mathematics

MDPI

Article

Computing Topological Indices and Polynomials for Line Graphs

Shahid Imran [1], **Muhammad Kamran Siddiqui** [2,*], **Muhammad Imran** [3,4] **and Muhammad Faisal Nadeem** [5]

[1] Govt Khawaja Rafique Shaheed College, Lahore 54000, Pakistan; imran_gon@yahoo.com
[2] Department of Mathematics, COMSATS University Islamabad, Sahiwal Campus, Sahiwal 57000, Pakistan
[3] Department of Mathematical Sciences, United Arab Emirates University, P. O. Box 15551, Al Ain, UAE; imrandhab@gmail.com
[4] Department of Mathematics, School of Natural Sciences (SNS), National University of Sciences and Technology (NUST), Sector H-12, Islamabad 44000, Pakistan
[5] Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore 54000, Pakistan; mfaisalnadeem@ymail.com
* Correspondence: kamransiddiqui75@gmail.com

Received: 16 July 2018; Accepted: 5 August 2018; Published: 10 August 2018

Abstract: A topological index is a number related to the atomic index that allows quantitative structure–action/property/toxicity connections. All the more vital topological indices correspond to certain physico-concoction properties like breaking point, solidness, strain vitality, and so forth, of synthetic mixes. The idea of the hyper Zagreb index, multiple Zagreb indices and Zagreb polynomials was set up in the substance diagram hypothesis in light of vertex degrees. These indices are valuable in the investigation of calming exercises of certain compound systems. In this paper, we computed the first and second Zagreb index, the hyper Zagreb index, multiple Zagreb indices and Zagreb polynomials of the line graph of wheel and ladder graphs by utilizing the idea of subdivision.

Keywords: hyper Zagreb index; first and second Zagreb index; multiple Zagreb indices; Zagreb polynomials; line graph; subdivision graph; tadpole; wheel; ladder

1. Introduction

Chemical graph theory is a branch of mathematical chemistry in which we apply apparatuses of the graph hypothesis to display the substance numerically. This hypothesis contributes noticeably to the synthetic sciences. A sub-atomic diagram is a straightforward limited graph in which vertices mean that the atoms and edges indicate concoction bonds in hidden compound structure. A topological index is actually a numerical amount related to the concoction constitution indicating the connection of the substance structure with numerous physio-synthetic properties, compound reactivity, and organic action. A decade ago, the diagram hypothesis found extensive use in research. The graph hypothesis has given physicists a variety of valuable apparatuses, such as topological files. Cheminformatics is a new subject that is a mix of science, arithmetic, and data science. It ponders quantitative structure–movement (QSAR) and structure–property (QSPR) connections that are utilized to anticipate the natural exercises and properties of synthetic mixes.

A graph G with vertex set V and edge set E are associated if there exists a connection between any combination of vertices in G. A network is just a connected diagram having no various edges and no self loops. For a graph G, the level of a vertex v is the quantity of edges occurrence to v and is indicated by $\xi(v)$.

A topological list $Top(G)$ of a graph G is a number with the property that for each chart H isomorphic to G, $Top(H) = Top(G)$. The idea of the topological file originated from the work done by Wiener [1], while at the same time, he was aiming to determine the breaking point of paraffin. He named this list as the way number. Later on, the way number was renamed as the Wiener index. The Wiener list is the first and most concentrated topological list, both from hypothetical and applications perspectives, and is characterized as the aggregate of separations between all sets of vertices in G (see [2] for details).

I. Gutman and N.Trinajstic [3] introduced the first and second Zagreb indices based on the degree of vertices as:

$$M_1(G) = \sum_{r_1r_2 \in E(G)} [\xi(r_1) + \xi(r_2)], \tag{1}$$

$$M_2(G) = \sum_{r_1r_2 \in E(G)} [\xi(r_1) \times \xi(r_2)]. \tag{2}$$

In 2013, Shirdel et al. [4] introduced the "hyper Zagreb index" as:

$$HM(G) = \sum_{r_1r_2 \in E(G)} \left[\xi(r_1) + \xi(r_2)\right]^2. \tag{3}$$

M. Ghorbani and N. Azimi defined [5] multiple Zagreb indices as:

$$PM_1(G) = \prod_{r_1r_2 \in E(G)} [\xi(r_1) + \xi(r_2)], \tag{4}$$

$$PM_2(G) = \prod_{r_1r_2 \in E(G)} [\xi(r_1) \times \xi(r_2)]. \tag{5}$$

The properties of $PM_1(G), PM_2(G)$ indices for some chemical structures have been studied in [6]. The first Zagreb polynomial $M_1(G,x)$) and second Zagreb polynomial $M_2(G,x)$ are defined as:

$$M_1(G,x) = \sum_{r_1r_2 \in E(G)} x^{[\xi(r_1)+\xi(r_2)]}, \tag{6}$$

$$M_2(G,x) = \sum_{r_1r_2 \in E(G)} x^{[\xi(r_1)\times\xi(r_2)]}. \tag{7}$$

There is now extensive research activity on $HM(G)$, $PM_1(G)$, $PM_2(G)$ indices and $M_1(G,x)$, $M_2(G,x)$ polynomials. See [7–9] for details.

