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Boundary Value Problem of the Operator $\oplus^k$ Related to the Biharmonic Operator and the Diamond Operator

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Abstract: This paper presents an alternative methodology for finding the solution of the boundary value problem (BVP) for the linear partial differential operator. We are particularly interested in the linear operator $\oplus^k$, where $\oplus^k = \heartsuit^k\diamondsuit^k$, $\heartsuit^k$ is the biharmonic operator iterated k-times and $\diamondsuit^k$ is the diamond operator iterated k-times. The solution is built on the Green's identity of the operators $\heartsuit^k$ and $\oplus^k$, in which their derivations are also provided. To illustrate our findings, the example with prescribed boundary conditions is exhibited.

Keywords: boundary value problem; Green's identity; Green's function; tempered distribution

1. Introduction

Boundary value problems (BVPs) for ordinary and partial differential equations have appeared in widespread applications ranging from cognitive science to engineering. Some examples include a vibrating string with time depending upon external force under the Dirichlet boundary conditions [1], Laplace's equation in polar coordinates with the Neumann boundary conditions [2], or the diffusion equation with the Robin boundary conditions [3]. Finally, the heat flow in a nonuniform rod without sources accompanied with initial—boundary conditions [4]. These types of problems inevitably associate with the partial differential operators—for example, the Laplace operator [5,6], the ultrahyperbolic operator [7,8], and the biharmonic operator [9,10].

One common choice to tackle such problems analytically is by using the method of separation of variables, which is somewhat limited. For instance, it must be applied to lower-order linear partial differential equations with a small number of variables. More sophisticated treatment for the BVPs was proposed by F. John [11], who utilizes the Laplace operator using the following Green's identity:

$$\int_\Omega v\triangle u dx = \int_\Omega u\triangle v dx + \int_{\partial\Omega}\left(v\frac{\partial u}{\partial\eta} - u\frac{\partial v}{\partial\eta}\right)dS,$$

where η is the exterior normal vector to a boundary $\partial\Omega$ and $\triangle$ is the Laplace operator defined by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

The solution, $u(\xi)$, then becomes

$$u(\xi) = \int_\Omega K(x,\xi)\triangle u dx - \int_{\partial\Omega}\left(K(x,\xi)\frac{\partial u(x)}{\partial\eta} - u(x)\frac{\partial K(x,\xi)}{\partial\eta}\right)dS_x,\ \ \xi\in\Omega,$$

where $K(x,\xi)$ is the Green's function of the Laplace operator.

C. Bunpog [12] subsequently studied BVPs of the diamond operator $\diamondsuit^k$ in which it was originally investigated by A. Kananthai [13] and later explored in more detail in [14,15]. It is denoted by

$$\begin{aligned}\diamondsuit^k &= \left[\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_p^2}\right)^2-\left(\frac{\partial^2}{\partial x_{p+1}^2}+\frac{\partial^2}{\partial x_{p+2}^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right)^2\right]^k\\ &= \triangle^k\square^k=\square^k\triangle^k, \qquad k=1,2,\ldots, \end{aligned} \tag{1}$$

where the Laplace operator iterated k-times, $\triangle^k$, can be expressed as

$$\triangle^k=\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right)^k, \tag{2}$$

and the ultrahyperbolic operator iterated k-times, $\square^k$, is represented by

$$\square^k=\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial x_{p+1}^2}-\frac{\partial^2}{\partial x_{p+2}^2}-\cdots-\frac{\partial^2}{\partial x_n^2}\right)^k. \tag{3}$$

The solution, $u(\xi)$, can be formulated with the following expression:

$$\begin{aligned}u(\xi) &= \int_\Omega D_k(x,\xi)\diamondsuit^k u(x,\xi)dx-\sum_{i=1}^{k-1}G(\diamondsuit^i u(x,\xi),D_{i+1}(x,\xi))\\ &- F(D_1(x,\xi),u(x,\xi)), \qquad k=2,3,\ldots,\end{aligned}$$

where $D_k(x,\xi)$ is the Green's function of the operator $\diamondsuit^k$. The functions F and G involve some boundary conditions on $\partial\Omega$.

The partial differential operator $\oplus^k$ has some qualitative properties which can be found in [16–20]. It associates with the operators $\diamondsuit^k$ and $\heartsuit^k$ such that

$$\begin{aligned}\oplus^k &= \left[\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_p^2}\right)^4-\left(\frac{\partial^2}{\partial x_{p+1}^2}+\frac{\partial^2}{\partial x_{p+2}^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right)^4\right]^k\\ &= \heartsuit^k\diamondsuit^k=\diamondsuit^k\heartsuit^k, \quad k=1,2,\ldots,\end{aligned} \tag{4}$$

where $\diamondsuit^k$ is defined by Equation (1) and $\heartsuit^k$ is the biharmonic operator iterated k-times:

$$\heartsuit^k=\left[\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_p^2}\right)^2+\left(\frac{\partial^2}{\partial x_{p+1}^2}+\frac{\partial^2}{\partial x_{p+2}^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right)^2\right]^k. \tag{5}$$

In this paper, the Green's identity of the operator $\oplus^k$ will be presented. Furthermore, the solution's existence under some suitable boundary conditions of the operator $\oplus^k$ is manifested by using Green's identity of the operators $\heartsuit$ and $\oplus^k$, as well as the BVP solution of the diamond operator $\diamondsuit$. Finally, applications connected to the BVP of the linear partial differential operators are shown.

2. Preliminaries

Let us begin by introducing some functions and lemmas that are occasionally referred to in the paper.

