof. Let $x \in X$. Then,

$$(\mu \tilde{\cup} \lambda)_{\vee}(x) = \bigvee\{\mu(x) \cup \lambda(x)\} \geq \bigvee\{\mu(x)\} \text{ (and } \bigvee\{\lambda(x)\})$$

and

$$(\mu \tilde{\cup} \lambda)_{\wedge}(x) = \bigwedge\{\mu(x) \cup \lambda(x)\} \leq \bigwedge\{\mu(x)\} \text{ (and } \bigwedge\{\lambda(x)\}).$$

It follows that

$$(\mu \tilde{\cup} \lambda)_{\vee}(x) \geq \bigvee\left\{\bigvee\{\mu(x)\},\ \bigvee\{\lambda(x)\}\right\}$$

and

$$(\mu \tilde{\cup} \lambda)_{\wedge}(x) \leq \bigwedge\left\{\bigwedge\{\mu(x)\},\ \bigwedge\{\lambda(x)\}\right\}.$$

Note that $\bigvee\{a, b\} + \bigvee\{c, d\} \geq \bigvee\{a + c,\ b + d\}$ for all $a, b, c, d \in [-1, 0]$. Hence,

$$\begin{aligned}(\mu \tilde{\cup} \lambda)_{\ell}(x) &= (\mu \tilde{\cup} \lambda)_{\vee}(x) - (\mu \tilde{\cup} \lambda)_{\wedge}(x) \\ &\geq \bigvee\left\{\bigvee\{\mu(x)\},\ \bigvee\{\lambda(x)\}\right\} - \bigwedge\left\{\bigwedge\{\mu(x)\},\ \bigwedge\{\lambda(x)\}\right\} \\ &\geq \bigvee\left\{\bigvee\{\mu(x)\},\ \bigvee\{\lambda(x)\}\right\} + \bigvee\left\{-\bigwedge\{\mu(x)\},\ -\bigwedge\{\lambda(x)\}\right\} \\ &\geq \bigvee\left\{\bigvee\{\mu(x)\} - \bigwedge\{\mu(x)\},\ \bigvee\{\lambda(x)\} - \bigwedge\{\lambda(x)\}\right\} \\ &= \bigvee\{\mu_{\ell}(x),\ \lambda_{\ell}(x)\},\end{aligned}$$

and so Equation (11) is valid. For any $x \in X$, we have

$$(\mu \tilde{\cap} \lambda)_{\vee}(x) = \bigvee\{\mu(x) \cap \lambda(x)\} \leq \bigvee\{\mu(x)\} \text{ (and } \bigvee\{\lambda(x)\})$$

and

$$(\mu \tilde{\cap} \lambda)_{\wedge}(x) = \bigwedge\{\mu(x) \cap \lambda(x)\} \geq \bigwedge\{\mu(x)\} \text{ (and } \bigwedge\{\lambda(x)\}),$$

which imply that

$$(\mu \tilde{\cap} \lambda)_{\vee}(x) \leq \bigwedge\left\{\bigvee\{\mu(x)\},\ \bigvee\{\lambda(x)\}\right\}$$

and

$$(\mu \tilde{\cap} \lambda)_{\wedge}(x) \geq \bigvee\left\{\bigwedge\{\mu(x)\},\ \bigwedge\{\lambda(x)\}\right\}.$$

Since $\bigwedge\{a, b\} + \bigwedge\{c, d\} \leq \bigwedge\{a + c,\ b + d\}$ for all $a, b, c, d \in [-1, 0]$, we have

$$
\begin{aligned}
(\mu \tilde{\cap} \lambda)_\ell(x) &= (\mu \tilde{\cap} \lambda)_\vee(x) - (\mu \tilde{\cap} \lambda)_\wedge(x) \\
&\leq \bigwedge \left\{ \bigvee\{\mu(x)\},\ \bigvee\{\lambda(x)\} \right\} - \bigvee \left\{ \bigwedge\{\mu(x)\},\ \bigwedge\{\lambda(x)\} \right\} \\
&= \bigwedge \left\{ \bigvee\{\mu(x)\},\ \bigvee\{\lambda(x)\} \right\} + \bigwedge \left\{ - \bigwedge\{\mu(x)\},\ - \bigwedge\{\lambda(x)\} \right\} \\
&\leq \bigwedge \left\{ \bigvee\{\mu(x)\} - \bigwedge\{\mu(x)\},\ \bigvee\{\lambda(x)\} - \bigwedge\{\lambda(x)\} \right\} \\
&= \bigwedge \{\mu_\ell(x), \lambda_\ell(x)\}.
\end{aligned}
$$

