mathematics

Article

Strong Convergence Theorems of Viscosity Iterative Algorithms for Split Common Fixed Point Problems

Peichao Duan *, **Xubang Zheng** and **Jing Zhao**

College of Science, Civil Aviation University of China, Tianjin 300300, China; xbzheng224@163.com (X.Z.); zhaojing200103@163.com (J.Z.)

* Correspondence: pcduancauc@126.com

Received: 19 November 2018; Accepted: 19 December 2018; Published: 24 December 2018

Abstract: In this paper, we propose a viscosity approximation method to solve the split common fixed point problem and consider the bounded perturbation resilience of the proposed method in general Hilbert spaces. Under some mild conditions, we prove that our algorithms strongly converge to a solution of the split common fixed point problem, which is also the unique solution of the variational inequality problem. Finally, we show the convergence and effectiveness of the algorithms by two numerical examples.

Keywords: strong convergence; viscosity iterative algorithm; split common fixed point problem; nonexpansive mapping; averaged mapping; bounded perturbation resilience; variational inequality problem

MSC: 47H05; 47H09; 47H10; 47J25

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. The *split feasibility problem* (SFP for short) is as follows:

$$\text{Find a point } x^* \in C \text{ such that } Ax^* \in Q, \tag{1}$$

where C and Q are nonempty closed convex subsets of H_1 and H_2, respectively, and A is a bounded linear operator of H_1 into H_2.

If the set of solutions of the problem (1) is nonempty, then x^* solving problem (1) is equivalent to

$$x^* = P_C(I - \tau A^*(I - P_Q)A)x^*, \tag{2}$$

where $\tau > 0$ and P_C denotes the metric projection of H_1 onto C and A^* is the corresponding adjoint operator of A.

Recently, many problems in engineering and technology can be modeled by problem (1) and many authors have shown that the SFP has many applications in our real life such as image reconstruction, signal processing and intensity-modulated radiation therapy (see [1–3]).

In 1994, Censor and Elfving [4] used their algorithm to solve the SFP in the finite-dimensional Euclidean space. In 2002, Byrne [5] improved the algorithm of Censor and Elfving and presented a new method called the *CQ algorithm* for solving the SFP (1) as follows:

$$x_{n+1} = P_C(I - \tau A^*(I - P_Q)A)x_n, \ \ \forall n \in \mathbb{N}. \tag{3}$$

The *split common fixed point problem* (shortly, SCFPP) is formulated as follows:

$$\text{Find a point } x^* \in Fix(U) \text{ such that } Ax^* \in Fix(T), \tag{4}$$

where $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are nonlinear mappings; here, $Fix(U)$ denotes the set of fixed points of the mapping U. We use S to denote the solution set of problem (4).

Note that, since every closed convex subset of a Hilbert space is the fixed point set of its associating projection if $U := P_C$ and $T := P_Q$, the SFP becomes a special case of the SCFPP.

In 2007, Censor and Segal [6] first studied the SCFPP and, to solve the SCFPP, they proposed the following iterative algorithm:

$$x_{n+1} = U(x_n - \tau A^*(I - T)Ax_n), \ \ \forall n \in \mathbb{N}, \tag{5}$$

where τ is a properly chosen stepsize. Algorithm (5) was originally designed to solve the problem (4) for directed mappings.

In 2010, Moudafi [7] proposed an iterative method to solve the SCFPP for quasi-nonexpansive mappings. In 2014, combining the Moudafi method with the Halpern iterative method, Kraikaew and Saejung [8] proposed a new iterative algorithm which does not involve the projection operator to solve the split common fixed point problem. More specifically, their algorithm generates a sequence $\{x_n\}$ via the recursions:

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)U(x_n - \tau A^*(I - T)Ax_n), \ \ \forall n \in \mathbb{N},$$

where $x_0 \in H$ is a fixed element, U and T are quasi-nonexpansive operators.

Recently, many authors have studied the SCFPP, the generalized SCFPP and some relative problems (see, for instance, refs. [3–5,9–13] and they have also proposed a lot of algorithms to solve the SCFPP (see [14–17] and the references therein).