Let $x=(x_1,x_2,\ldots,x_n)$ be a point of $\mathbb{R}^n$ and $v(x)=x_1^2+x_2^2+\cdots+x_n^2$. The elliptic kernel of Marcel Riesz defined by Riesz [21] has the following expression

$$R_\alpha^e(v) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right) v^{\frac{\alpha-n}{2}}}{2^\alpha \pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})}, \tag{6}$$

where α is any complex number and Γ is the Gamma function. It is an ordinary function if $Re(\alpha) \geq n$ and is a distribution of α if $Re(\alpha) < n$. In addition, $(-1)^k R_{2k}^e(v)$ is the Green's function of the operator $\triangle^k$ defined by Equation (2) (see [13]).

Let $y(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_n^2$ be a nondegenerated quadratic form. The interior of the forward cone is denoted by $\mathcal{F} = \{x \in \mathbb{R}^n \mid x_1 > 0, y > 0\}$. The ultrahyperbolic kernel of Marcel Riesz presented by Nozaki [22] is expressed as

$$R_\beta^H(y) \begin{cases} \frac{y^{\frac{\beta-n}{2}}}{K_n(\beta)}, & \text{for } x \in \mathcal{F}; \\ 0, & \text{for } x \notin \mathcal{F}, \end{cases} \tag{7}$$

where

$$K_n(\beta) = \frac{\pi^{\frac{n,1}{2}} \Gamma\left(\frac{2+\beta-n}{2}\right) \Gamma\left(\frac{1-\beta}{2}\right) \Gamma(\beta)}{\Gamma\left(\frac{2+\beta-p}{2}\right) \Gamma\left(\frac{p-\beta}{2}\right)},$$

and β is a complex number. Note that $R_\beta^H(y)$ is an ordinary function if $Re(\beta) \geq n$ and is a distribution of β if $Re(\beta) < n$. Furthermore, $R_{2k}^H(y)$ is the Green's function of the operator $\square^k$ in the form of Equation (3) (see [23]).

Let $w(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \cdots + x_n^2)$ and $z(x) = x_1^2 + x_2^2 + \cdots + x_p^2 + i(x_{p+1}^2 + x_{p+2}^2 + \cdots + x_n^2)$, where $i = \sqrt{-1}$. Functions $S_\gamma(w)$ and $T_\eta(z)$ are defined by

$$S_\gamma(w) = \frac{\Gamma\left(\frac{n-\gamma}{2}\right) w^{\frac{\gamma-n}{2}}}{2^\gamma \pi^{\frac{n}{2}} \Gamma(\frac{\gamma}{2})}, \tag{8}$$

$$T_\kappa(z) = \frac{\Gamma\left(\frac{n-\kappa}{2}\right) z^{\frac{\kappa-n}{2}}}{2^\kappa \pi^{\frac{n}{2}} \Gamma(\frac{\kappa}{2})}, \tag{9}$$

for any complex numbers γ and κ. The convolution $S_{2k}(w) * T_{2k}(z)$ is a tempered distribution (or a distribution of slow growth, [24]) and the Green's function of the operator $\heartsuit^k$ defined by Equation (5), that is,

$$\heartsuit^k \left(S_{2k}(w) * T_{2k}(z)\right) = \delta(x), \tag{10}$$

where $\delta(x)$ is the Dirac delta distribution [18].

We modify these functions by introducing the following definitions.

Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be a point of $\mathbb{R}^n$ and $x - \xi = (x_1 - \xi_1, x_2 - \xi_2, \ldots, x_n - \xi_n)$. We define

$$R_\alpha^e(x,\xi) \equiv R_\alpha^e(v(x-\xi)),$$

$$R_\beta^H(x,\xi) \equiv R_\beta^H(y(x-\xi)),$$

$$S_\gamma(x,\xi) \equiv S_\gamma(w(x-\xi)),$$

$$T_\kappa(x,\xi) \equiv T_\kappa(z(x-\xi)),$$

where $R_\alpha^e(v)$, $R_\beta^H(y)$, $S_\gamma(w)$ and $T_\kappa(z)$ are defined by Equations (6)–(9), respectively. We let

$$D_k(x,\xi) \equiv (-1)^k R_{2k}^e(x,\xi) * R_{2k}^H(x,\xi), \; k = 1, 2, \ldots. \tag{11}$$

$$M_k(x,\xi) \equiv (-1)^k R_{2k}^e(x,\xi) * R_{2k}^H(x,\xi) * S_{2k}(x,\xi) * T_{2k}(x,\xi), \; k = 1, 2, \ldots. \tag{12}$$

Note that functions $D_k(x,\xi)$ and $M_k(x,\xi)$ are tempered distributions [13,16], which can be written in the form of functions $R_2^e(x,\xi)$, $R_2^H(x,\xi)$, $S_2(x,\xi)$, and $T_2(x,\xi)$. Equations (11) and (12) can thus be computed via [25]:

$$D_k(x,\xi) = \frac{(-1)^{2k-1}\,(v(x,\xi))^{k-1}\,R_2^e(x,\xi) * (y(x,\xi))^{k-1}\,R_2^H(x,\xi)}{2^{2(k-1)}\,((k-1)!)^2\prod_{l=1}^{k-1}(n-2(l+1))^2},\ k=2,3,\ldots,$$

$$M_k(x,\xi) = \frac{(-1)^k D_k(x,\xi) * (w(x,\xi))^{k-1}\,S_2(x,\xi) * (z(x,\xi))^{k-1}\,T_2(x,\xi)}{2^{2(k-1)}\,((k-1)!)^2\prod_{l=1}^{k-1}(n-2(l+1))^2},\ k=2,3,\ldots.$$