This completes the proof. □

Proposition 2. *If μ is an $\mathcal{N}$-hyper set over an initial universe set X, then*

$$(\forall x \in X) \left((\mu \tilde{\cup} \mu')_\ell(x) \geq \bigvee \{\mu_\ell(x),\ \mu'_\ell(x)\} \right). \tag{13}$$

Proof. Note that

$$(\mu \tilde{\cup} \mu')(x) = \mu(x) \cup \mu'(x) = \mu(x) \cup ([-1, 0] \setminus \mu(x)) = [-1, 0]$$

for all $x \in X$. It follows that

$$(\mu \tilde{\cup} \mu')_\ell(x) = (\mu \tilde{\cup} \mu')_\vee(x) - (\mu \tilde{\cup} \mu')_\wedge(x) = 1 \geq \bigvee \{\mu_\ell(x),\ \mu'_\ell(x)\}$$

for all $x \in X$. □

Proposition 3. *If μ and λ are $\mathcal{N}$-hyper sets over an initial universe set X, then*

$$(\forall x \in X) \left((\mu \setminus \lambda)_\ell(x) \leq \mu_\ell(x) \right). \tag{14}$$

Proof. Note that $(\mu \setminus \lambda)(x) = \mu(x) \setminus \lambda(x) \subseteq \mu(x)$ for all $x \in X$. Hence,

$$(\mu \setminus \lambda)_\vee(x) \leq \mu_\vee(x) \text{ and } (\mu \setminus \lambda)_\wedge(x) \geq \mu_\wedge(x).$$

It follows that

$$
\begin{aligned}
(\mu \setminus \lambda)_\ell(x) &= (\mu \setminus \lambda)_\vee(x) - (\mu \setminus \lambda)_\wedge(x) \\
&\leq \mu_\vee(x) - \mu_\wedge(x) = \mu_\ell(x),
\end{aligned}
$$

proving the proposition. □

Given $\mathcal{N}$-hyper sets μ and λ over an initial universe set X, we define

$$\mu_\xi : X \to [-1, 0],\ \ x \mapsto \mu_\wedge(x) - \mu_\vee(x), \tag{15}$$

$$\mu \tilde{\vee} \lambda : X \to \mathcal{P}^*([-1, 0]),\ \ x \mapsto \left\{ \bigvee\{a, b\} \in [-1, 0] \mid a \in \mu(x), b \in \lambda(x) \right\}, \tag{16}$$

$$\mu \tilde{\wedge} \lambda : X \to \mathcal{P}^*([-1, 0]),\ \ x \mapsto \left\{ \bigwedge\{a, b\} \in [-1, 0] \mid a \in \mu(x), b \in \lambda(x) \right\}. \tag{17}$$

Example 4. *Let μ and λ be $\mathcal{N}$-hyper sets over $X = \{a, b, c, d\}$ defined by Table 5.*

Table 5. $\mathcal{N}$-hyper sets μ and λ.

X	***a***	***b***	***c***	***d***
μ	$[-0.5, 0]$	$\{-1, -0.6\}$	$[-0.4, -0.2)$	$(-1, -0.8]$
λ	$[-0.6, -0.3]$	$\{-1, -0.8, -0.5\}$	$[-0.5, -0.3)$	$(-0.9, -0.7]$

Then, μ_ξ is given as Table 6

Table 6. $\mathcal{N}$-function (X, μ_ξ).

X	***a***	***b***	***c***	***d***
$\mu_\wedge$	-0.5	-1	-0.4	-1
$\mu_\vee$	0	-0.6	-0.2	-0.8
μ_ξ	-0.5	-0.4	-0.2	-0.2

and

$$\begin{aligned}(\mu \tilde{\vee} \lambda)(b) = \Big\{ & \bigvee\{-1,-1\}, \bigvee\{-1,-0.8\}, \bigvee\{-1,-0.5\}, \\ & \bigvee\{-0.6,-1\}, \bigvee\{-0.6,-0.8\}, \bigvee\{-0.6,-0.5\}\Big\} \\ = \{-1,-0.8,-0.5,-0.6\}&, \\ (\mu \tilde{\wedge} \lambda)(b) = \Big\{ & \bigwedge\{-1,-1\}, \bigwedge\{-1,-0.8\}, \bigwedge\{-1,-0.5\}, \\ & \bigwedge\{-0.6,-1\}, \bigwedge\{-0.6,-0.8\}, \bigwedge\{-0.6,-0.5\}\Big\} \\ = \{-1,-0.8,-0.6\}&.\end{aligned}$$

Thus, $(\mu \tilde{\vee} \lambda)_\vee(b) = -0.5$, $(\mu \tilde{\wedge} \lambda)_\vee(b) = -0.6$ and $(\mu \tilde{\vee} \lambda)_\wedge(b) = -1 = (\mu \tilde{\wedge} \lambda)_\wedge(b)$.