On the other hand, the bounded perturbation resilience and superiorization of iterative methods have been studied by some authors (see [18–23]). These problems have received much attention because of their applications in convex feasibility problems [24], image reconstruction [25] and inverse problems of radiation therapy [26] and so on.

Let $\mathbf{P}$ denote an algorithm operator. If the iteration $x_{n+1} = \mathbf{P}x_n$ is replaced by

$$x_{n+1} = \mathbf{P}(x_n + \beta_n \nu_n),$$

where β_n is a sequence of nonnegative real numbers and $\{\nu_n\}$ is a sequence in H such that

$$\sum_{n=0}^{\infty} \beta_n < \infty \ \text{ and } \ \|\nu_n\| \le M. \tag{6}$$

Then, the algorithm is still convergent and so the algorithm $\mathbf{P}$ is the bounded perturbation resilient [19].

In 2016, Jin, Censor and Jiang [21] introduced the projected scaled gradient method (PSG for short) with bounded perturbations for solving the following minimization problem:

$$\min_{x \in C} f(x), \tag{7}$$

where f is a continuous differentiable, convex function. The method PSG generates a sequence $\{x_n\}$ defined by

$$x_{n+1} = P_C(x_n - \gamma_n D(x_n)\nabla f(x_n) + e(x_n)), \ \ \forall n \ge 0, \tag{8}$$

where $D(x_n)$ is a diagonal scaling matrix and $e(x_n)$ denotes the sequence of outer perturbations satisfying $\sum_{n=0}^{\infty} \|e(x_n)\| < \infty$.

Recently, Xu [22] projected the superiorization techniques for the relaxed PSG as follows:

$$x_{n+1} = (1-\tau_n)x_n + \tau_n P_C(x_n - \gamma_n D(x_n)\nabla f(x_n) + e(x_n)), \ \forall n \ge 0, \tag{9}$$

where τ_n is a sequence in $[0,1]$.

Recently, for solving minimization problem of the combination of two convex functions $\min_{x\in H} f(x) + g(x)$, Guo and Cui [20] considered the modified proximal gradient method:

$$x_{n+1} = \alpha_n h(x_n) + (1-\alpha_n) prox_{\lambda_n g}(I - \lambda_n \nabla f)(x_n) + e(x_n), \ \forall n \ge 0, \tag{10}$$

and, under suitable conditions, they proved some strong convergence theorems of the method. The definition of proximal operator $prox_{\lambda\varphi}$ is as follows.

Definition 1 (see [27]). *Let $\Gamma_0(H)$ be the space of functions on a real Hilbert space H that are proper, lower semicontinuous and convex. The proximal operator of $\varphi \in \Gamma_0(H)$ is defined by*

$$prox_{\varphi}(x) = arg\min_{\nu\in H}\{\varphi(\nu) + \frac{1}{2}\|\nu - x\|^2\}, \quad x \in H.$$

The proximal operator of φ of order $\lambda > 0$ is defined as the proximal operator of $\lambda\varphi$, that is,

$$prox_{\lambda\varphi}(x) = arg\min_{\nu\in H}\{\varphi(\nu) + \frac{1}{2\lambda}\|\nu - x\|^2\}, \quad x \in H.$$

Now, we propose a viscosity method for the problem (4) as follows:

$$x_{n+1} = \alpha_n h(x_n + e(x_n)) + (1-\alpha_n)U(x_n - \tau_n A^*(I-T)Ax_n + e(x_n)), \ \forall n \ge 0. \tag{11}$$

If we treat the above algorithm as the basic algorithm **P**, the bounded perturbation of it is a sequence $\{x_n\}$ generated by the iterative process:

$$\begin{cases} y_n = x_n + \beta_n \nu_n, \\ x_{n+1} = \alpha_n h(y_n + e(y_n)) \\ \qquad\quad + (1-\alpha_n)U(y_n - \tau_n A^*(I-T)Ay_n + e(y_n)), \ \forall n \ge 0. \end{cases} \tag{12}$$

In this paper, mainly based on the above works [6,20,22], we prove that our main iterative method (11) is the bounded perturbation resilient and, under some mild conditions, our algorithms strongly converge to a solution of the split common fixed point problem, which is also the unique solution of the variational inequality problem (13). Finally, we give two numerical examples to demonstrate the effectiveness of our iterative schemes.