Moreover, the function $M_k(x,\xi)$ satisfies

$$\oplus^m M_k(x,\xi) = M_{k-m}(x,\xi),\ 0 \le m < k,\ k=1,2,\ldots. \tag{13}$$

Lemma 1 (Gauss divergence theorem)**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$, $\partial\Omega$ is the boundary of Ω, and $u \in C^1(\overline{\Omega})$, $\overline{\Omega} = \Omega \cup \partial\Omega$. Then*

$$\int_\Omega \frac{\partial u(x)}{\partial x_k}dx = \int_{\partial\Omega} u(x)\frac{\partial x_k}{\partial \eta}dS_x = \int_{\partial\Omega} u(x)\eta_k dS_x, \tag{14}$$

where $\frac{\partial}{\partial\eta}$ denotes a differentiation in the direction of the exterior unit normal $\eta = (\eta_1,\ldots,\eta_n)$ of $\partial\Omega$, $dx = dx_1\cdots dx_n$ and dS_x is a surface element with integration on x.

Proof of Lemma 1. (see [26]). □

Lemma 2 (Green's identity of the biharmonic operator)**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$, $\partial\Omega$ be the boundary of Ω and $u,v \in C^4(\overline{\Omega})$, $\overline{\Omega} = \Omega \cup \partial\Omega$. Then, the Green's identity of the biharmonic operator $\heartsuit$ is*

$$\int_\Omega v\heartsuit u dx - \int_\Omega u\heartsuit v dx = \int_\Omega (L_2 u L_1 v - L_2 v L_1 u)\,dx + H(u,v), \tag{15}$$

where $H(u,v)$ is given by

$$H(u,v) = \int_{\partial\Omega}\left[\left(v\frac{\partial(L_2u)}{\partial\eta_\star} - u\frac{\partial(L_2v)}{\partial\eta_\star}\right) + \left(L_2v\frac{\partial u}{\partial\eta_\star} - L_2u\frac{\partial v}{\partial\eta_\star}\right)\right]dS_x, \tag{16}$$

$\eta_\star = (\eta_1,\eta_2,\ldots,\eta_p,i\eta_{p+1},i\eta_{p+2},\ldots,i\eta_n)$ denotes the complex-transversal to $\partial\Omega$, and $\frac{\partial}{\partial\eta_\star}$ denotes the derivative in the complex-transversal direction.

Proof of Lemma 2. From Equation (5) with $k=1$, we can write

$$\heartsuit = L_1 L_2, \tag{17}$$

where L_1 and L_2 are defined by

$$L_1 = \sum_{l=1}^{p}\frac{\partial^2}{\partial x_l^2} + i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}$$

and

$$L_2 = \sum_{l=1}^{p}\frac{\partial^2}{\partial x_l^2} - i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2},\quad i=\sqrt{-1}.$$

Since

$$\frac{\partial(vu)}{\partial x_k} = v\frac{\partial u}{\partial x_k} + u\frac{\partial v}{\partial x_k},$$

thus

$$\int_\Omega v\frac{\partial u}{\partial x_k}dx = \int_\Omega \frac{\partial(vu)}{\partial x_k}dx - \int_\Omega u\frac{\partial v}{\partial x_k}dx.$$

By Equation (14), we obtain

$$\int_\Omega v\frac{\partial u}{\partial x_k}dx = \int_{\partial\Omega}(vu)\eta_k dS_x - \int_\Omega u\frac{\partial v}{\partial x_k}dx,$$

and

$$\int_\Omega v\frac{\partial^2 u}{\partial x_k^2}dx = \int_{\partial\Omega} v\frac{\partial u}{\partial x_k}\eta_k dS_x - \int_\Omega \frac{\partial u}{\partial x_k}\cdot\frac{\partial v}{\partial x_k}dx.$$

Therefore

$$\begin{aligned}\int_\Omega vL_1u dx &= \int_{\partial\Omega} v\left(\sum_{l=1}^{p}\frac{\partial u}{\partial x_l}\eta_l + i\sum_{j=p+1}^{p+q}\frac{\partial u}{\partial x_j}\eta_j\right)dS_x \\ &\quad - \int_\Omega\left(\sum_{l=1}^{p}\frac{\partial u}{\partial x_l}\cdot\frac{\partial v}{\partial x_l} + i\sum_{j=p+1}^{p+q}\frac{\partial u}{\partial x_j}\cdot\frac{\partial v}{\partial x_j}\right)dx \\ &= \int_{\partial\Omega} v\frac{\partial u}{\partial\eta_\star}dS_x - \int_\Omega\left(\sum_{l=1}^{p}\frac{\partial u}{\partial x_l}\cdot\frac{\partial v}{\partial x_l} + i\sum_{j=p+1}^{p+q}\frac{\partial u}{\partial x_j}\cdot\frac{\partial v}{\partial x_j}\right)dx\end{aligned}$$

which follows

$$\int_\Omega uL_1v dx = \int_{\partial\Omega} u\frac{\partial v}{\partial\eta_\star}dS_x - \int_\Omega\left(\sum_{l=1}^{p}\frac{\partial v}{\partial x_l}\cdot\frac{\partial u}{\partial x_l} + i\sum_{j=p+1}^{p+q}\frac{\partial v}{\partial x_j}\cdot\frac{\partial u}{\partial x_j}\right)dx.$$

