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Complex Symmetric Formulation of Maxwell Equations for Fields and Potentials

George Livadiotis

Southwest Research Institute, Space Science & Engineering, San Antonio, TX 78238, USA; glivadiotis@swri.edu; Tel.: +1-210-274-4028

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Abstract: Maxwell equations have two types of asymmetries between the electric and magnetic fields. The first asymmetry is the inhomogeneity induced by the absence of magnetic charge sources. The second asymmetry is due to parity. We show how both asymmetries are naturally resolved under an alternative formulation of Maxwell equations for fields or potentials that uses a compact complex vector operator representation. The developed complex symmetric operator formalism can be easily applied to performing the continuity equation, the field wave equations, the Maxwell equations for potentials, the gauge transformations, and the 4-momentum representation; in general, the developed formalism constitutes a simple way of unfolding the Maxwell theory. Finally, we provide insights for extending the presented analysis within the context of (i) bicomplex numbers and tessarine algebra; and (ii) L^p-spaces in nonlinear Maxwell equations.

Keywords: Maxwell equations; complex representation; E/M waves; gauge transformation; gravitomagnetism; L^p norms

1. Introduction

There are various representations of Maxwell equations. Some examples are the following: standard complex representation [1,2], spinor form [3], Silberstein–Bateman–Majorana form [4–6], Kemmer–Duffin–Petiau form (also known as the meson algebra) [4,7], matrix representation [8], Dirac form [9–11], Poincaré algebra [12], Debye sources [13,14], Penrose's transformation presented in terms of integral geometry [15,16], integral representation [17], and multipolar presentation [18].

This paper uses the complex vector representation of Maxwell equations in order to develop the presented *complex operator formalism*. This developed formalism: (i) emerges naturally from the symmetry between electric and magnetic fields; and (ii) exhibits a compact set of equations for the fields and their potentials.

Most importantly, the presented formulation of Maxwell equations constitutes a much simpler and compact way of unfolding the Maxwell theory compared to previous complex formulations (e.g., continuity equation, wave equations, Maxwell equations for potentials, gauge symmetry).

The presented formulation of Maxwell equations constitutes a much simpler way of unfolding the Maxwell theory compared with previous complex formulations (e.g., continuity equation, wave equations, Maxwell equations for potentials, gauge symmetry). The analysis can trigger several theoretical developments and applications different from the standard Maxwell equations. Indeed, in the last section, we expose two examples where the presented analysis can be applied and extended, that is, within the context of (i) bicomplex numbers and tessarine algebra; and (ii) L^p-spaces in nonlinear Maxwell equations.

Next, in Section 2, we present the compact complex formalism of Maxwell equations. In Section 3, we apply this formalism in the derivations of the basic concepts of (i) continuity equation; (ii) wave equations; (iii) Maxwell equations for potentials; (iv) gauge transformation; and (v) 4-momentum of electromagnetic field. In Section 4, we summarize the conclusions, while in Section 5, we discuss what's next for further theoretical developments and applications of this formalism.

2. Compact Complex Representation of Maxwell Equations

The differential formalism of Maxwell equations for the electric $\vec{E}$ and magnetic $\vec{B}$ fields in the presence of electric charge sources with density $\boldsymbol{\rho_e}$ and current $\vec{\mathbf{J}}_{\boldsymbol{e}}$, are written as:

$$\vec{\nabla} \cdot \vec{E} = \frac{\boldsymbol{\rho_e}}{\varepsilon_0} \tag{1a}$$

$$\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B} - \mu_0 \vec{\mathbf{J}}_{\boldsymbol{e}} \tag{1b}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1c}$$

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E} \tag{1d}$$

which is consistent [19] with the sources continuity equation (by applying $\partial/\partial t$ and $-\vec{\nabla}$ to (1a) and (1b), respectively, then, summing):

$$\frac{\partial \boldsymbol{\rho_e}}{\partial t} + \vec{\nabla} \cdot \vec{\mathbf{J}}_{\boldsymbol{e}} = 0. \tag{2}$$

(Note: Electric/magnetic charge sources are denoted in bold letters throughout the letter.) The first two equations are inhomogeneous due to the electric charge sources. The source-free equations are homogeneous, but still suffer from the units and parity asymmetry:

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{3a}$$

$$\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B} \tag{3b}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{3c}$$

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E} \tag{3d}$$

The units asymmetry can be easily resolved by setting the spatial coordinates (ct,x,y,z) and the magnetic field $c\vec{B}$. In this way, the:

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{4a}$$

$$\frac{\partial \vec{E}}{\partial ct} = \vec{\nabla} \times c\vec{B} \tag{4b}$$

$$\vec{\nabla} \cdot c\vec{B} = 0 \tag{4c}$$

$$\frac{\partial c\vec{B}}{\partial ct} = -\vec{\nabla} \times \vec{E} \tag{4d}$$

(where we used $c^2 \mu_0 \varepsilon_0 = 1$).