2. Applications of Topological Indices

A ago, graph hypothesis had found an amazing use in research. Compound graph speculation has given researchers a variety of important gadgets (e.g., topological files). The Zagreb index is a topological descriptor that is related to a considerable measure of fabricated attributes of the particles, and has been discovered parallel to setting up the limit and Kovats constants of the particles [10]. The particle bond arranged hyper Zagreb index has a superior relationship with the security of direct dendrimers, besides the expanded medication stores and for setting up the strain criticalness of cycloalkanes [11–15]. To relate with certain physico-mix properties, different Zagreb indices have particularly needed insightful control over the farsighted essentialness of the dendrimers [16,17]. The first and second Zagreb polynomials were found to aid in the calculation of the aggregate π-electron imperativeness of the particles inside particular brutal verbalizations [18,19].

3. Topological Indices for Line Graph of Subdivided Graph $L(S(G))$

The subdivision graph [20] $S(G)$ is the diagram acquired from G by supplanting every one of its edges by a way of length 2, or equivalently, by embedding an extra vertex into each edge of G. The line diagram of the chart G, composed $L(G)$, is the basic diagram whose vertices are the edges of G, with $ef \in E(L(G))$ when e and f have a typical end point in G. Likewise, the line chart of the subdivided diagram is indicated by $L(S(G))$.

The tadpole graph $T_{n,k}$ is the diagram acquired by joining a cycle diagram C_n to a way of length k. By beginning with a disjoint association of two charts G_1 and G_2 and including edges joining every vertex of G_1 to that of G_2, one gets the whole $G_1 + G_2$ of G_1 and G_2. The total $C_n + K_1$ of a cycle C_n and a solitary vertex is alluded to as a wheel chart W_{n+1} with arrange n. The Cartesian product $G_1 \times G_2$ of charts G_1 and G_2 is a diagram with vertex set $V_1 \times V_2$, and two vertices r_1, s_1 and r_2, s_2 are nearby in $G_1 \times G_2$ if and only if either $r_1 = r_2$ and $s_1 s_2 \in E_2$, or $s_1 = s_2$ and $r_1 r_2 \in E_1$. The stepping stool diagram L_n is given by $L_n = K_2 \times P_n$, where P_n is the way of length n. It is along these lines proportionate to the framework chart $G_{2,n}$. The diagram acquired by means of this definition resembles a stepping stool, having two rails and n rungs between them.

In 2011, Ranjini et al. figured the unequivocal articulations for the Schultz lists of the subdivision diagrams of the tadpole, wheel, steerage, and stepping stool charts. They additionally contemplated the Zagreb records of the line diagrams of the tadpole, haggle charts with subdivision in [21,22]. Ali et al. [23] registered the topological lists for the line diagram of the sparkler chart, and Sardar et al. [24] computed the topological files of the line diagrams of Banana tree and Firecracker diagrams. Ahmad et al. [25] discuss the m-polynomials and degree-based topological indices for the line graph of the Firecracker graph. Soleimani et al. [26] discuss the topological properties of nanostructures. In 2015, Su and Xu figured the general aggregate availability records and co-lists of the line diagrams of the tadpole and haggle charts with subdivision in [27]. Nadeem et al. [28,29] registered ABC_4 and GA_5 records of the line charts of the tadpole, wheel, stepping stool, $2D-$lattice, nanotube, and nanotorus of $TUC_4C_8[p,q]$ diagrams.

3.1. Zagreb Indices and Zagreb Polynomials of the Line Graph of the Tadpole Graph $T_{n,k}$

Theorem 1. *Let R be the line graph of the tadpole graph $T_{n,k}$. Then*

$$\begin{aligned}
M_1(R) &= 8n + 8k + 12, \\
M_2(R) &= 8n + 8k + 23, \\
HM(R) &= 32n + 32k + 96, \\
PM_1(R) &= 3 \times 4^{(2n+2k-6)} \times 5^3 \times 6^3, \\
PM_2(R) &= 2 \times 4^{(2n+2k-6)} \times 6^3 \times 9^3, \\
M_1(R,x) &= x^3 + (2n+2k-6)x^4 + 3x^5 + 3x^6, \\
M_2(R,x) &= x^2 + (2n+2k-6)x^4 + 3x^6 + 3x^9.
\end{aligned}$$

Proof. The subdivision diagram of $T_{n,k}$ and the related line chart R appear individually in Figure 1a,b. The subdivision chart $S(T_{n,k})$ contains $2n+2k$ edges, so its line diagram contains $2n+2k$ vertices, out of which 3 vertices are of degree 3 and one vertex is of degree 1. The rest of the $2n+2k-4$ vertices are all of degree 2. It is easy to see that the aggregate number of edges of R is $2n+2k+1$. The edge set $E(R)$ separates into our edge segments in view of degrees of end vertices:

$E_{12}(R) = \{r_1 r_2 \in E(R) \mid \xi(r_1) = 1, \xi(r_2) = 2\}$,

$E_{22}(R) = \{r_1 r_2 \in E(R) \mid \xi(r_1) = 2, \xi(r_2) = 2\}$,

$E_{23}(R) = \{r_1 r_2 \in E(R) \mid \xi(r_1) = 2, \xi(r_2) = 3\}$,

$E_{33}(R) = \{r_1 r_2 \in E(R) \mid \xi(r_1) = 3, \xi(r_2) = 3\}$.

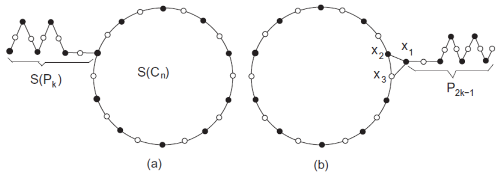

Figure 1. (**a**) Subdivision graph of the tadpole graph $T_{n,k}$; (**b**) Line graph of the subdivision graph of $(T_{n,k})$.