Hence

$$\int_\Omega vL_1u dx = \int_\Omega uL_1v dx + \int_{\partial\Omega}\left(v\frac{\partial u}{\partial\eta_\star} - u\frac{\partial v}{\partial\eta_\star}\right)dS_x. \tag{18}$$

From Equations (17) and (18), we derive

$$\int_\Omega v\heartsuit u dx = \int_\Omega L_2uL_1v dx + \int_{\partial\Omega}\left(v\frac{\partial(L_2u)}{\partial\eta_\star} - L_2u\frac{\partial v}{\partial\eta_\star}\right)dS_x. \tag{19}$$

Similarly, we find that

$$\int_\Omega u\heartsuit v dx = \int_\Omega L_2vL_1u dx + \int_{\partial\Omega}\left(u\frac{\partial(L_2v)}{\partial\eta_\star} - L_2v\frac{\partial u}{\partial\eta_\star}\right)dS_x. \tag{20}$$

By Equations (19) and (20), it can be concluded that

$$\int_\Omega v\heartsuit u dx - \int_\Omega u\heartsuit v dx = \int_\Omega (L_2uL_1v - L_2vL_1u)\,dx + H(u,v),$$

where $H(u,v)$ is defined by Equation (16). The proof is completed. □

Lemma 3. *Let Ω be a bounded open subset of the Euclidian space $\mathbb{R}^n$, $\partial\Omega$ be the boundary of Ω, and $D_1(x,\xi)$ be a function which is given by Equation (11) with $k=1$. Accordingly, the BVP solution of the diamond operator $\Diamond$ becomes*

$$u(\xi) = \int_\Omega D_1(x,\xi)\Diamond u(x,\xi)dx - F(D_1(x,\xi), u(x,\xi)),\ \ \xi\in\Omega, \tag{21}$$

where

$$F(D_1,u) = \int_{\partial\Omega}\left(D_1\frac{\partial(\triangle u)}{\partial\eta_*} - \triangle u\frac{\partial D_1}{\partial\eta_*}\right)dS_x + \int_{\partial\Omega}\left(\Box D_1\frac{\partial u}{\partial\eta} - u\frac{\partial(\Box D_1)}{\partial\eta}\right)dS_x, \tag{22}$$

$\eta_* = (\eta_1,\eta_2,\ldots,\eta_p,-\eta_{p+1},-\eta_{p+2},\ldots,-\eta_n)$ *denotes the transversal to* $\partial\Omega$*, and* $\frac{\partial}{\partial\eta_*}$ *denotes the derivative in the transversal direction* [27].

Proof of Lemma 3. (see [12]). □

3. Results

In this section, the Green's identity along with the solution of the BVP of the operator $\oplus^k$ are described. The results stated in the previous section are used to show the existence of a solution.

Theorem 1 (Green's identity of the operator $\oplus^k$)**.** *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$*,* $\partial\Omega$ *be the boundary of* Ω *and* $u,v \in C^{8k}(\overline{\Omega})$*,* $\overline{\Omega} = \Omega\cup\partial\Omega$*. Then, the Green's identity of the operator* $\oplus^k$ *defined by Equation (4) is*

$$\begin{aligned}
\int_\Omega v\oplus^k u dx - \int_\Omega u\oplus^k v dx &= \int_\Omega\left[\diamondsuit\left(\oplus^{k-1}u\right)\heartsuit v - \diamondsuit\left(\oplus^{k-1}v\right)\heartsuit u\right]dx \\
&+ \int_\Omega\left[L_2\diamondsuit\left(\oplus^{k-1}u\right)L_1 v - L_2 v L_1\diamondsuit\left(\oplus^{k-1}u\right)\right]dx \\
&+ \int_\Omega\left[L_2 u L_1\diamondsuit\left(\oplus^{k-1}v\right) - L_2\diamondsuit\left(\oplus^{k-1}v\right)L_1 u\right]dx \\
&+ H\left(\diamondsuit\left(\oplus^{k-1}u\right),v\right) + H\left(u,\diamondsuit\left(\oplus^{k-1}v\right)\right).
\end{aligned} \tag{23}$$

Proof of Theorem 1. Since $u \in C^{8k}(\overline{\Omega})$, replacing it by $\diamondsuit\left(\oplus^{k-1}u\right)$ in Equation (15), we have

$$\begin{aligned}
\int_\Omega v\oplus^k u dx - \int_\Omega\diamondsuit\left(\oplus^{k-1}u\right)\heartsuit v dx &= \int_\Omega\left[L_2\diamondsuit\left(\oplus^{k-1}u\right)L_1 v - L_2 v L_1\diamondsuit\left(\oplus^{k-1}u\right)\right]dx \\
&+ H(\diamondsuit\left(\oplus^{k-1}u\right),v).
\end{aligned} \tag{24}$$

Likewise, since $v \in C^{8k}(\overline{\Omega})$, replacing it by $\diamondsuit\left(\oplus^{k-1}v\right)$ in Equation (15), we have

$$\begin{aligned}
\int_\Omega u\oplus^k v dx - \int_\Omega\diamondsuit\left(\oplus^{k-1}v\right)\heartsuit u dx &= \int_\Omega\left[L_2\diamondsuit\left(\oplus^{k-1}v\right)L_1 u - L_2 u L_1\diamondsuit\left(\oplus^{k-1}v\right)\right]dx \\
&- H(u,\diamondsuit\left(\oplus^{k-1}v\right)).
\end{aligned} \tag{25}$$