The parity transformation flips the sign of spatial coordinates, $\hat{P}\vec{r} = -\vec{r}$. The electric field has parity -1 (as any vector), while the magnetic field has parity +1 (as any axial vector, defined by a curl of a vector). Using the complex Riemann–Silberstein vector field $\vec{G} \equiv \vec{E} + ic\vec{B}$ [20] and Minkowski metric [21], the spacetime is represented by four equivalent spatial components (*ict*,*x*,*y*,*z*) and the parity of $ic\vec{B}$ is -1 (as any regular vector),

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{5a}$$

$$\frac{\partial \vec{E}}{\partial ict} = -\vec{\nabla} \times ic\vec{B} \tag{5b}$$

$$\vec{\nabla} \cdot ic\vec{B} = 0 \tag{5c}$$

$$\frac{\partial ic\vec{B}}{\partial ict} = -\vec{\nabla} \times \vec{E} \tag{5d}$$

The Maxwell equations in (5) are symmetric. In fact, they can be compacted to:

$$\vec{\nabla} \cdot \vec{G} = 0 \tag{6a}$$

$$\left(\frac{\partial}{\partial ict} + \vec{\nabla}\times\right)\vec{G} = 0 \tag{6b}$$

by setting:

$$\vec{G} \equiv \vec{E} + ic\vec{B}, \tag{7}$$

(where $ic\vec{B}$ and $\vec{G}$ are vectors having parity -1).

The corresponding equations with electric charge sources are:

$$\vec{\nabla} \cdot \vec{E} = \frac{\boldsymbol{\rho_e}}{\varepsilon_0} \tag{8a}$$

$$\frac{\partial \vec{E}}{\partial ict} = -\vec{\nabla} \times ic\vec{B} + \mu_0 ic\vec{\mathbf{J}}_{\boldsymbol{e}} \tag{8b}$$

$$\vec{\nabla} \cdot ic\vec{B} = 0 \tag{8c}$$

$$\frac{\partial ic\vec{B}}{\partial ict} = -\vec{\nabla} \times \vec{E} \tag{8d}$$

which can be compacted, analogously to Equation (6):

$$\vec{\nabla} \cdot \vec{G} = \frac{\boldsymbol{\rho_e}}{\varepsilon_0} \tag{9a}$$

$$\left(\frac{\partial}{\partial ict} + \vec{\nabla}\times\right)\vec{G} = \mu_0 ic\vec{\mathbf{J}}_{\boldsymbol{e}} \tag{9b}$$

The Maxwell equations in Equation (8) are still asymmetric due to the absence of magnetic charge sources (monopoles). If there were monopoles, Maxwell equations would be written in the symmetric form:

$$\vec{\nabla} \cdot \vec{E} = (\mu_0 c) \cdot (c\boldsymbol{\rho_e}) \tag{10a}$$

$$\frac{\partial \vec{E}}{\partial (ict)} = -\vec{\nabla} \times ic\vec{B} - (\mu_0 c) \cdot (\vec{\mathbf{J}}_{\boldsymbol{e}}/i) \tag{10b}$$

$$\vec{\nabla} \cdot (ic\vec{B}) = (i\mu_{0m} c) \cdot (c\boldsymbol{\rho_m}) \tag{10c}$$

$$\frac{\partial (ic\vec{B})}{\partial (ict)} = -\vec{\nabla} \times \vec{E} - (i\mu_{0m} c) \cdot (\vec{\mathbf{J}}_{\boldsymbol{m}}/i) \tag{10d}$$

where the magnetic permeability caused by moving electric charges μ_0 or by magnetic monopoles μ_{0m} can be symbolized by the complex $\boldsymbol{\mu} = \mu_0 + i\mu_{0m}$. The unified complex charge density and current can be set by:

$$(c\boldsymbol{\rho}) \equiv (\mu_0 c)(c\boldsymbol{\rho_e}) + (i\mu_{0m}c)(c\boldsymbol{\rho_m}) \tag{11a}$$

$$(\vec{\mathbf{J}}/i) \equiv (\mu_0 c)(\vec{\mathbf{J}}_{\boldsymbol{e}}/i) + (i\mu_{0m}c)(\vec{\mathbf{J}}_{\boldsymbol{m}}/i) \tag{11b}$$