□

These four partitions of the edge set correspond to their degree sum of neighbors of end vertices. The number of edges in $E_{12}(R)$ is 1, in $E_{22}(R)$ there are $2n+2k-6$, in $E_{23}(R)$ there are 3, and in $E_{33}(R)$ there are 3. Now, using Equations (1)–(7), we have

$$\begin{aligned}
M_1(G) &= \sum_{r_1r_2 \in E(G)} [\xi(r_1)+\xi(r_2)], \\
M_1(R) &= \sum_{r_1r_2 \in E_{12}(R)} [\xi(r_1)+\xi(r_2)] + \sum_{r_1r_2 \in E_{22}(R)} [\xi(r_1)+\xi(r_2)] + \sum_{r_1r_2 \in E_{23}(R)} [\xi(r_1)+\xi(r_2)] \\
&+ \sum_{r_1r_2 \in E_{33}(R)} [\xi(r_1)+\xi(r_2)] \\
&= 3|E_{12}(R)| + 4|E_{22}(R)| + 5|E_{23}(R)| + 6|E_{33}(R)| \\
&= 3(1) + 4((2n+2k-6)) + 5(3) + 6(3) = 8n + 8k + 12,
\end{aligned}$$

$$\begin{aligned}
M_2(G) &= \sum_{r_1r_2 \in E(G)} [\xi(r_1)\times\xi(r_2)] \\
M_2(R) &= \sum_{r_1r_2 \in E_{12}(R)} [\xi(r_1)\times\xi(r_2)] + \sum_{r_1r_2 \in E_{22}(R)} [\xi(r_1)\times\xi(r_2)] + \sum_{r_1r_2 \in E_{23}(R)} [\xi(r_1)\times\xi(r_2)] \\
&+ \sum_{r_1r_2 \in E_{33}(R)} [\xi(r_1)\times\xi(r_2)] \\
&= 2|E_{12}(R)| + 4|E_{22}(R)| + 6|E_{23}(R)| + 9|E_{33}(R)| \\
&= 2(1) + 4((2n+2k-6)) + 6(3) + 9(3) = 8n + 8k + 23,
\end{aligned}$$

$$\begin{aligned}
HM(G) &= \sum_{r_1r_2 \in E(G)} [\xi(r_1)+\xi(r_2)]^2, \\
HM(R) &= \sum_{r_1r_2 \in E_{12}(R)} [\xi(r_1)+\xi(r_2)]^2 + \sum_{r_1r_2 \in E_{22}(R)} [\xi(r_1)+\xi(r_2)]^2 + \sum_{r_1r_2 \in E_{23}(R)} [\xi(r_1)+\xi(r_2)]^2 \\
&+ \sum_{r_1r_2 \in E_{33}(R)} [\xi(r_1)+\xi(r_2)]^2 \\
&= 9|E_{12}(R)| + 16|E_{22}(R)| + 25|E_{23}(R)| + 36|E_{33}(R)| \\
&= 9(1) + 16(2n+2k-6) + 25(3) + 36(3) = 2n + 32k + 96,
\end{aligned}$$

$$\begin{aligned}
PM_1(G) &= \prod_{r_1r_2 \in E(G)} [\xi(r_1)+\xi(r_2)], \\
PM_1(R) &= \prod_{r_1r_2 \in E_{12}(R)} [\xi(r_1)+\xi(r_2)] \times \prod_{r_1r_2 \in E_{22}(R)} [\xi(r_1)+\xi(r_2)] \times \prod_{r_1r_2 \in E_{23}(R)} [\xi(r_1)+\xi(r_2)] \\
&\times \prod_{r_1r_2 \in E_{33}(R)} [\xi(r_1)+\xi(r_2)] \\
&= 3^{|E_{12}(R)|} \times 4^{|E_{22}(R)|} \times 5^{|E_{23}(R)|} \times 6^{|E_{33}(R)|} = 3 \times 4^{(2n+2k-6)} \times 5^3 \times 6^3,
\end{aligned}$$

$$\begin{aligned}
PM_2(G) &= \prod_{r_1r_2\in E(G)} [\xi(r_1)\times\xi(r_2)],\\
PM_2(R) &= \prod_{r_1r_2\in E_{12}(R)} [\xi(r_1)\times\xi(r_2)] \times \prod_{r_1r_2\in E_{22}(R)} [\xi(r_1)\times\xi(r_2)] \times \prod_{r_1r_2\in E_{23}(R)} [\xi(r_1)\times\xi(r_2)]\\
&\times \prod_{r_1r_2\in E_{33}(R)} [\xi(r_1)\times\xi(r_2)]\\
&= 2^{|E_{12}(R)|}\times 4^{|E_{22}(R)|}\times 6^{|E_{23}(R)|}\times 9^{|E_{33}(R)|} = 2\times 4^{(2n+2k-6)}\times 6^3\times 9^3,
\end{aligned}$$