2. Preliminaries

Let $\{x_n\}$ be a sequence in the real Hilbert space H. We adopt the following notations:

(1) Denote $\{x_n\}$ converging weakly to x by $x_n \rightharpoonup x$ and $\{x_n\}$ converging strongly to x by $x_n \to x$.
(2) Denote the weak ω-limit set of $\{x_n\}$ by $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$.

Definition 2. *A mapping $F : H \to H$ is said to be:*

(i) Lipschitz if there exists a positive constant L such that

$$\|Fx - Fy\| \le L\|x - y\|, \ \forall x, y \in H.$$

In particular, if $L = 1$, then we say that F is nonexpansive, namely,

$$\|Fx - Fy\| \leq \|x - y\|, \ \forall x, y \in H.$$

If $L \in [0, 1)$, then we say F is contractive.

(ii) *α-averaged mapping (shortly, α-av) if*

$$F = (1 - \alpha)I + \alpha T,$$

where $\alpha \in [0, 1)$ and $T : H \to H$ is nonexpansive.

Definition 3. *A mapping $B : H \to H$ is said to be:*

(i) *monotone if*

$$\langle Bx - By, x - y\rangle \geq 0, \ \forall x, y \in H.$$

(ii) *η-strongly monotone if there exists a positive constant η such that*

$$\langle Bx - By, x - y\rangle \geq \eta\|x - y\|^2, \ \forall x, y \in H.$$

(iii) *α-inverse strongly monotone (shortly, α-ism) if there exists a positive constant α such that*

$$\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2, \ \forall x, y \in H.$$

In particular, if $\alpha = 1$, then we say B is firmly nonexpansive, namely,

$$\langle Bx - By, x - y\rangle \geq \|Bx - By\|^2, \ \forall x, y \in H.$$

Using the Cauchy–Schwartz inequality, it is easy to deduce that B is $\frac{1}{\alpha}-$Lipschitz if it is α-ism. Now, we give the following lemmas and propositions needed in the proof of the main results.

Lemma 1 ([28])**.** *Let H be a real Hilbert space. Then, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle x + y, y\rangle, \ \forall x, y \in H.$$

Lemma 2 ([29])**.** *Let $h : H \to H$ be a ρ-contraction with $\rho \in (0, 1)$ and $T : H \to H$ be a nonexpansive mapping. Then,*

(i) *$I - h$ is $(1 - \rho)$-strongly monotone, that is,*

$$\langle (I - h)x - (I - h)y, x - y\rangle \geq (1 - \rho)\|x - y\|^2, \ \forall x, y \in H.$$

(ii) *$I - T$ is monotone, that is,*

$$\langle (I - T)x - (I - T)y, x - y\rangle \geq 0, \ \forall x, y \in H.$$

Proposition 1 ([30])**.**

(i) *If $T_1, T_2, \cdots, T_n$ are averaged mappings, then we can get that $T_nT_{n-1}\cdots T_1$ is averaged. In particular, if T_i is α_i-av for each $i = 1, 2$, where $\alpha_i \in (0, 1)$, then T_2T_1 is $(\alpha_2 + \alpha_1 - \alpha_2\alpha_1)$-av.*

(ii) *If the mappings* $\{T_i\}_{i=1}^N$ *are averaged and have a common fixed point, then we have*

$$\bigcap_{i=1}^{N} Fix(T_i) = Fix(T_1 \dots T_N).$$

(iii) *A mapping* T *is nonexpansive if and only if* $I - T$ *is* $\frac{1}{2}-$*ism.*

(iv) *If* T *is* ν*-ism, then, for any* $\tau > 0$, τT *is* $\frac{\nu}{\tau}$*-ism.*