Proposition 4. *Let X be an initial universe set with a binary operation $*$. If μ and λ are $\mathcal{N}$-hyper sets over X, then*

$$(\forall x \in X)\left((\mu \tilde{\vee} \lambda)_\vee(x) = \bigvee\{\mu_\vee(x), \lambda_\vee(x)\}\right) \tag{18}$$

and

$$(\forall x \in X)\left((\mu \tilde{\vee} \lambda)_\wedge(x) = \bigvee\{\mu_\wedge(x), \lambda_\wedge(x)\}\right). \tag{19}$$

Proof. For any $x \in X$, let $\alpha := \mu_\vee(x)$ and $\beta := \lambda_\vee(x)$. Then,

$$\begin{aligned}(\mu \tilde{\vee} \lambda)_\vee(x) &= \bigvee\{(\mu \tilde{\vee} \lambda)(x)\} \\ &= \bigvee\Big\{\bigvee\{a,b\} \in [-1,0] \mid a \in \mu(x), b \in \lambda(x)\Big\} \\ &= \bigvee\Big\{\bigvee\{\alpha, b \mid b \in \lambda(x)\}, \bigvee\{a, b \mid a \in \mu(x), b \in \lambda(x)\}, \\ &\qquad \bigvee\{a, \beta \mid a \in \mu(x)\}, \bigvee\{\alpha, \beta\}\Big\} \\ &= \bigvee\{\alpha, \beta\} = \bigvee\{\mu_\vee(x), \lambda_\vee(x)\}.\end{aligned}$$

Thus, Equation (18) is valid. Similarly, we can prove Equation (19). □

Similarly, we have the following property.

Proposition 5. *Let X be an initial universe set with a binary operation $*$. If μ and λ are $\mathcal{N}$-hyper sets over X, then*

$$(\forall x \in X) \left((\mu \,\tilde{\curlywedge}\, \lambda)_\vee(x) = \bigwedge \{\mu_\vee(x), \lambda_\vee(x)\} \right) \tag{20}$$

and

$$(\forall x \in X) \left((\mu \,\tilde{\curlywedge}\, \lambda)_\wedge(x) = \bigwedge \{\mu_\wedge(x), \lambda_\wedge(x)\} \right). \tag{21}$$

Definition 5. *Let X be an initial universe set with a binary operation $*$. An $\mathcal{N}$-hyper set $\mu : X \to \mathcal{P}^*([-1,0])$ is called: an $\mathcal{N}$-hyper subset of $(X,*)$ with type (i,j) for $i,j \in \{1,2,3,4\}$ (briefly, $\mathcal{N}_{(i,j)}$-substructure of $(X,*)$) if $(X,\mu_\vee)$ is an $\mathcal{N}_i$-substructure of $(X,*)$ and $(X,\mu_\wedge)$ is an $\mathcal{N}_j$-substructure of $(X,*)$.*

Given an $\mathcal{N}$-hyper set $\mu : X \to \mathcal{P}^*([-1,0])$, we consider the set

$$\mathcal{N}_{(i,j)}(\mu) := \{(X,*) \in \mathcal{H}(X) \mid \mu \text{ is an } \mathcal{N}_{(i,j)}\text{-substructure of } (X,*)\}$$

for $i,j \in \{1,2,3,4\}$.

Theorem 3. *Let X be an initial universe set with a binary operation $*$. For any $\mathcal{N}$-hyper set $\mu : X \to \mathcal{P}^*([-1,0])$, we have*

$$(X,*) \in \mathcal{N}_{(3,4)}(\mu) \;\Rightarrow (X,*) \in \mathcal{N}_4(\mu_\xi). \tag{22}$$

Proof. Let $(X,*) \in \mathcal{N}_{(3,4)}(\mu)$. Then, $(X,\mu_\vee)$ is an $\mathcal{N}_3$-substructure of $(X,*)$ and $(X,\mu_\wedge)$ is an $\mathcal{N}_4$-substructure of $(X,*)$, that is,

$$\mu_\vee(x*y) \geq \bigvee\{\mu_\vee(x), \mu_\vee(y)\}$$

and

$$\mu_\wedge(x*y) \leq \bigwedge\{\mu_\wedge(x), \mu_\wedge(y)\}$$

for all $x,y \in X$. It follows that

$$\mu_\xi(x*y) = \mu_\wedge(x*y) - \mu_\vee(x*y) \leq \mu_\wedge(x) - \mu_\vee(x) = \mu_\xi(x).$$