$$\begin{aligned}
M_1(G,x) &= \sum_{r_1r_2\in E(G)} x^{[\xi(r_1)+\xi(r_2)]},\\
M_1(R,x) &= \sum_{r_1r_2\in E_1(R)} x^{[\xi(r_1)+\xi(r_2)]} + \sum_{r_1r_2\in E_2(R)} x^{[\xi(r_1)+\xi(r_2)]} + \sum_{r_1r_2\in E_3(R)} x^{[\xi(r_1)+\xi(r_2)]}\\
&+ \sum_{r_1r_2\in E_4(R)} x^{[\xi(r_1)+\xi(r_2)]}\\
&= \sum_{r_1r_2\in E_1(R)} x^3 + \sum_{r_1r_2\in E_2(R)} x^4 + \sum_{r_1r_2\in E_3(R)} x^5 + \sum_{r_1r_2\in E_4(R)} x^6\\
&= |E_{12}(R)|x^3 + |E_{22}(R)|x^4 + |E_{23}(R)|x^5 + |E_{33}(R)|x^6 = x^3 + (2n+2k-6)x^4 + 3x^5 + 3x^6,
\end{aligned}$$

$$\begin{aligned}
M_2(G,x) &= \sum_{r_1r_2\in E(G)} x^{[\xi(r_1)\times\xi(r_2)]},\\
M_2(R,x) &= \sum_{r_1r_2\in E_1(R)} x^{[\xi(r_1)\times\xi(r_2)]} + \sum_{r_1r_2\in E_2(R)} x^{[\xi(r_1)\times\xi(r_2)]} + \sum_{r_1r_2\in E_3(R)} x^{[\xi(r_1)\times\xi(r_2)]}\\
&+ \sum_{r_1r_2\in E_4(R)} x^{[\xi(r_1)\times\xi(r_2)]}\\
&= \sum_{r_1r_2\in E_1(R)} x^2 + \sum_{r_1r_2\in E_2(R)} x^4 + \sum_{r_1r_2\in E_3(R)} x^6 + \sum_{r_1r_2\in E_4(R)} x^9\\
&= |E_{12}(R)|x^2 + |E_{22}(R)|x^4 + |E_{23}(R)|x^6 + |E_{33}(R)|x^9 = x^2 + (2n+2k-6)x^4 + 3x^6 + 3x^9.
\end{aligned}$$

Theorem 2. *Let H be the line graph of the wheel graph W_{n+1}. Then*

$$\begin{aligned}
M_1(H) &= n^3 + 27n,\\
M_2(H) &= \frac{n^4 - n^3 + 6n^2 + 72n}{2},\\
HM(H) &= 2n^4 - n^3 + 6n^2 + 45n,\\
PM_1(H) &= 6^{4n}\times(3+n)^n\times(2n)^{\left(\frac{n^2-n}{2}\right)},\\
PM_2(H) &= 9^{4n}\times(3n)^n\times n^{n^2-n},\\
M_1(H,x) &= 4nx^4 + nx^{3+n} + \left(\frac{n^2-n}{2}\right)x^{2n},\\
M_2(H,x) &= 4nx^9 + nx^{3n} + \left(\frac{n^2-n}{2}\right)x^{n^2}.
\end{aligned}$$

Proof. The subdivision chart of wheel W_{n+1} and the relating line diagram H appear separately in Figure 2a,b. The line chart H contains $4n$ vertices, of which $3n$ vertices are of degree 3 and the others are of degree n. It is simple to determine that the aggregate number of edges in the line diagram H are $\frac{n^2+9n}{2}$. To demonstrate the above proclamation, the edge set $E(H)$ isolates into three edge segments in light of the degrees of end vertices:

$E_{33}(H) = \{r_1r_2 \in E(H) \mid \xi(r_1) = 3, \xi(r_2) = 3\}$,

$E_{3n}(H) = \{r_1r_2 \in E(H) \mid \xi(r_1) = 3, \xi(r_2) = n\}$,

$$E_{nn}(H) = \{r_1r_2 \in E(H) \mid \xi(r_1) = n, \xi(r_2) = n\}.$$

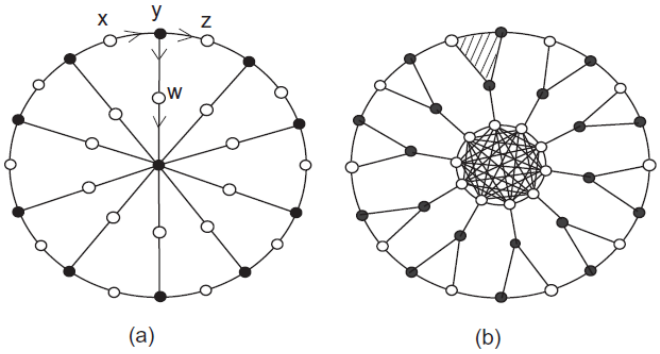

Figure 2. (**a**) Subdivision graph of wheel graph W_{n+1}; (**b**) Line graph of the subdivision graph of the wheel graph W_{n+1}.