(v) T *is averaged if and only if* $I - T$ *is* ν*-ism for some* $\nu > \frac{1}{2}$*. Indeed, for any* $0 < \alpha < 1$, T *is* $\alpha-$*averaged if and only if* $I - T$ *is* $\frac{1}{2\alpha}$*-ism.*

Lemma 3 ([31])**.** *Let* H *be a real Hilbert space and* $T : H \to H$ *be a nonexpansive mapping with* $Fix(T) \neq \emptyset$*. If* $\{x_n\}$ *is a sequence in* H *weakly converging to* x *and* $\{(I - T)x_n\}$ *converges strongly to* y*, then* $(I - T)x = y$*. In particular, if* $y = 0$*, then* $x \in Fix(T)$*.*

Lemma 4 ([32] or [33])**.** *Assume that* $\{s_n\}$ *is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n\delta_n, \;\; s_{n+1} \leq s_n - \eta_n + \varphi_n, \;\; \forall n \geq 0,$$

where $\{\gamma_n\}$ *is a sequence in* $(0, 1)$*,* $\{\eta_n\}$ *is a sequence of nonnegative real numbers,* $\{\delta_n\}$ *and* $\{\varphi_n\}$ *are two sequences in* $\mathbb{R}$ *such that*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\lim_{n\to\infty} \varphi_n = 0$;

(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ *for any subsequence* $\{n_k\} \subset \{n\}$*.*

Then, $\lim_{n\to\infty} s_n = 0$*.*

Lemma 5. *Assume that* $A : H_1 \to H_2$ *is a bounded linear operator and* A^* *is the corresponding adjoint operator of* A*. Let* $T : H_2 \to H_2$ *be a nonexpansive mapping. If there exists a point* $z \in H_1$ *such that* $Az \in Fix(T)$*, then*

$$(I - T)Ax = 0 \Longleftrightarrow A^*(I - T)Ax = 0, \;\; \forall x \in H_1.$$

Proof. It is clear that $(I - T)Ax = 0$ implies $A^*(I - T)Ax = 0$ for all $x \in H_1$.

To see the converse, let $x \in H$ such that $A^*(I - T)Ax = 0$. Take $Az \in Fix(T)$. Since T is nonexpansive, we have

$$\|TAx - Az\|^2 = \|TAx - TAz\|^2 \leq \|Ax - Az\|^2$$

and

$$\|TAx - Az\|^2 = \|Ax - TAx - (Ax - Az)\|^2 = \|Ax - TAx\|^2 - 2\langle Ax - TAx, Ax - Az\rangle + \|Ax - Az\|^2.$$

Combine the above two formulas, we have

$$\|Ax - TAx\|^2 \leq 2\langle Ax - TAx, Ax - Az\rangle = 2\langle A^*(I - T)Ax, x - z\rangle = 0.$$

This completes the proof. □

3. The Main Results

In 2000, Moudafi [34] proposed the viscosity approximation method:

$$x_{n+1} = \alpha_n h(x_n) + (1 - \alpha_n)Nx_n, \;\; \forall n \geq 0,$$

which converges strongly to a fixed point x^* of the nonexpansive mapping N (see [35,36]). In 2004, Xu [29] further proved that $x^* \in Fix(N)$ is also the unique solution of the following variational inequality problem:

$$\langle (I-h)x^*, \tilde{x}-x^* \rangle \geq 0, \ \ \forall \tilde{x} \in Fix(N), \tag{13}$$

where $h : H \to H$ is a ρ-contraction. By Lemma 2, we get $I-h$ is strongly monotone, hence the solution of problem (13) is unique.

In this section, we present a viscosity iterative method for solving problem (4). Meanwhile, the algorithm approximates the unique fixed point of variational inequality problem (13).