The compact equations become:

$$\vec{\nabla} \cdot \vec{G} = c\boldsymbol{\rho} \tag{12a}$$

$$\left(\frac{\partial}{\partial ict} + \vec{\nabla}\times\right)\vec{G} = i\,\vec{\mathbf{J}} \tag{12b}$$

Moreover, we define the 4-current:

$$\mathbf{j} = \begin{pmatrix} c\rho \\ i\,\vec{\mathbf{J}} \end{pmatrix} = (\mu_0 c)\mathbf{j}_{\boldsymbol{e}} + (i\mu_{0m}c)\mathbf{j}_{\boldsymbol{m}} = (\mu_0 c)\begin{pmatrix} c\rho_e \\ i\,\vec{\mathbf{J}}_e \end{pmatrix} + (i\mu_{0m}c)\begin{pmatrix} c\rho_m \\ i\,\vec{\mathbf{J}}_m \end{pmatrix} \text{ with } \mathbf{j}_e = \begin{pmatrix} c\rho_e \\ i\,\vec{\mathbf{J}}_e \end{pmatrix},\ \mathbf{j}_m = \begin{pmatrix} c\rho_m \\ i\,\vec{\mathbf{J}}_m \end{pmatrix}, \tag{13}$$

and the 4-E/M-operator:

$$\hat{\mathrm{L}}_{\mathrm{E/M}} \equiv \begin{pmatrix} \vec{\nabla}\cdot \\ \frac{\partial}{\partial ict} + \vec{\nabla}\times \end{pmatrix}, \tag{14}$$

that acts on three-dimensional (3D) vectors to produce 4-vectors, i.e., $\left(\hat{\mathrm{L}}_{\mathrm{E/M}}\vec{a}\right)_{1\mathrm{D}} = \vec{\nabla}\cdot\vec{a}$ and $\left(\hat{\mathrm{L}}_{\mathrm{E/M}}\vec{a}\right)_{3\mathrm{D}} = \partial\vec{a}/\partial ict + \vec{\nabla}\times\vec{a}$. Then, the compact and symmetric Maxwell Equations (12) can be written as:

$$\begin{pmatrix} \vec{\nabla}\cdot \\ \frac{\partial}{\partial ict} + \vec{\nabla}\times \end{pmatrix}\vec{G} = \begin{pmatrix} c\rho \\ i\,\vec{\mathbf{J}} \end{pmatrix} \text{ or } \hat{\mathrm{L}}_{\mathrm{E/M}}\vec{G} = \mathbf{j}, \tag{15}$$

which in the absence of magnetic monopoles become:

$$\begin{pmatrix} \vec{\nabla}\cdot \\ \frac{\partial}{\partial ict} + \vec{\nabla}\times \end{pmatrix}\vec{G} = (\mu_0 c)\cdot\begin{pmatrix} c\boldsymbol{\rho_e} \\ i\,\vec{\mathbf{J}}_{\boldsymbol{e}} \end{pmatrix} \text{ or } \hat{\mathrm{L}}_{\mathrm{E/M}}\vec{G} = (\mu_0 c)\cdot\mathbf{j}_{\boldsymbol{e}}. \tag{16}$$

The action of the 4-Laplace operator ∇_4 on $\hat{\mathrm{L}}_{\mathrm{E/M}}\vec{G}$ is:

$$\nabla_4\cdot\hat{\mathrm{L}}_{\mathrm{E/M}} = \left(-\frac{\partial}{\partial ict}, \vec{\nabla}\cdot\right)\begin{pmatrix} \vec{\nabla}\cdot \\ \frac{\partial}{\partial ict} + \vec{\nabla}\times \end{pmatrix} = -\frac{\partial}{\partial ict}(\vec{\nabla}\cdot) + (\vec{\nabla}\cdot)\frac{\partial}{\partial ict} + \vec{\nabla}\cdot(\vec{\nabla}\times) = 0, \tag{17}$$

leading to the continuity equation:

$$0 = \nabla_4\cdot\mathbf{j} = \left(-\frac{\partial}{\partial ict}, \vec{\nabla}\cdot\right)\begin{pmatrix} c\rho \\ i\,\vec{\mathbf{J}} \end{pmatrix} = -\frac{\partial c\rho}{\partial ict} + \vec{\nabla}\cdot i\,\vec{\mathbf{J}} = i\left(\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{\mathbf{J}}\right) \text{ or } \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{\mathbf{J}} = 0. \tag{18}$$

Given Equation (11), the continuity equation can be decomposed for the electric and magnetic charge, separately:

$$\frac{\partial\rho_e}{\partial t} + \vec{\nabla}\cdot\vec{\mathbf{J}}_e = 0 \text{ and } \frac{\partial\rho_m}{\partial t} + \vec{\nabla}\cdot\vec{\mathbf{J}}_m = 0. \tag{19}$$

3. Applications

Next, we examine how the symmetric form of Equation (15) can be applied to the wave equations of fields and gauge transformations. We also obtain the respective equations for potentials.