Similarly, we get $\mu_\xi(x*y) \leq \mu_\xi(y)$. Hence, $\mu_\xi(x*y) \leq \bigwedge\{\mu_\xi(x), \mu_\xi(y)\}$, and so $(X,*) \in \mathcal{N}_4(\mu_\xi)$. □

Corollary 1. *Let X be an initial universe set with a binary operation $*$. For any $\mathcal{N}$-hyper set $\mu : X \to \mathcal{P}^*([-1,0])$, we have*

$$(X,*) \in \mathcal{N}_{(3,4)}(\mu) \;\Rightarrow (X,*) \in \mathcal{N}_1(\mu_\xi).$$

Theorem 4. *Let X be an initial universe set with a binary operation $*$. For any $\mathcal{N}$-hyper set $\mu : X \to \mathcal{P}^*([-1,0])$, we have*

$$(X,*) \in \mathcal{N}_{(4,3)}(\mu) \;\Rightarrow (X,*) \in \mathcal{N}_3(\mu_\xi). \tag{23}$$

Proof. It is similar to the proof of Theorem 3. □

Corollary 2. *Let X be an initial universe set with a binary operation $*$. For any $\mathcal{N}$-hyper set $\mu : X \to \mathcal{P}^*([-1,0])$, we have*

$$(X,*) \in \mathcal{N}_{(4,3)}(\mu) \ \Rightarrow (X,*) \in \mathcal{N}_2(\mu_\xi).$$

Theorem 5. *Let X be an initial universe set with a binary operation $*$. For any $\mathcal{N}$-hyper set $\mu : X \to \mathcal{P}^*([-1,0])$, we have*

$$(X,*) \in \mathcal{N}_{(1,3)}(\mu) \ \Rightarrow (X,*) \in \mathcal{N}_3(\mu_\xi). \tag{24}$$

Proof. Let $(X,*) \in \mathcal{N}_{(1,3)}(\mu)$. Then, $(X,\mu_\vee)$ is an $\mathcal{N}_1$-substructure of $(X,*)$ and $(X,\mu_\wedge)$ is an $\mathcal{N}_3$-substructure of $(X,*)$, that is,

$$\mu_\vee(x*y) \leq \bigvee\{\mu_\vee(x),\mu_\vee(y)\} \tag{25}$$

and

$$\mu_\wedge(x*y) \geq \bigvee\{\mu_\wedge(x),\mu_\wedge(y)\}$$

for all $x,y \in X$. Equation (25) implies that

$$\mu_\vee(x*y) \leq \mu_\vee(x) \text{ or } \mu_\vee(x*y) \leq \mu_\vee(y).$$

If $\mu_\vee(x*y) \leq \mu_\vee(x)$, then

$$\mu_\xi(x*y) = \mu_\wedge(x*y) - \mu_\vee(x*y) \geq \mu_\wedge(x) - \mu_\vee(x) = \mu_\xi(x).$$

If $\mu_\vee(x*y) \leq \mu_\vee(y)$, then

$$\mu_\xi(x*y) = \mu_\wedge(x*y) - \mu_\vee(x*y) \geq \mu_\wedge(y) - \mu_\vee(y) = \mu_\xi(y).$$

It follows that $\mu_\xi(x*y) \geq \bigvee\{\mu_\xi(x),\mu_\xi(y)\}$, and so $(X,*) \in \mathcal{N}_3(\mu_\xi)$. □

Corollary 3. *Let X be an initial universe set with a binary operation $*$. For any $\mathcal{N}$-hyper set $\mu : X \to \mathcal{P}^*([-1,0])$, we have*

$$(X,*) \in \mathcal{N}_{(1,3)}(\mu) \ \Rightarrow (X,*) \in \mathcal{N}_2(\mu_\xi). \tag{26}$$

Theorem 6. *Let X be an initial universe set with a binary operation $*$. For any $\mathcal{N}$-hyper set $\mu : X \to \mathcal{P}^*([-1,0])$, we have*

$$(X,*) \in \mathcal{N}_{(3,1)}(\mu) \ \Rightarrow (X,*) \in \mathcal{N}_1(\mu_\xi). \tag{27}$$