Putting $e_n := e(x_n)$, we can rewrite the iteration (11) as follows:

$$\begin{aligned} x_{n+1} &= \alpha_n h(x_n+e_n) + (1-\alpha_n)U(x_n - \tau_n A^*(I-T)Ax_n + e_n) \\ &= \alpha_n h(x_n) + (1-\alpha_n)U(x_n - \tau_n A^*(I-T)Ax_n) + \tilde{e}_n, \ \ \forall n \geq 0, \end{aligned} \tag{14}$$

where

$$\begin{aligned} \tilde{e}_n &= \alpha_n (h(x_n+e_n) - h(x_n)) \\ &\quad + (1-\alpha_n)(U(x_n - \tau_n A^*(I-T)Ax_n + e_n) - U(x_n - \tau_n A^*(I-T)Ax_n)). \end{aligned}$$

Since U is nonexpansive and h is contractive, it is easy to get

$$\begin{aligned} \|\tilde{e}_n\| &\leq \alpha_n \|h(x_n+e_n) - h(x_n)\| \\ &\quad + (1-\alpha_n)\|U(x_n - \tau_n A^*(I-T)Ax_n + e_n) - U(x_n - \tau_n A^*(I-T)Ax_n)\| \\ &\leq (\alpha_n \rho + 1 - \alpha_n)\|e_n\| \\ &\leq \|e_n\|. \end{aligned} \tag{15}$$

Theorem 1. *Let H_1,H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator with $L = \|A^*A\|$, where A^* is the adjoint of A. Suppose that $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two averaged mappings with the coefficients γ_1 and γ_2, respectively. Assume that the problem (4) is consistent (i.e., $S \neq \emptyset$). Let $h : H_1 \to H_1$ be a ρ-contraction with $0 \leq \rho < 1$. For any $x_0 \in H_1$, define the sequence $\{x_n\}$ by (14). If the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \tau_n \leq \limsup_{n\to\infty} \tau_n < \frac{1}{\gamma_2 L}$;
(iii) $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $x^* \in S$, which is also the unique solution of the variational inequality problem (13).

Proof. Set $V_{\tau_n} := U(I - \tau_n A^*(I-T)A)$. Then, by Proposition 1, it follows that $U(I - \tau_n A^*(I-T)A)$ is $(\gamma_1 + (1-\gamma_1)\gamma_2\tau_n L)$-*av* as $0 < \tau_n < \frac{1}{\gamma_2 L}$.

Step 1. Show that $\{x_n\}$ is bounded. For any $z \in S$, we have

$$\begin{aligned} &\|x_{n+1} - z\| \\ &= \|\alpha_n h(x_n) + (1-\alpha_n)V_{\tau_n}x_n + \tilde{e}_n - z\| \\ &= \|\alpha_n (h(x_n) - z) + (1-\alpha_n)(V_{\tau_n}x_n - z) + \tilde{e}_n\| \\ &\leq \alpha_n \|h(x_n) - h(z)\| + \alpha_n\|h(z) - z\| + (1-\alpha_n)\|V_{\tau_n}x_n - z\| + \|\tilde{e}_n\| \\ &\leq \alpha_n \rho \|x_n - z\| + \alpha_n \|h(z)-z\| + (1-\alpha_n)\|x_n - z\| + \|\tilde{e}_n\| \\ &= (1 - \alpha_n(1-\rho))\|x_n - z\| + \alpha_n(1-\rho)\frac{\|h(z)-z\| + \|\tilde{e}_n\|/\alpha_n}{1-\rho}. \end{aligned} \tag{16}$$

Note that the condition (iii) and (15) imply that $\sum_{n=0}^{\infty} \|\tilde{e}_n\| < \infty$ and, from the conditions (i), (iii) and $\alpha_n > 0$, it is easy to show that $\{\|\tilde{e}_n\|/\alpha_n\}$ is bounded. Therefore, there exists $M_1 > 0$, such that $M_1 := \sup_{n\in\mathbb{N}}\{\|h(z) - z\| + \|\tilde{e}_n\|/\alpha_n\}$. Thus, since the induction argument shows that

$$\|x_n - z\| \le \max\left\{\|x_0 - z\|, \frac{M_1}{1-\rho}\right\},$$

it turns out that the sequence $\{x_n\}$ is bounded and so are $\{h(x_n)\}$, $\{V_{\tau_n}x_n\}$ and $\{A^*(I-T)Ax_n\}$.