Equation (23) can be obtained according to Equations (24) and (25). □

Theorem 2. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$*,* $\partial\Omega$ *be the boundary of* Ω*,* $u \in C^{8k}(\overline{\Omega})$*,* $\overline{\Omega} = \Omega\cup\partial\Omega$ *and* $M_k(x,\xi)$ *be a function which is given by Equation (12). Consequently*

(1) the BVP solution of the operator $\oplus$ *becomes*

$$\begin{aligned}
u(\xi) &= \int_\Omega M_1(x,\xi)\oplus u(x,\xi)dx \\
&- \int_\Omega\left[L_2\diamondsuit u(x,\xi)L_1 M_1(x,\xi) - L_2 M_1(x,\xi)L_1\diamondsuit u(x,\xi)\right]dx \\
&- H(\diamondsuit u(x,\xi),M_1(x,\xi)) - F(D_1(x,\xi),u(x,\xi)),\;\; \xi\in\Omega,
\end{aligned}$$

(2) the BVP solution of the operator $\oplus^k$, for $k \geq 2$, is

$$
\begin{aligned}
u(\xi) &= \int_\Omega M_k(x,\xi) \oplus^k u(x,\xi) dx \\
&- \int_\Omega \left[\diamondsuit \oplus^{k-1} u(x,\xi) \heartsuit M_k(x,\xi) - \diamondsuit M_2(x,\xi) \heartsuit \oplus u(x,\xi) \right] dx \\
&- \int_\Omega \left[L_2 \diamondsuit \oplus^{k-1} u(x,\xi) L_1 M_k(x,\xi) - L_2 M_k(x,\xi) L_1 \diamondsuit \oplus^{k-1} u(x,\xi) \right] dx \\
&- \int_\Omega \left[L_2 \oplus u(x,\xi) L_1 \diamondsuit M_2(x,\xi) - L_2 \diamondsuit M_2(x,\xi) L_1 \oplus u(x,\xi) \right] dx \\
&- \int_\Omega \left[L_2 \diamondsuit u(x,\xi) L_1 M_1(x,\xi) - L_2 M_1(x,\xi) L_1 \diamondsuit u(x,\xi) \right] dx \\
&- H\left(\diamondsuit \oplus^{k-1} u(x,\xi), M_k(x,\xi) \right) - H\left(\oplus u(x,\xi), \diamondsuit M_2(x,\xi) \right) \\
&- H(\diamondsuit u(x,\xi), M_1(x,\xi)) - F(D_1(x,\xi), u(x,\xi)), \;\; \xi \in \Omega,
\end{aligned}
$$

where H and F are defined by Equations (16) and (22), respectively.

Proof of Theorem 2. (1) By Equation (15), u and v are replaced by $\diamondsuit u$ and M_1, respectively. It follows that

$$
\begin{aligned}
\int_\Omega M_1 \oplus u dx - \int_\Omega \diamondsuit u \heartsuit M_1 dx &= \int_\Omega \left[L_2 \diamondsuit u L_1 M_1 - L_2 M_1 L_1 \diamondsuit u \right] dx \\
&+ H(\diamondsuit u, M_1).
\end{aligned}
$$

By Equations (10)–(12), with $k = 1$, we have $\heartsuit M_1 = D_1$. Therefore

$$
\begin{aligned}
\int_\Omega \diamondsuit u D_1 dx &= \int_\Omega M_1 \oplus u dx - \int_\Omega \left[L_2 \diamondsuit u L_1 M_1 - L_2 M_1 L_1 \diamondsuit u \right] dx \\
&- H(\diamondsuit u, M_1).
\end{aligned} \tag{26}
$$

According to Equations (21) and (26), the solution $u(\xi)$ becomes

$$
\begin{aligned}
u(\xi) &= \int_\Omega M_1(x,\xi) \oplus u(x,\xi) dx \\
&- \int_\Omega \left[L_2 \diamondsuit u(x,\xi) L_1 M_1(x,\xi) - L_2 M_1(x,\xi) L_1 \diamondsuit u(x,\xi) \right] dx \\
&- H(\diamondsuit u(x,\xi), M_1(x,\xi)) - F(D_1(x,\xi), u(x,\xi)), \;\; \xi \in \Omega.
\end{aligned} \tag{27}
$$

(2) By Equation (23) k, u, and v are replaced by $k-1$, $\oplus u$, and M_k, respectively. This leads to

$$
\begin{aligned}
\int_\Omega M_k \oplus^k u dx &- \int_\Omega \oplus u \oplus^{k-1} M_k dx \\
&= \int_\Omega \left[\diamondsuit \left(\oplus^{k-1} u \right) \heartsuit M_k - \diamondsuit \left(\oplus^{k-2} M_k \right) \heartsuit \oplus u \right] dx \\
&- \int_\Omega \left[L_2 \diamondsuit \left(\oplus^{k-1} u \right) L_1 M_k - L_2 M_k L_1 \diamondsuit \left(\oplus^{k-1} u \right) \right] dx \\
&- \int_\Omega \left[L_2 \oplus u L_1 \diamondsuit \left(\oplus^{k-2} M_k \right) - L_2 \diamondsuit \left(\oplus^{k-2} M_k \right) L_1 \oplus u \right] dx \\
&- H\left(\diamondsuit \left(\oplus^{k-1} u \right), M_k \right) - H\left(\oplus u, \diamondsuit \left(\oplus^{k-2} M_k \right) \right), \;\; k = 2, 3, \ldots.
\end{aligned}
$$