Proof. It is similar to the proof of Theorem 5. □

Theorem 7. *Let X be an initial universe set with a binary operation $*$. For any $\mathcal{N}$-hyper set $\mu : X \to \mathcal{P}^*([-1,0])$, we have*

$$(X,*) \in \mathcal{N}_{(2,4)}(\mu) \ \Rightarrow (X,*) \in \mathcal{N}_1(\mu_\xi). \tag{28}$$

Proof. Let $(X, *) \in \mathcal{N}_{(2,4)}(\mu)$. Then, $(X, \mu_\vee)$ is an $\mathcal{N}_2$-substructure of $(X, *)$ and $(X, \mu_\wedge)$ is an $\mathcal{N}_4$-substructure of $(X, *)$, that is,

$$\mu_\vee(x * y) \geq \bigwedge\{\mu_\vee(x), \mu_\vee(y)\} \tag{29}$$

and

$$\mu_\wedge(x * y) \leq \bigwedge\{\mu_\wedge(x), \mu_\wedge(y)\}$$

for all $x, y \in X$. Then, $\mu_\vee(x * y) \geq \mu_\vee(x)$ or $\mu_\vee(x * y) \geq \mu_\vee(y)$ by Equation (29). If $\mu_\vee(x * y) \geq \mu_\vee(x)$, then

$$\mu_\xi(x * y) = \mu_\wedge(x * y) - \mu_\vee(x * y) \leq \mu_\wedge(x) - \mu_\vee(x) = \mu_\xi(x).$$

If $\mu_\vee(x * y) \geq \mu_\vee(y)$, then

$$\mu_\xi(x * y) = \mu_\wedge(x * y) - \mu_\vee(x * y) \leq \mu_\wedge(y) - \mu_\vee(y) = \mu_\xi(y).$$

It follows that $\mu_\xi(x * y) \leq \bigvee\{\mu_\xi(x), \mu_\xi(y)\}$, that is, $(X, *) \in \mathcal{N}_1(\mu_\xi)$. □

Theorem 8. *Let X be an initial universe set with a binary operation $*$. For any $\mathcal{N}$-hyper set $\mu : X \to \mathcal{P}^*([-1, 0])$, if $(X, *) \in \mathcal{N}_{(4,2)}(\mu)$, then*

$$(\forall x, y \in X) \left(\mu_\ell(x * y) \leq \bigvee\{\mu_\ell(x), \mu_\ell(y)\}\right). \tag{30}$$

Proof. If $(X, *) \in \mathcal{N}_{(4,2)}(\mu)$, then $(X, \mu_\vee)$ is an $\mathcal{N}_4$-substructure of $(X, *)$ and $(X, \mu_\wedge)$ is an $\mathcal{N}_2$-substructure of $(X, *)$, that is,

$$\mu_\vee(x * y) \leq \bigwedge\{\mu_\vee(x), \mu_\vee(y)\}$$

and

$$\mu_\wedge(x * y) \geq \bigwedge\{\mu_\wedge(x), \mu_\wedge(y)\} \tag{31}$$

for all $x, y \in X$. Then, $\mu_\wedge(x * y) \geq \mu_\wedge(x)$ or $\mu_\wedge(x * y) \geq \mu_\wedge(y)$ by Equation (31). If $\mu_\wedge(x * y) \geq \mu_\wedge(x)$, then

$$\mu_\ell(x * y) = \mu_\vee(x * y) - \mu_\wedge(x * y) \leq \mu_\vee(x) - \mu_\wedge(x) = \mu_\ell(x).$$

If $\mu_\wedge(x * y) \geq \mu_\wedge(y)$, then

$$\mu_\ell(x * y) = \mu_\vee(x * y) - \mu_\wedge(x * y) \leq \mu_\vee(y) - \mu_\wedge(y) = \mu_\ell(y).$$

It follows that $\mu_\ell(x * y) \leq \bigvee\{\mu_\ell(x), \mu_\ell(y)\}$ for all $x, y \in X$. □

4. Conclusions

Fuzzy set theory has been considered by Zadeh as one of the mathematical tools to deal with the uncertainties. Because fuzzy set theory could not deal with negative information, Jun et al. have introduced a new function, which is called negative-valued function, and constructed $\mathcal{N}$-structures in 2009 as a mathematical tool to deal with negative information. Since then, $\mathcal{N}$-structures have been applied to algebraic structures and soft sets, etc. Using the $\mathcal{N}$-structures, in this article, we have studied the notions of (extended) $\mathcal{N}$-hyper sets, $\mathcal{N}$-substructures of type 1, 2, 3 and 4, and have been investigated several related properties.

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