First, we define the 4-operator $\hat{\hat{L}}_{E/M}$ that acts on 4-vectors to produce a 3D vector.

$$\hat{\hat{L}}_{E/M} \equiv \left(\vec{\nabla}, \frac{\partial}{\partial ict} - \vec{\nabla}\times\right),\ \hat{\hat{L}}_{E/M}\begin{pmatrix} a_0 \\ \vec{a} \end{pmatrix} = \vec{\nabla}a_0 + \frac{\partial \vec{a}}{\partial ict} - \vec{\nabla} \times \vec{a}. \tag{20}$$

3.1. Identities

Below, we examine the action of this 4-operator on two specific 4-vectors, as well the action of the 4-E/M-operator $\hat{L}_{E/M}$ on $\hat{\hat{L}}_{E/M}$:

The 4-operator $\hat{\hat{L}}_{E/M}$ acts on the E/M 4-vector operator $\hat{L}_{E/M}$ and produces the wave d'Alembert operator that acts on the 3D vectors:

$$\begin{aligned}\hat{\hat{L}}_{E/M} \cdot \hat{L}_{E/M} = \left(\vec{\nabla}, \tfrac{\partial}{\partial ict} - \vec{\nabla}\times\right)\begin{pmatrix} \vec{\nabla}\cdot \\ \tfrac{\partial}{\partial ict} + \vec{\nabla}\times \end{pmatrix} &= \vec{\nabla}(\vec{\nabla}\cdot) + \tfrac{\partial^2}{\partial(ict)^2} + \tfrac{\partial}{\partial ict}(\vec{\nabla}\times) - \vec{\nabla}\times\tfrac{\partial}{\partial ict} - \vec{\nabla}\times(\vec{\nabla}\times) \\ &= \nabla^2 + \tfrac{\partial^2}{\partial(ict)^2} \equiv \Box^2\end{aligned} \tag{21}$$

(because $\vec{\nabla} \times (\vec{\nabla}\times) = \vec{\nabla}(\vec{\nabla}\cdot) - \nabla^2$).

- The 4-operator $\hat{\hat{L}}_{E/M}$ acts on the gauge transformation 4-vector and vanishes:

$$\left(\vec{\nabla}, \frac{\partial}{\partial ict} - \vec{\nabla}\times\right)\begin{pmatrix} -\frac{\partial}{\partial ict} \\ \vec{\nabla} \end{pmatrix} = -\vec{\nabla}\frac{\partial}{\partial ict} + \frac{\partial}{\partial ict}\vec{\nabla} - (\vec{\nabla}\times)\vec{\nabla} = 0. \tag{22}$$

- The action of the 4-E/M-operator $\hat{L}_{E/M}$ on $\hat{\hat{L}}_{E/M}$,

$$\hat{L}_{E/M} \cdot \hat{\hat{L}}_{E/M} = \begin{pmatrix} \vec{\nabla}\cdot \\ \frac{\partial}{\partial ict} + \vec{\nabla}\times \end{pmatrix}\left(\vec{\nabla}, \frac{\partial}{\partial ict} - \vec{\nabla}\times\right) = \begin{pmatrix} \nabla^2 & \frac{\partial}{\partial ict}\vec{\nabla}\cdot \\ \frac{\partial}{\partial ict}\vec{\nabla} & \frac{\partial^2}{\partial(ict)^2} - \vec{\nabla}\times(\vec{\nabla}\times) \end{pmatrix}. \tag{23}$$