□

These three partitions of the edge set correspond to their degree sum of neighbors of end vertices. The number of edges in $E_{33}(H)$ are $4n$, in $E_{3n}(HG)$ there are n, and in $E_{nn}(H)$ there are $\frac{n^2-n}{2}$. Now, using Equations (1)–(7), we have:

$$\begin{aligned}
M_1(G) &= \sum_{r_1r_2 \in E(G)} [\xi(r_1) + \xi(r_2)], \\
M_1(H) &= \sum_{r_1r_2 \in E_{33}(H)} [\xi(r_1) + \xi(r_2)] + \sum_{r_1r_2 \in E_{3n}(H)} [\xi(r_1) + \xi(r_2)] + \sum_{r_1r_2 \in E_{nn}(H)} [\xi(r_1) + \xi(r_2)] \\
&= 6|E_{33}(H)| + (3+n)|E_{3n}(H)| + 2n|E_{nn}(H)| \\
&= 6(4n) + (3+n)(n) + 2n(\frac{n^2-n}{2}) = n^3 + 27n, \\
M_2(G) &= \sum_{r_1r_2 \in E(G)} [\xi(r_1) \times \xi(r_2)], \\
M_2(H) &= \sum_{r_1r_2 \in E_{33}(H)} [\xi(r_1) \times \xi(r_2)] + \sum_{r_1r_2 \in E_{3n}(H)} [\xi(r_1) \times \xi(r_2)] + \sum_{r_1r_2 \in E_{nn}(H)} [\xi(r_1) \times \xi(r_2)] \\
&= 9|E_{33}(H)| + 3n|E_{3n}(H)| + n^2|E_{nn}(H)| \\
&= 9(4n) + 3n(n) + n^2(\frac{n^2-n}{2}) = \frac{n^4 - n^3 + 6n^2 + 72n}{2}, \\
HM(G) &= \sum_{r_1r_2 \in E(G)} [\xi(r_1) + \xi(r_2)]^2, \\
HM(H) &= \sum_{r_1r_2 \in E_{33}(H)} [\xi(r_1) + \xi(r_2)]^2 + \sum_{r_1r_2 \in E_{3n}(H)} [\xi(r_1) + \xi(r_2)]^2 + \sum_{r_1r_2 \in E_{nn}(H)} [\xi(r_1) + \xi(r_2)]^2 \\
&= 9|E_{33}(H)| + (3+n)^2|E_{3n}(H)| + 4n^2|E_{nn}(H)| \\
&= 9(4n) + n(3+n)^2 + 4n^2\left(\frac{n^2-n}{2}\right) = 2n^4 - n^3 + 6n^2 + 45n, \\
PM_1(G) &= \prod_{r_1r_2 \in E(G)} [\xi(r_1) + \xi(r_2)], \\
PM_1(H) &= \prod_{r_1r_2 \in E_{33}(H)} [\xi(r_1) + \xi(r_2)] \times \prod_{r_1r_2 \in E_{3n}(H)} [\xi(r_1) + \xi(r_2)] \times \prod_{r_1r_2 \in E_{nn}(H)} [\xi(r_1) + \xi(r_2)] \\
&= 6^{|E_{33}(H)|} \times (3+n)^{|E_{3n}(H)|} \times 2n^{|E_3(H)|} \\
&= 6^{4n} \times (3+n)^n \times (2n)^{\left(\frac{n^2-n}{2}\right)},
\end{aligned}$$

$$
\begin{aligned}
PM_2(G) &= \prod_{r_1r_2 \in E(G)} [\xi(r_1) \times \xi(r_2)], \\
PM_2(H) &= \prod_{r_1r_2 \in E_{33}(H)} [\xi(r_1) \times \xi(r_2)] \times \prod_{r_1r_2 \in E_{3n}(H)} [\xi(r_1) \times \xi(r_2)] \times \prod_{r_1r_2 \in E_{nn}(H)} [\xi(r_1) \times \xi(r_2)] \\
&= 9^{|E_{33}(H)|} \times (3n)^{|E_{3n}(H)|} \times (n^2)^{|E_{nn}(H)|} \\
&= 9^{4n} \times (3n)^n \times n^{n^2-n},
\end{aligned}
$$

$$
\begin{aligned}
M_1(G,x) &= \sum_{r_1r_2 \in E(G)} x^{[\xi(r_1)+\xi(r_2)]}, \\
M_1(H,x) &= \sum_{r_1r_2 \in E_1(H)} x^{[\xi(r_1)+\xi(r_2)]} + \sum_{r_1r_2 \in E_{3n}(H)} x^{[\xi(r_1)+\xi(r_2)]} + \sum_{r_1r_2 \in E_{nn}(H)} x^{[\xi(r_1)+\xi(r_2)]} \\
&= \sum_{r_1r_2 \in E_1(H)} x^6 + \sum_{r_1r_2 \in E_2(H)} x^{3+n} + \sum_{r_1r_2 \in E_3(H)} x^{2n} \\
&= |E_{33}(H)|x^6 + |E_{3n}(H)|x^{3+n} + |E_{nn}(H)|x^{2n} \\
&= 4nx^4 + nx^{3+n} + \Big(\frac{n^2-n}{2}\Big)x^{2n}, \\
M_2(G,x) &= \sum_{r_1r_2 \in E(G)} x^{[\xi(r_1)\times\xi(r_2)]}, \\
M_2(H,x) &= \sum_{r_1r_2 \in E_1(H)} x^{[\xi(r_1)\times\xi(r_2)]} + \sum_{r_1r_2 \in E_{3n}(H)} x^{[\xi(r_1)\times\xi(r_2)]} + \sum_{r_1r_2 \in E_{nn}(H)} x^{[\xi(r_1)\times\xi(r_2)]} \\
&= \sum_{r_1r_2 \in E_1(H)} x^9 + \sum_{r_1r_2 \in E_2(H)} x^{3n} + \sum_{r_1r_2 \in E_3(H)} x^{n^2} \\
&= |E_{33}(H)|x^9 + |E_{3n}(H)|x^{3n} + +|E_{nn}(H)|x^{n^2} \\
&= 4nx^9 + nx^{3n} + \Big(\frac{n^2-n}{2}\Big)x^{n^2}.
\end{aligned}
$$