We have $\oplus^{k-1}M_k = M_1$ and $\oplus^{k-2}M_k = M_2$ resulting from Equation (13). Thus

$$\begin{aligned}
\int_\Omega \oplus u M_1 dx &= \int_\Omega M_k \oplus^k u dx - \int_\Omega \left[\Diamond\left(\oplus^{k-1}u\right)\heartsuit M_k - \Diamond M_2 \heartsuit \oplus u\right] dx \\
&- \int_\Omega \left[L_2 \Diamond\left(\oplus^{k-1}u\right) L_1 M_k - L_2 M_k L_1 \Diamond\left(\oplus^{k-1}u\right)\right] dx \\
&- \int_\Omega \left[L_2 \oplus u L_1 \Diamond M_2 - L_2 \Diamond M_2 L_1 \oplus u\right] dx \\
&- H\left(\Diamond\left(\oplus^{k-1}u\right), M_k\right) - H\left(\oplus u, \Diamond M_2\right), \ \ k = 2, 3, \ldots.
\end{aligned} \tag{28}$$

From Equations (27) and (28), we obtain

$$\begin{aligned}
u(\xi) &= \int_\Omega M_k \oplus^k u dx - \int_\Omega \left[\Diamond\left(\oplus^{k-1}u\right)\heartsuit M_k - \Diamond M_2 \heartsuit \oplus u\right] dx \\
&- \int_\Omega \left[L_2 \Diamond\left(\oplus^{k-1}u\right) L_1 M_k - L_2 M_k L_1 \Diamond\left(\oplus^{k-1}u\right)\right] dx \\
&- \int_\Omega \left[L_2 \oplus u L_1 \Diamond M_2 - L_2 \Diamond M_2 L_1 \oplus u\right] dx \\
&- \int_\Omega \left[L_2 \Diamond u L_1 M_1 - L_2 M_1 L_1 \Diamond u\right] dx \\
&- H\left(\Diamond\left(\oplus^{k-1}u\right), M_k\right) - H\left(\oplus u, \Diamond M_2\right) \\
&- H(\Diamond u, M_1) - F(D_1, u), \ \ k = 2, 3, \ldots.
\end{aligned} \tag{29}$$

Our claim is now completely proved. □

3.1. Example 1

To illustrate the results, let us consider an equation

$$\oplus u(\xi) = f(\xi), \ \ \xi \in \Omega, \tag{30}$$

where f is any tempered distribution on Ω. The boundary conditions on $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ are given by

$$\Diamond u(x,\xi) = M_1(x,\xi), \ \ x \in \partial\Omega_1, \tag{31}$$

$$u(x,\xi) = 0, \ \ x \in \partial\Omega_2. \tag{32}$$

From Equation (30) and [18] (p. 226), we have

$$L_2 \Diamond u(x,\xi) = -(-i)^{\frac{n-p}{2}} S_2(x,\xi) * f(x,\xi), \ \ \xi \in \Omega \tag{33}$$

$$L_1 M_1(x,\xi) = -R_2^e(x,\xi) * R_2^H(x,\xi) * \left[-(i)^{\frac{n-p}{2}} T_2(x,\xi)\right], \ \ \xi \in \Omega \tag{34}$$

$$L_2 M_1(x,\xi) = -R_2^e(x,\xi) * R_2^H(x,\xi) * \left[-(-i)^{\frac{n-p}{2}} S_2(x,\xi)\right], \ \ \xi \in \Omega \tag{35}$$

$$L_1 \Diamond u(x,\xi) = -(i)^{\frac{n-p}{2}} T_2(x,\xi) * f(x,\xi), \ \ \xi \in \Omega. \tag{36}$$

By taking the convolution operator $D_1(x,\xi)$ on both sides of Equation (31), it follows that

$$\begin{aligned}
D_1(x,\xi) * \Diamond u(x,\xi) &= D_1(x,\xi) * M_1(x,\xi), \\
\Diamond D_1(x,\xi) * u(x,\xi) &= D_1(x,\xi) * M_1(x,\xi), \\
\delta(x,\xi) * u(x,\xi) &= D_1(x,\xi) * M_1(x,\xi), \\
u(x,\xi) &= D_1(x,\xi) * M_1(x,\xi), \ \ x \in \partial\Omega_1.
\end{aligned} \tag{37}$$

By Equations (31) and (32), we get

$$H(\diamondsuit u, M_1) = 0, \;\; x \in \partial\Omega. \tag{38}$$

According to Equations (22), (32), and (37), this leads to

$$\begin{aligned} F(D_1, u) &= F(D_1, D_1 * M_1) \\ &= \int_{\partial\Omega} \left(D_1 \frac{\partial(\triangle D_1 * M_1)}{\partial\eta_*} - \triangle D_1 * M_1 \frac{\partial D_1}{\partial\eta_*} \right) dS_x \\ &+ \int_{\partial\Omega} \left(\Box D_1 \frac{\partial D_1 * M_1}{\partial\eta} - D_1 * M_1 \frac{\partial(\Box D_1)}{\partial\eta} \right) dS_x. \end{aligned} \tag{39}$$