3.2. Wave Equations

The first identity, in Equation (21), can be applied to producing the field wave equations:

$$\Box^2\vec{G} = -\hat{\hat{L}}_{E/M}\begin{pmatrix} c\boldsymbol{\rho} \\ i\,\vec{\mathbf{J}} \end{pmatrix}, \tag{24a}$$

expanded as follows:

$$\Box^2\vec{G} = \left(\vec{\nabla}, -\frac{\partial}{\partial ict} + \vec{\nabla}\times\right)\begin{pmatrix} c\boldsymbol{\rho} \\ i\,\vec{\mathbf{J}} \end{pmatrix} = \vec{\nabla}(c\boldsymbol{\rho}) - \frac{\partial(i\,\vec{\mathbf{J}})}{\partial ict} + \vec{\nabla} \times (i\,\vec{\mathbf{J}}), \tag{24b}$$

which can be further decomposed to the specific wave equations of the electric field:

$$\Box^2\vec{E} = (\mu_0 c)\vec{\nabla}(c\boldsymbol{\rho_e}) - (\mu_0 c)\frac{\partial\vec{\mathbf{J}}_{\boldsymbol{e}}}{\partial ct} - (\mu_{0m}c)\vec{\nabla} \times \vec{\mathbf{J}}_{\boldsymbol{m}}, \text{ or} \tag{25}$$

$$\Box^2\vec{E} = \frac{1}{\varepsilon_0}\vec{\nabla}\boldsymbol{\rho_e} - \mu_0\frac{\partial\vec{\mathbf{J}}_{\boldsymbol{e}}}{\partial t} - \mu_{0m}c\,\vec{\nabla} \times \vec{\mathbf{J}}_{\boldsymbol{m}}. \tag{26}$$

and the magnetic field:

$$\Box^2(c\vec{B}) = (\mu_{0m}c)\vec{\nabla}(c\boldsymbol{\rho_m}) - (\mu_{0m}c)\frac{\partial\vec{\mathbf{J}}_{\boldsymbol{m}}}{\partial ct} + (\mu_0 c)\vec{\nabla} \times \vec{\mathbf{J}}_{\boldsymbol{e}}, \text{ or} \tag{27}$$

$$\square^2 \vec{B} = \frac{1}{\varepsilon_{0m}} \vec{\nabla} \boldsymbol{\rho_m} - \mu_{0m} \frac{\partial \vec{\mathbf{J}}_{\boldsymbol{m}}}{\partial t} + \mu_0 c \vec{\nabla} \times \vec{\mathbf{J}}_{\boldsymbol{e}}. \tag{28}$$

3.3. Gauge Transformations

The identity in Equation (22) can be applied to producing the gauge transformations. First, the potential representation is written as:

$$\vec{G} = -\hat{\hat{\mathrm{L}}}_{\mathrm{E/M}} \mathrm{A}_{\boldsymbol{e}} = -\left(\vec{\nabla}, \frac{\partial}{\partial ict} - \vec{\nabla} \times \right) \mathrm{A}_{\boldsymbol{e}},\ \mathrm{A}_{\boldsymbol{e}} \equiv \begin{pmatrix} \Phi_{\boldsymbol{e}} \\ ic\vec{\mathrm{A}}_{\boldsymbol{e}} \end{pmatrix}. \tag{29}$$

Note that in the presence of magnetic monopoles, the 4-potential becomes:

$$\vec{G} = -\hat{\hat{\mathrm{L}}}_{\mathrm{E/M}} \mathrm{A} = -\left(\vec{\nabla}, \frac{\partial}{\partial ict} - \vec{\nabla} \times \right) \mathrm{A},\ \text{with} \tag{30}$$

$$\mathrm{A} \equiv \mathrm{A}_{\boldsymbol{e}} + i\,\mathrm{A}_m = \begin{pmatrix} \Phi \\ ic\vec{\mathrm{A}} \end{pmatrix} = \begin{pmatrix} \Phi_{\boldsymbol{e}} \\ ic\vec{\mathrm{A}}_{\boldsymbol{e}} \end{pmatrix} + i \begin{pmatrix} \Phi_m \\ ic\vec{\mathrm{A}}_m \end{pmatrix} = \begin{pmatrix} \Phi_{\boldsymbol{e}} + i\,\Phi_m \\ ic\vec{\mathrm{A}}_{\boldsymbol{e}} - c\vec{\mathrm{A}}_m \end{pmatrix}. \tag{31}$$

which can be decomposed to the respective potential equations of the electric field:

$$\vec{E} = -\vec{\nabla}\Phi_{\boldsymbol{e}} - \frac{\partial \vec{\mathrm{A}}_{\boldsymbol{e}}}{\partial t} - c\,\vec{\nabla} \times \vec{\mathrm{A}}_m, \tag{32}$$

and the magnetic field:

$$c\vec{B} = -\vec{\nabla}\Phi_m - \frac{\partial \vec{\mathrm{A}}_m}{\partial t} + c\,\vec{\nabla} \times \vec{\mathrm{A}}_{\boldsymbol{e}}. \tag{33}$$