Theorem 3. *Let P_n be the line graph of the ladder graph L_n of order n. Then,*

$$
\begin{aligned}
M_1(P_n) &= 154n - 76, \\
M_2(P_n) &= 81n - 132, \\
HM(P_n) &= 324n - 524, \\
PM_1(P_n) &= 4^6 \times 5^4 \times 6^{(9n-20)}, \\
PM_2(P_n) &= 4^6 \times 6^4 \times 9^{(9n-20)}, \\
M_1(P_n,x) &= 6x^4 + 4x^5 + (9n-20)x^6, \\
M_2(P_n,x) &= 6x^4 + 4x^6 + (9n-20)x^9.
\end{aligned}
$$

Proof. The subdivision diagram of the stepping stool chart L_n and the comparing line chart P_n appear in Figure 3a,b, respectively. The quantity of vertices in the line chart P_n are $6n-4$, among which 8 vertices are of degree 2 and the rest of the $6n-12$ vertices are of degree 3. It is simple to process that the aggregate number of edges in the line chart P_n is $9n-10$. To demonstrate the above proclamation, the edge set $E(P_n)$ isolates into three edge parcels in light of the degrees of end vertices:
$E_{22}(P_n) = \{r_1r_2 \in E(P_n) \mid \xi(r_1) = 2, \xi(r_2) = 2\}$,

$E_{23}(P_n) = \{r_1r_2 \in E(P_n) \mid \xi(r_1) = 2, \xi(r_2) = 3\}$,

$E_{33}(P_n) = \{r_1r_2 \in E(P_n) \mid \xi(r_1) = 3, \xi(r_2) = 3\}$.

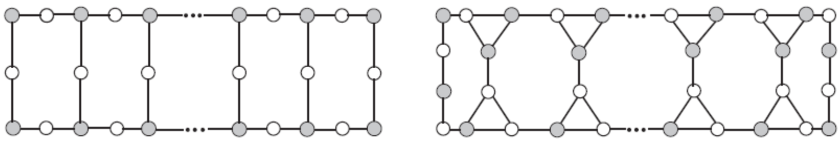

Figure 3. (**a**) Subdivision graph of the ladder graph L_n; (**b**) Line graph of the subdivision graph of the ladder graph L_n.

□

These three partitions of the edge set correspond to their degree sum of neighbors of end vertices. The number of edges in $E_{22}(P_n)$ are 6, in $E_{23}(P_n)$ there are 4, and in $E_{33}(P_n)$ there are $9n-20$. Now, using Equations (1)–(7), we have:

$$\begin{aligned}
M_1(G) &= \sum_{r_1r_2\in E(G)} [\xi(r_1)+\xi(r_2)], \\
M_1(P_n) &= \sum_{r_1r_2\in E_{22}(P_n)} [\xi(r_1)+\xi(r_2)] + \sum_{r_1r_2\in E_{23}(P_n)} [\xi(r_1)+\xi(r_2)] + \sum_{r_1r_2\in E_{33}(P_n)} [\xi(r_1)+\xi(r_2)] \\
&= 4|E_{22}(P_n)| + 5|E_{23}(P_n)| + 6|E_{33}(P_n)| \\
&= 4(6)+5(4)+6(9n-20) = 154n-76,
\end{aligned}$$

$$\begin{aligned}
M_2(G) &= \sum_{r_1r_2\in E(G)} [\xi(r_1)\times\xi(r_2)], \\
M_2(P_n) &= \sum_{r_1r_2\in E_{22}(P_n)} [\xi(r_1)\times\xi(r_2)] + \sum_{r_1r_2\in E_{23}(P_n)} [\xi(r_1)\times\xi(r_2)] + \sum_{r_1r_2\in E_{33}(P_n)} [\xi(r_1)\times\xi(r_2)] \\
&= 4|E_{22}(P_n)| + 6|E_{23}(P_n)| + 9|E_{33}(P_n)| \\
&= 4(6)+6(4)+9(9n-20) = 81n-132,
\end{aligned}$$

$$\begin{aligned}
HM(G) &= \sum_{r_1r_2\in E(G)} [\xi(r_1)+\xi(r_2)]^2, \\
HM(P_n) &= \sum_{r_1r_2\in E_{22}(P_n)} [\xi(r_1)+\xi(r_2)]^2 + \sum_{r_1r_2\in E_{23}(P_n)} [\xi(r_1)+\xi(r_2)]^2 + \sum_{r_1r_2\in E_{33}(P_n)} [\xi(r_1)+\xi(r_2)]^2 \\
&= 16|E_{22}(P_n)| + 25|E_{23}(P_n)| + 36|E_{33}(P_n)| \\
&= 16(6)+25(4)+36(9n-20) = 324n-524,
\end{aligned}$$

$$\begin{aligned}
PM_1(G) &= \prod_{r_1r_2\in E(G)} [\xi(r_1)+\xi(r_2)], \\
PM_1(P_n) &= \prod_{r_1r_2\in E_{22}(P_n)} [\xi(r_1)+\xi(r_2)] \times \prod_{r_1r_2\in E_{23}(P_n)} [\xi(r_1)+\xi(r_2)] \times \prod_{r_1r_2\in E_{33}(P_n)} [\xi(r_1)+\xi(r_2)] \\
&= 4^{|E_{22}(P_n)|} \times 5^{|E_{23}(P_n)|} \times 6^{|E_{33}(P_n)|} = 4^6 \times 5^4 \times 6^{(9n-20)},
\end{aligned}$$