Step 2. Show that, for any sequence $\{n_k\} \subset \{n\}$, if $\eta_{n_k} \to 0$, then $\lim_{k\to\infty} \|x_{n_k} - V_{\tau_{n_k}}x_{n_k}\| = 0$. First, if $z \in S$, then we have

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&= \|\alpha_n h(x_n) + (1-\alpha_n)V_{\tau_n}x_n + \tilde{e}_n - z\|^2 \\
&= \|\alpha_n h(x_n) + (1-\alpha_n)V_{\tau_n}x_n - z\|^2 + 2\langle \alpha_n h(x_n) + (1-\alpha_n)V_{\tau_n}x_n - z, \tilde{e}_n\rangle + \|\tilde{e}_n\|^2 \\
&\le \alpha_n^2\|h(x_n) - z\|^2 + (1-\alpha_n)^2\|V_{\tau_n}x_n - z\|^2 + 2\alpha_n(1-\alpha_n)\langle h(x_n) - z, V_{\tau_n}x_n - z\rangle \\
&\quad + (2\alpha_n\|h(x_n) - z\| + 2(1-\alpha_n)\|x_n - z\| + \|\tilde{e}_n\|)\|\tilde{e}_n\| \\
&\le 2\alpha_n^2(\|h(x_n) - h(z)\|^2 + \|h(z) - z\|^2) + (1-\alpha_n)^2\|V_{\tau_n}x_n - z\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle h(x_n) - z, V_{\tau_n}x_n - z\rangle + M_2\|\tilde{e}_n\| \\
&\le 2\alpha_n^2(\|h(x_n) - h(z)\|^2 + \|h(z) - z\|^2) + (1-\alpha_n)^2\|V_{\tau_n}x_n - z\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)(\|h(x_n) - h(z)\|\|x_n - z\| + \langle h(z) - z, V_{\tau_n}x_n - z\rangle) + M_2\|\tilde{e}_n\| \\
&\le (1 - \alpha_n(2 - \alpha_n(1+2\rho^2) - 2(1-\alpha_n)\rho))\|x_n - z\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle h(z) - z, V_{\tau_n}x_n - z\rangle + 2\alpha_n^2\|h(z) - z\|^2 + M_2\|\tilde{e}_n\|,
\end{aligned} \tag{17}$$

where $M_2 := \sup_{n\in\mathbb{N}}\{2\alpha_n\|h(x_n) - z\| + 2(1-\alpha_n)\|x_n - z\| + \|\tilde{e}_n\|\}$.

Second, we can rewrite V_{τ_n} as

$$V_{\tau_n} = U(I - \tau_n A^*(I-T)A) = (1-w_n)I + w_n W_n, \tag{18}$$

where $w_n = \gamma_1 + (1-\gamma_1)\gamma_2\tau_n L$ and W_n is nonexpansive. By the condition (ii), we get $\gamma_1 < \liminf_{n\to\infty} w_n \le \limsup_{n\to\infty} w_n < 1$. Thus, it follows from (14), (17) and (18) that

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&= \|\alpha_n h(x_n) + (1-\alpha_n)V_{\tau_n}x_n + \tilde{e}_n - z\|^2 \\
&\le \|\alpha_n h(x_n) + (1-\alpha_n)V_{\tau_n}x_n - z\|^2 + M_2\|\tilde{e}_n\| \\
&= \|V_{\tau_n}x_n - z + \alpha_n(h(x_n) - V_{\tau_n}x_n)\|^2 + M_2\|\tilde{e}_n\| \\
&= \|V_{\tau_n}x_n - z\|^2 + {\alpha_n}^2\|h(x_n) - V_{\tau_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\tau_n}x_n - z, h(x_n) - V_{\tau_n}x_n\rangle + M_2\|\tilde{e}_n\| \\
&= \|(1-w_n)x_n + w_n W_n x_n - z\|^2 + {\alpha_n}^2\|h(x_n) - V_{\tau_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\tau_n}x_n - z, h(x_n) - V_{\tau_n}x_n\rangle + M_2\|\tilde{e}_n\| \\
&= (1-w_n)\|x_n - z\|^2 + w_n\|W_n x_n - W_n z\|^2 - w_n(1-w_n)\|W_n x_n - x_n\|^2 \\
&\quad + {\alpha_n}^2\|h(x_n) - V_{\tau_n}x_n\|^2 + 2\alpha_n\langle V_{\tau_n}x_n - z, h(x_n) - V_{\tau_n}x_n\rangle + M_2\|\tilde{e}_n\| \\
&\le \|x_n - z\|^2 - w_n(1-w_n)\|W_n x_n - x_n\|^2 + {\alpha_n}^2\|h(x_n) - V_{\tau_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\tau_n}x_n - z, h(x_n)V_{\tau_n}x_n\rangle + M_2\|\tilde{e}_n\|.
\end{aligned} \tag{19}$$