By substituting Equations (30), (33)–(36), (38), and (39) into Equation (27), the solution becomes

$$\begin{aligned} u(\xi) &= \int_\Omega [M_1 f + ((-i)^{\frac{n-p}{2}} S_2 * f)(R_2^e * R_2^H * (i)^{\frac{n-p}{2}} T_2) \\ &- (R_2^e * R_2^H * (-i)^{\frac{n-p}{2}} S_2)((i)^{\frac{n-p}{2}} T_2 * f)]dx \\ &- \int_{\partial\Omega} \left(D_1 \frac{\partial(\triangle D_1 * M_1)}{\partial\eta_*} - \triangle D_1 * M_1 \frac{\partial D_1}{\partial\eta_*} \right) dS_x \\ &- \int_{\partial\Omega} \left(\Box D_1 \frac{\partial D_1 * M_1}{\partial\eta} - D_1 * M_1 \frac{\partial(\Box D_1)}{\partial\eta} \right) dS_x, \;\; \xi \in \Omega. \end{aligned} \tag{40}$$

Since all terms within the integrand are tempered distribution, the solution $u(\xi)$ therefore exists. Generally speaking, if we consider

$$\oplus^{k_1,k_2,k_3} = \triangle^{k_1} \Box^{k_2} \heartsuit^{k_3}, \tag{41}$$

where k_1, k_2 and k_3 are nonnegative integers. The operator $\oplus^{k_1,k_2,k_3}$ can reduce to the diamond operator iterated k-times, the Laplace operator iterated k-times, the ultrahyperbolic operator iterated k-times and the biharmonic operator iterated k-times, defined by Equations (1), (2), (3) and (5), respectively. For example, if we put $k_1 = k, k_2 = k_3 = 0$, the operator $\oplus^{k_1,k_2,k_3}$ becomes the Laplace operator iterated k-times $\triangle^k$.

3.2. Example 2 (Potential on Sphere with Dirichlet Boundary)

In the case that the operator $\oplus^k$ reduces to the Laplace operator iterated k-times $\triangle^k$,

$$\triangle^k u(\xi) = f(\xi), \;\; \xi \in \Omega, \tag{42}$$

where f is any tempered distribution and $\Omega = B(0, a) = \{\xi, |\xi| < a\}$ is a ball of radius a. The boundary conditions on Ω are given by

$$\triangle^i u(x, \xi) = R_{2i}^e(x, \xi), \; i = 1, 2, \ldots, k-1; \;\; x \in \partial\Omega, \tag{43}$$

and

$$u(x, \xi) = g(x, \xi), \;\; x \in \partial\Omega, \tag{44}$$

where g is a given tempered distribution. The solution of Equation (42) is

$$\begin{aligned} u(\xi) &= \int_\Omega R_{2k}^e \triangle^k u dx - \int_{\partial\Omega} \left(R_{2k}^e \frac{\partial u}{\partial\eta} - u \frac{\partial(R_{2k}^e)}{\partial\eta} \right) dS_x \\ &- \sum_{i=1}^{k-1} \int_{\partial\Omega} \left(R_{2i}^e \frac{\partial(\triangle^i u)}{\partial\eta} - (\triangle^i u) \frac{\partial R_{2i}^e}{\partial\eta} \right) dS_x, \;\; k \in \mathbb{N}. \end{aligned} \tag{45}$$

The sphere $\partial\Omega$ is the locus of point x for which the ratio of distances $r = |x - \xi|$ and $r^* = |x - \xi^*|$ from certain points is constant. Here we can choose any point $\xi \in \Omega$, then ξ^* is the point obtained from ξ by reflection with respect to the sphere $\partial\Omega$.

That is, $\xi^* = \frac{a^2}{|\xi|^2}\xi$, such that $R^e_{2k}(x,\xi) = \frac{\Gamma\left(\frac{n-2k}{2}\right)r^{2k-n}}{2^{2k}\pi^{\frac{n}{2}}\Gamma(k)}$ and $R^e_{2k}(x,\xi^*) = \frac{\Gamma\left(\frac{n-2k}{2}\right)r^{*2k-n}}{2^{2k}\pi^{\frac{n}{2}}\Gamma(k)}$ are the Green's functions of the Laplace operator with poles ξ and ξ^* respectively. Thus, for $x \in \partial\Omega$,

$$R^e_{2k}(x,\xi^*) = \left(\frac{a}{|\xi|}\right)^{2k-n} R^e_{2k}(x,\xi).$$

Define the function

$$G_k(x,\xi) = R^e_{2k}(x,\xi) - \left(\frac{|\xi|}{a}\right)^{2k-n} R^e_{2k}(x,\xi^*), \tag{46}$$

we have that $G_k(x,\xi)$ is the Green's function of Laplace operator and $G_k(x,\xi) = 0$ for $x \in \partial\Omega$.

By substituting Equations (42)–(44) and (46) into Equation (45), the solution becomes

$$\begin{aligned} u(\xi) &= \int_\Omega \left[R^e_{2k}(x,\xi) - \left(\frac{|\xi|}{a}\right)^{2k-n} R^e_{2k}(x,\xi^*)\right] f(x)dx \\ &\quad + \int_{\partial\Omega} g(x,\xi) \frac{\partial\left[R^e_{2k}(x,\xi) - \left(\frac{|\xi|}{a}\right)^{2k-n} R^e_{2k}(x,\xi^*)\right]}{\partial\eta} dS_x. \end{aligned}$$

In the special case when $k = 1$, it is the potential on the sphere $\partial\Omega$ of the problem (42) with the Dirichlet boundary condition (44).

3.3. Remark

In general, suppose that we consider equation $Lu = f$ where L is any linear partial differential operator. The solution to this problem can be found provided that L can be written in terms of two linear operators M and N (i.e., $L = MN$). Moreover, the solution to the equation $Mu = f$ as well as the Green's identity of the operator N are required.