The gauge transformations can be set as:

$$\mathrm{A}' = \mathrm{A} + \begin{pmatrix} -\frac{\partial}{\partial ict} \\ \vec{\nabla} \end{pmatrix} \cdot ic\Psi,\ \text{with}\ \vec{G}' = -\hat{\hat{\mathrm{L}}}_{\mathrm{E/M}} \mathrm{A}'\ \text{and}\ \vec{G} = -\hat{\hat{\mathrm{L}}}_{\mathrm{E/M}} \mathrm{A}. \tag{34}$$

Hence:

$$\vec{G}' = \vec{G} - \left(\vec{\nabla}, \frac{\partial}{\partial ict} - \vec{\nabla} \times \right) \begin{pmatrix} -\frac{\partial}{\partial ict} \\ \vec{\nabla} \end{pmatrix} \cdot ic\Psi = \vec{G}. \tag{35}$$

3.4. Symmetric Complex Maxwell Equations for Potentials

Finally, using the third identity in Equation (23), we derive the Maxwell equations for the potentials:

$$\hat{\mathrm{L}}_{\mathrm{E/M}} \vec{G} = -\left(\hat{\mathrm{L}}_{\mathrm{E/M}} \cdot \hat{\hat{\mathrm{L}}}_{\mathrm{E/M}}\right) \mathrm{A} = \mathbf{J},\ \text{i.e.,} \tag{36}$$

$$-\begin{pmatrix} \nabla^2 & \frac{\partial}{\partial ict} \vec{\nabla} \cdot \\ \frac{\partial}{\partial ict} \vec{\nabla} & \frac{\partial^2}{\partial (ict)^2} - \vec{\nabla} \times (\vec{\nabla} \times) \end{pmatrix} \begin{pmatrix} \Phi \\ ic\vec{\mathrm{A}} \end{pmatrix} = \begin{pmatrix} c\boldsymbol{\rho} \\ i\,\vec{\mathbf{J}} \end{pmatrix}, \tag{37}$$

which can be decomposed to:

$$\begin{gathered} -\nabla^2 \Phi_{\boldsymbol{e}} - \frac{\partial}{\partial t} \vec{\nabla} \vec{\mathrm{A}}_{\boldsymbol{e}} = \frac{1}{\varepsilon_0} \boldsymbol{\rho}_{\boldsymbol{e}} \\ \frac{\partial}{\partial t} \vec{\nabla} \Phi_{\boldsymbol{e}} + \frac{\partial^2 \vec{\mathrm{A}}_{\boldsymbol{e}}}{\partial t^2} - c^2 \vec{\nabla} \times (\vec{\nabla} \times \vec{\mathrm{A}}_{\boldsymbol{e}}) = \mu_0 c^2 \vec{\mathbf{J}}_{\boldsymbol{e}} \end{gathered} \tag{38}$$

and:

$$\begin{aligned} -\nabla^2\Phi_m - \frac{\partial}{\partial t}\vec{\nabla}\vec{\mathrm{A}}_m &= \frac{1}{\varepsilon_{0m}}\boldsymbol{\rho}_m \\ \frac{\partial}{\partial t}\vec{\nabla}\Phi_m + \frac{\partial^2\vec{\mathrm{A}}_m}{\partial t^2} + c^2\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathrm{A}}_m) &= \mu_{0m}c^2\vec{\mathbf{J}}_m \end{aligned} \tag{39}$$

3.5. Energy–Momentum

The energy and momentum of the electromagnetic field can be expressed in terms of the vector $\vec{G} \equiv \vec{E} + ic\vec{B}$. Indeed, the energy (density) is given by $\frac{1}{2}\varepsilon_0\vec{E}^2$ and $\frac{1}{2}\mu_0{}^{-1}\vec{B}^2 = \frac{1}{2}\varepsilon_0(c\vec{B})^2$ for the electric and magnetic fields, respectively, that is, $\frac{1}{2}\varepsilon_0[\vec{E}^2 + (c\vec{B})^2]$, or:

$$\mathrm{E} = \frac{1}{2}\varepsilon_0[\vec{E}^2 + (c\vec{B})^2] = \frac{1}{2}\varepsilon_0\|\vec{G}\|^2 = \frac{1}{2}\varepsilon_0(\vec{G}\cdot\vec{G}^*). \tag{40a}$$