$$\begin{aligned}
PM_2(G) &= \prod_{r_1r_2\in E(G)} [\xi(r_1)\times\xi(r_2)], \\
PM_2(P_n) &= \prod_{r_1r_2\in E_{22}(P_n)} [\xi(r_1)\times\xi(r_2)] \times \prod_{r_1r_2\in E_{23}(P_n)} [\xi(r_1)\times\xi(r_2)] \times \prod_{r_1r_2\in E_{33}(P_n)} [\xi(r_1)\times\xi(r_2)] \\
&= 4^{|E_{22}(P_n)|} \times 6^{|E_{23}(P_n)|} \times 9^{|E_{33}(P_n)|} = 4^6 \times 6^4 \times 9^{(9n-20)},
\end{aligned}$$

$$
\begin{aligned}
M_1(G,x) &= \sum_{r_1r_2\in E(G)} x^{[\xi(r_1)+\xi(r_2)]},\\
M_1(P_n,x) &= \sum_{r_1r_2\in E_1(P_n)} x^{[\xi(r_1)+\xi(r_2)]} + \sum_{r_1r_2\in E_2(P_n)} x^{[\xi(r_1)+\xi(r_2)]} + \sum_{r_1r_2\in E_3(P_n)} x^{[\xi(r_1)+\xi(r_2)]}\\
&= \sum_{r_1r_2\in E_1(P_n)} x^4 + \sum_{r_1r_2\in E_2(P_n)} x^5 + \sum_{r_1r_2\in E_3(P_n)} x^6\\
&= |E_{22}(P_n)|x^4 + |E_{23}(P_n)|x^5 + +|E_{33}(P_n)|x^6 = 6x^4 + 4x^5 + (9n-20)x^6,\\
M_2(G,x) &= \sum_{r_1r_2\in E(G)} x^{[\xi(r_1)\times\xi(r_2)]},\\
M_2(P_n,x) &= \sum_{r_1r_2\in E_1(P_n)} x^{[\xi(r_1)\times\xi(r_2)]} + \sum_{r_1r_2\in E_2(P_n)} x^{[\xi(r_1)\times\xi(r_2)]} + \sum_{r_1r_2\in E_3(P_n)} x^{[\xi(r_1)\times\xi(r_2)]}\\
&= \sum_{r_1r_2\in E_1(P_n)} x^4 + \sum_{r_1r_2\in E_2(P_n)} x^6 + \sum_{r_1r_2\in E_3(P_n)} x^9\\
&= |E_{22}(P_n)|x^4 + |E_{23}(P_n)|x^6 + +|E_{33}(P_n)|x^9 = 6x^4 + 4x^6 + (9n-20)x^9.
\end{aligned}
$$

4. Conclusions

In this paper we determined first and second Zagreb record, Hyper Zagreb index, first numerous Zagreb index, second different Zagreb index, and Zagreb polynomials of the line chart of tadpole and haggle diagrams by utilizing the idea of subdivision.

In the past couple of decades, investigations of the topological indices in view of end-vertex degrees of edges have seen a significant increase. The issue of determining the estimations of some outstanding degree-based topological indices is completely addressed for the line diagram of the subdivision graphs. This provides a path forward in this field of research. Also, in future we are intrigued to register these records for the line diagrams of some outstanding graphs.

Author Contributions: S.I and M.I. contributed to supervision, project administration, and funding, and analyzed the data curation. M.I. and M.K.S. contributed to designing the experiments, validation, conceptualization, formal analysis of experiments, resources, software, and some computations. M.F.N. contributed to investigation and methodology, and wrote the initial draft of the paper (which were investigated and approved by M.K.S.), and wrote the final draft. All authors read and approved the final version of the paper.

Funding: This research is supported by the Start-Up Research Grant 2016 of United Arab Emirates University (UAEU), Al Ain, United Arab Emirates via Grant No. G00002233, UPAR Grant of UAEU via Grant No. G00002590 and by the Summer Undergraduate Research Experience (SURE) plus 2017 research Grant via Grant No. G00002412.

Acknowledgments: The authors are grateful to the anonymous referees for their valuable comments and suggestions that improved this paper.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Wiener, H. Structural determination of paraffin boiling points. *J. Am. Chem. Soc.* **1947**, *69*, 17–20. [CrossRef] [PubMed]
2. Dobrynin, A.A.; Entringer, R.; Gutman, I. Wiener index of trees: Theory and applications. *Acta Appl. Math.* **2001**, *66*, 211–249. [CrossRef]
3. Gutman, I.; Trinajstic, N. Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons. *Chem. Phys. Lett.* **1972**, *17*, 535–538. [CrossRef]
4. Shirdel, G.H.; Reza Pour, H.; Sayadi, A.M. The Hyper Zagreb Index of Graph Operations. *Iran. J. Math. Chem.* **2013**, *42*, 213–220.
5. Ghorbani, M.; Azimi, N. Note on multiple Zagreb indices. *Iran. J. Math. Chem.* **2012**, *32*, 137–143.
6. Eliasi, M.; Iranmanesh, A.; Gutman, I. Multiplicative version of first Zagreb index. *Match Commun. Math. Comput. Chem.* **2012**, *68*, 217–230.