4. Conclusions

This paper focuses on finding the Green's identity together with the solution of the BVP for the operator $\oplus^k$ which can be formulated in terms of the biharmonic and diamond operators. We first consider the solution for the case where $k = 1$ by employing the solution of the diamond operator and the Green's identity of the biharmonic operator. The solution for $k > 2$ is subsequently derived using the solution for the case $k = 1$ as well as the Green's identity for the operator $\oplus^k$. The solution for all k consists of the boundary terms satisfying Equations (16) and (22).

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Abbreviations

The following abbreviations are used in this manuscript:

BVP Boundary value problem

References

1. Pinsky, M.A. *Partial Ditterential Equations and Boundary Value Problems with Applications*; McGraw-Hill, Inc.: New York, NY, USA, 1991; p. 144.
2. McOwen, R.C. *Partial Ditterential Equations Method and Applications*; Pearson Education, Inc.: Cranbury, NJ, USA, 2003; p. 105.
3. Ozisik, M.N. *Heat Conduction*; John Wiley & Sons, Inc.: New York, NY, USA, 1993; pp. 7–17.
4. Haberman, R. *Applied Partial Ditterential Equations with Fourier Series and Boundary Value Problems*; Pearson Education, Inc.: Cranbury, NJ, USA, 2004; p. 170.
5. Gunzburger, M.; Lehoucq, R.B. A nonlocal vector calculus with application to nonlocal boundary value problems. *Multiscale Model. Simul.* **2010**, *8*, 1581–1598. [CrossRef]
6. Yosra, H.; Fadili, J.; Elmoataz, A. Nonlocal p-Laplacian evolution problems on graphs. *SIAM J. Numer. Anal.* **2018**, *56*, 1064–1090.
7. Papakhao, J.; Nonlaopon, K. On the general solution of the operator $\square_c^k$ related to the ultrahyperbolic operator. *Appl. Math. Sci.* **2011**, *5*, 2629–2638.
8. Liangprom, A.; Nonlaopon, K. On the convolution equation related to the diamond Klein-Gordon operator. *Abstr. Appl. Anal.* **2011**, *2011*, 908491. [CrossRef]
9. Zhang, L.; Zhao, Y. The lower bounds of the first eigenvalues for the biharmonic operator on manifolds. *J. Inequal. Appl.* **2016**, *2016*, 5. [CrossRef]
10. Choudhury, A.P.; Heck, H. Stability of the inverse boundary value problem for the biharmonic operator: Logarithmic estimates. *J. Inverse Ill-Posed Probl.* **2017**, *25*, 251–263. [CrossRef]
11. John, F. *Partial Differential Equations*; Springer: New York, NY, USA, 1982; pp. 94–101.
12. Bunpog, C. Cauchy problem of the diamond operator iterated k-times. *Int. J. Math. Anal.* **2013**, *7*, 2161–2170. [CrossRef]
13. Kananthai, A. On the solution of the n-dimensional diamond operator. *Appl. Math. Comput.* **1997**, *88*, 27–37. [CrossRef]
14. Trione, S.E. On the solutions of the causal and anticausal n-dimensional diamond operator. *Integral Transforms Spec. Funct.* **2002**, *13*, 49–57. [CrossRef]
15. Bupasiri, S.; Nonlaopon, K. On the solution of the n-dimensional operator related to the diamond operater. *FJMS* **2010**, *45*, 69–80.
16. Kananthai, A.; Suantai, S. On the convolution product of the distributional kernel $K_{\alpha,\beta,\gamma,\nu}$. *IJMMS* **2003**, *2003*, 153–158.
17. Kananthai, A.; Suantai, S. On the inversion of the kernel $K_{\alpha,\beta,\gamma,\nu}$, related to the operator $\oplus^k$. *Indian J. Pure Appl. Math.* **2003**, *34*, 1419–1429.
18. Kananthai, A.; Suantai, S.; Longani, V. On the operator $\oplus^k$ related to the wave equation and Laplacian. *Appl. Math. Comput.* **2002**, *132*, 219–229. [CrossRef]
19. Tariboon, J.; Kananthai, A. On the Green function of the $(\oplus + m^2)^k$ operator. *Integral Transforms Spec. Funct.* **2007**, *18*, 297–304. [CrossRef]
20. Satsanit, W. Green function and Fourier transform for o-plus operators. *Electron. J. Differ. Equ.* **2010**, *2010*, 1–14.
21. Riesz, M. Integrale de Riemann-Liouville et le probleme de Cauchy. *Acta Math.* **1949**, *81*, 1–223. [CrossRef]
22. Nozaki, Y. On Riemann-Liouville integral of ultra-hyperbolic type. *Kodai Math. Semin. Rep.* **1964**, *6*, 69–87. [CrossRef]
23. Trione, S.E. On Marcel Riesz's Ultra-hyperbolic kernel. *Stud. Appl. Math.* **1988**, *6*, 185–191. [CrossRef]
24. Zemanian, A.H. *Distribution Theory and Transform Analysis*; Mc-Graw Hill: New York, NY, USA, 1964; p. 102.
25. Bunpog, C. Distribution solutions of some PDEs related to the wave equation and the diamond operator. *Appl. Math. Sci.* **2013**, *7*, 5515–5524. [CrossRef]
26. Treves, F. *Basic Linear Partial Differential Equations*; Academic Press: New York, NY, USA, 1975.
27. Stakgold, I. *Boundary Value Problems of Mathematical Physics*; The Macmillan Company: New York, NY, USA, 1968.