The momentum is proportional to the pointing vector, $\vec{\mathrm{p}} = c^{-2}\cdot\vec{S}$, and can be written as:

$$\vec{\mathrm{p}} = \varepsilon_0(\vec{E}\times\vec{B}) = \frac{1}{2}\frac{i\varepsilon_0}{c}(\vec{G}\times\vec{G}^*). \tag{40b}$$

Hence, the 4-momentum (in Minkowski space) is expressed by:

$$\left(i\frac{E}{c}, \vec{\mathrm{p}}\right) = \frac{1}{2}\frac{i\varepsilon_0}{c}(\vec{G}\cdot\vec{G}^*, \vec{G}\times\vec{G}^*). \tag{40c}$$

4. Conclusions

In this paper, we presented the complex representation of Maxwell equations, indicating the symmetry between electric and magnetic fields, and concluding with a compact form of equations for the fields and their potentials. Using these compact symmetric forms, the wave equations and gauge transformation of the electric and magnetic fields were derived.

The complex symmetric operator formulation presented here can be used as a different way to express the physical content of Maxwell equations. Nevertheless, the developed formulation of Maxwell equations constitutes a much simpler way of unfolding the Maxwell theory rather than previous complex formulations. The simplicity of the presented formulation was shown in the derivations of the basic concepts of (i) continuity equation; (ii) wave equations; (iii) Maxwell equations for potentials; (iv) gauge transformation; and (v) 4-momentum of E/M field, which are briefly summarized below:

By setting,

- Definitions:

$$\vec{G} \equiv \vec{E} + ic\vec{B},\ \mathrm{A} \equiv \begin{pmatrix} \Phi \\ ic\vec{\mathrm{A}} \end{pmatrix},\ \mathbf{j} \equiv \begin{pmatrix} c\boldsymbol{\rho} \\ i\vec{\mathbf{J}} \end{pmatrix}. \tag{41}$$

- Maxwell equations:

$$\begin{pmatrix} \vec{\nabla}\cdot \\ \frac{\partial}{\partial ict} + \vec{\nabla}\times \end{pmatrix}\vec{G} = \mathbf{j}. \tag{42}$$

- Potential representation:

$$\vec{G} = -\left(\vec{\nabla},\ \frac{\partial}{\partial ict} - \vec{\nabla}\times\right)\mathrm{A}. \tag{43}$$

we obtain the derivations:

(i) Continuity equation: Action of $\left(-\frac{\partial}{\partial ict}, \vec{\nabla}\cdot\right)$ on Equation (42):

$$\left(-\frac{\partial}{\partial ict}, \vec{\nabla}\cdot\right) \mathbf{j} = 0. \tag{44}$$

(ii) Wave equations: Action of $\left(\vec{\nabla}, \frac{\partial}{\partial ict} - \vec{\nabla}\times\right)$ on Equation (42):

$$\Box^2 \vec{G} = \left(\vec{\nabla}, -\frac{\partial}{\partial ict} + \vec{\nabla}\times\right) \cdot \mathbf{j}. \tag{45}$$

(iii) Maxwell equations for potentials: Action of $\begin{pmatrix} \vec{\nabla}\cdot \\ \frac{\partial}{\partial ict} + \vec{\nabla}\times \end{pmatrix}$ on $\left(\vec{\nabla}, \frac{\partial}{\partial ict} - \vec{\nabla}\times\right)$ in Equation (43):

$$-\begin{pmatrix} \nabla^2 & \frac{\partial}{\partial ict}\vec{\nabla}\cdot \\ \frac{\partial}{\partial ict}\vec{\nabla} & \frac{\partial^2}{\partial (ict)^2} - \vec{\nabla}\times(\vec{\nabla}\times) \end{pmatrix} \mathrm{A} = \begin{pmatrix} \Phi \\ ic\vec{\mathrm{A}} \end{pmatrix} = \begin{pmatrix} c\rho \\ i\,\vec{\mathbf{J}} \end{pmatrix}. \tag{46}$$

(iv) Gauge transformation: Action of $\left(\vec{\nabla}, \frac{\partial}{\partial ict} - \vec{\nabla}\times\right)$ in Equation (43) on $\begin{pmatrix} -\frac{\partial}{\partial ict} \\ \vec{\nabla} \end{pmatrix}$:

$$\mathrm{A}' = \mathrm{A} + \begin{pmatrix} -\frac{\partial}{\partial ict} \\ \vec{\nabla} \end{pmatrix} \cdot ic\Psi,\ \vec{G}' = \vec{G}. \tag{47}$$

(v) 4-momentum (energy and momentum) of electromagnetic field:

$$\left(i\frac{E}{c}, \vec{\mathrm{P}}\right) = \frac{1}{2}\frac{i\varepsilon_0}{c}(\vec{G}\cdot\vec{G}^*, \vec{G}\times\vec{G}^*). \tag{48}$$

5. What's Next

The presented analysis can trigger theoretical developments and applications that differ from the standard Maxwell equations. For example, it will be very exciting to extend the presented analysis within the context of (i) bicomplex numbers and tessarine algebra, that is a four-dimensional vector space over the reals, two-dimensional over the complex numbers [22]; and (ii) L^p-spaces [23,24] in nonlinear Maxwell equations [25].