7. Liu, J.B.; Siddiqui, M.K.; Zahid, M.A.; Naeem, M.; Baig, A.Q. Topological Properties of Crystallographic Structure of Molecules. *Symmetry* **2018**, *10*, 265. [CrossRef]
8. Shao, Z.; Siddiqui, M.K.; Muhammad, M.H. Computing Zagreb Indices and Zagreb Polynomials for Symmetrical Nanotubes. *Symmetry* **2018**, *10*, 244. [CrossRef]
9. Siddiqui, M.K.; Imran, M.; Ahmad, A. On Zagreb indices, Zagreb polynomials of some nanostar dendrimers. *Appl. Math. Comput.* **2016**, 280, 132–139. [CrossRef]
10. Gao, W.; Siddiqui, M.K.; Naeem, M.; Rehman, N.A. Topological Characterization of Carbon Graphite and Crystal Cubic Carbon Structures. *Molecules* **2017**, *22*, 1496. [CrossRef] [PubMed]
11. Bača, M.; Horváthová, J.; Mokrišová, M.; Suhányiová, A. On topological indices of fullerenes. *Appl. Math. Comput.* **2015**, *251*, 154–161. [CrossRef]
12. Bača, M.; Horváthová, J.; Mokrišová, M.; Semanicová-Fenovcíková, A.; Suhányiová, A. On topological indices of carbon nanotube network. *Can. J. Chem.* **2015**, *93*, 1–4. [CrossRef]
13. Gao, W.; Farahani, M.R.; Siddiqui, M.K.; Jamil, M.K. On the First and Second Zagreb and First and Second Hyper-Zagreb Indices of Carbon Nanocones CNCk[n]. *J. Comput. Theor. Nanosci.* **2016**, *13*, 7475–7482. [CrossRef]
14. Gao, W.; Farahani, M.R.; Jamil, M.K.; Siddiqui, M.K. The Redefined First, Second and Third Zagreb Indices of Titania Nanotubes TiO_2[m,n]. *Open Biotechnol. J.* **2016**, *10*, 272–277. [CrossRef]
15. Gao, W.; Siddiqui, M.K.; Imran, M.; Jamil, M.K.; Farahani, M.R. Forgotten Topological Index of Chemical Structure in Drugs. *Saudi Pharm. J.* **2016**, *24*, 258–267. [CrossRef] [PubMed]
16. Imran, M.; Hayat, S.; Mailk, M.Y.H. On topological indices of certain interconnection networks. *Appl. Math. Comput.* **2014**, *244*, 936–951. [CrossRef]
17. Imran, M.; Siddiqui, M.K.; Naeem, M.; Iqbal, M.A. On Topological Properties of Symmetric Chemical Structures. *Symmetry* **2018**, *10*, 173. [CrossRef]
18. Siddiqui, M.K.; Gharibi, W. On Zagreb Indices, Zagreb Polynomials of Mesh Derived Networks. *J. Comput. Theor. Nanosci.* **2016**, *13*, 8683–8688. [CrossRef]
19. Siddiqui, M.K.; Naeem, M.; Rahman, N.A.; Imran, M. Computing topological indicesof certain networks. *J. Optoelectron. Adv. Mater.* **2016**, *18*, 884–892.
20. Rajan, M.A.; Lokesha, V.; Ranjini, P.S. A Study on Series Edge Graph Transformation. In Proceedings of the 23rd Joint Congress Iran-South Korea Jangjeon Mathematical Society, Ahvaz, Iran, 6–11 February 2010.
21. Ranjini, P.S.; Lokesha, V.; Cangül, I.N. On the Zagreb indices of the line graphs of the subdivision graphs. *Appl. Math. Comput.* **2011**, *218*, 699–702. [CrossRef]
22. Ranjini, P.S.; Lokesha, V.; Rajan, M.A. On the Shultz index of the subdivision graphs. *Adv. Stud. Contemp. Math.* **2011**, *213*, 279–290.
23. Ali, A.; Iqbal, H.; Nazeer, W.; Kang, S.M. on topological indices for the line graph of firecracker graph. *Int. J. Pure Appl. Math.* **2017**, *116*, 103–1042.
24. Sardar, M.S.; Zafar, S.; Zahid, Z. Computing topological indices of the line graphs of Banana tree graph and Firecracker graph. *Appl. Math. Nonlinear Sci.* **2017**, *2*, 83–92. [CrossRef]
25. Ahmad, M.S.; Nazeer, W.; Kang, S.M.; Jung, C.Y. M-polynomials and Degree based Topological Indices for the Line Graph of Firecracker Graph. *Global J. Pure Appl. Math.* **2017**, *13*, 2749–2776.
26. Soleimani, N.; Mohseni, E.; Rezaei, F.; Khati, F. Some Formulas for the Polynomials and Topological Indices of Nanostructures. *Acta Chem. Iasi* **2016**, *24*, 122–138. [CrossRef]
27. Su, G.; Xu, L. Topological indices of the line graph of subdivision graphs and their Schur-bounds. *Appl. Math. Comput.* **2015**, *253*, 395–401. [CrossRef]
28. Nadeem, M.F.; Zafar, S.; Zahid, Z. Certain topological indicies of the line graph of subdivsion graphs. *Appl. Math. Comput.* **2015**, *271*, 790–794.
29. Nadeem, M.F.; Zafar, S.; Zahid, Z. On topological properties of the line graphs of subdivision graphs of certain nanostructures. *Appl. Math. Comput.* **2016**, *273*, 125–130. [CrossRef]