In particular, the linearization of general relativity (weak field limit approximation) [26] makes two fields, the gravitoelectric (that is simply the conventional gravity), and the gravitomagnetic (caused by twist of spacetime, e.g., spinning massive objects) [27] that appears in a frame of reference different from that of a freely moving inertial body.

$$\begin{array}{cc} \vec{\nabla}\cdot\vec{E}_g = -\frac{\rho_g}{\varepsilon_{0g}} & \vec{\nabla}\cdot\vec{E} = \frac{\rho_e}{\varepsilon_0} \\ \mu_{0g}\varepsilon_{0g}\frac{\partial\vec{E}_g}{\partial t} = \vec{\nabla}\times\vec{B}_g + \mu_{0g}\vec{\mathbf{J}}_g & \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \vec{\nabla}\times\vec{B} - \mu_0\vec{\mathbf{J}}_e \\ \vec{\nabla}\cdot\vec{B}_g = 0 & \vec{\nabla}\cdot\vec{B} = 0 \\ \frac{\partial\vec{B}_g}{\partial t} = -\vec{\nabla}\times\vec{E}_g & \frac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E} \end{array} \tag{49}$$

where E_g is the static gravitational or gravitoelectric field and B_g is the gravitomagnetic field; and sources ρ_g and J_g are the mass density and current density, respectively. The involved constants are defined by:

$$1/(4\pi\,\varepsilon_{0\,g}) \equiv G \text{ and } \mu_{0g} = 1/(\,c^2\varepsilon_{0\,g}) \equiv 4\pi G/c^2, \tag{50}$$

where G is the conventional gravitational constant; the quasi-particle involved in the gravitational Maxwell equations—the graviton—is characterized by the speed of light at vacuum (similar to a photon). The presented formalism can be applied to both sets of Maxwell equations:

$$\begin{pmatrix} \vec{\nabla}\cdot \\ \frac{\partial}{\partial ict} + \vec{\nabla}\times \end{pmatrix} \vec{G}_g = \begin{pmatrix} c\rho_g \\ i\,\vec{\mathbf{J}}_g \end{pmatrix}, \begin{pmatrix} \vec{\nabla}\cdot \\ \frac{\partial}{\partial ict} + \vec{\nabla}\times \end{pmatrix} \vec{G} = \begin{pmatrix} c\rho_e \\ i\,\vec{\mathbf{J}}_e \end{pmatrix} \tag{51}$$

with:

$$\vec{G}_g \equiv \vec{E}_g + ic\vec{B}_g,\ \vec{G} \equiv \vec{E} + ic\vec{B}. \tag{52}$$

The algebra of bicomplex numbers can be used to unify the two sets of Maxwell equations. Lastly, it will be interesting to investigate whether gravitons can be involved in large-scale quantization constants (e.g., see the work of Carneiro [28] and Livadiotis & McComas [29]).

In the example that uses L^p-norms, we can generalize Equation (15) to:

$$\begin{pmatrix} \vec{\nabla}_p\cdot \\ \frac{\partial}{\partial ict} + \vec{\nabla}_p\times \end{pmatrix} \vec{G} = \begin{pmatrix} c\rho \\ i\,\vec{\mathbf{J}} \end{pmatrix}, \tag{53a}$$

by using a well-defined L^p-normed divergence and curl [30]:

$$\vec{\nabla}\cdot\vec{u} \to \vec{\nabla}_p\cdot\vec{u} \equiv \left|\vec{\nabla}\cdot\vec{u}\right|^{p-2}(\vec{\nabla}\cdot\vec{u}),\ \vec{\nabla}\times\vec{u} \to \vec{\nabla}_p\times\vec{u} \equiv \left|\vec{\nabla}\times\vec{u}\right|^{p-2}(\vec{\nabla}\times\vec{u}). \tag{53b}$$

Then, we may investigate the generated Maxwell equations.

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