Furthermore, set

$$s_n = \|x_n - z\|^2, \quad \gamma_n = \alpha_n(2 - \alpha_n(1+2\rho^2) - 2(1-\alpha_n)\rho),$$

$$\delta_n = \frac{1}{2-\alpha_n(1+2\rho^2)-2(1-\alpha_n)\rho}[2\alpha_n\|h(z)-z\|^2 + M_2\frac{\|\tilde{e}_n\|}{\alpha_n} + 2(1-\alpha_n)\langle h(z)-z, V_{\tau_n}x_n - z\rangle],$$

$$\eta_n = w_n(1-w_n)\|W_n x_n - x_n\|^2,$$

$$\varphi_n = {\alpha_n}^2\|h(x_n) - V_{\tau_n}x_n\|^2 + 2\alpha_n\langle V_{\tau_n}x_n - z, h(x_n) - V_{\tau_n}x_n\rangle + M_2\|\tilde{e}_n\|.$$

Using the condition (i), it is easy to get $\gamma_n \to 0$, $\Sigma_{n=0}^{\infty}\gamma_n = \infty$ and $\varphi_n \to 0$. In order to complete the proof, from Lemma 4, it suffices to verify that $\eta_{n_k} \to 0$ as $k \to \infty$, which implies that

$$\limsup_{k\to\infty}\delta_{n_k} \le 0$$

for any subsequence $\{n_k\} \subset \{n\}$. Indeed, $\eta_{n_k} \to 0$ as $k \to \infty$ implies that $\|W_{n_k}x_{n_k} - x_{n_k}\| \to 0$ as $k \to \infty$ from the condition (iii). Thus, from (18), it follows that

$$\|x_{n_k} - V_{\tau_{n_k}}x_{n_k}\| = w_{n_k}\|x_{n_k} - W_{n_k}x_{n_k}\| \to 0. \tag{20}$$

Step 3. Show that

$$\omega_w(x_{n_k}) \subset S, \tag{21}$$

where $\omega_w(x_{n_k})$ is the set of all weak cluster points of $\{x_{n_k}\}$. To see (21), we prove the following:

Take $\tilde{x} \in \omega_w\{x_{n_k}\}$ and assume that $\{x_{n_{k_j}}\}$ is a subsequence of $\{x_{n_k}\}$ weakly converging to $\tilde{x}$. Without loss of generality, we still use $\{x_{n_k}\}$ to denote $\{x_{n_{k_j}}\}$. Assume $\tau_{n_k} \to \tau$. Then, we have $0 < \tau < \frac{1}{\gamma_2 L}$. Setting $V = U(I - \tau A^*(I-T)A)$, we deduce that

$$\begin{aligned}
&\|V_{\tau_{n_k}}x_{n_k} - Vx_{n_k}\| \\
&= \|U(x_{n_k} - \tau_{n_k}A^*(I-T)Ax_{n_k}) - U(x_{n_k} - \tau A^*(I-T)Ax_{n_k})\| \\
&\le |\tau_{n_k} - \tau|\|A^*(I-T)Ax_{n_k}\|.
\end{aligned} \tag{22}$$

Since $\tau_{n_k} \to \tau$ as $k \to \infty$, it follows immediately from (22) that

$$\|V_{\tau_{n_k}}x_{n_k} - Vx_{n_k}\| \to 0$$

as $k \to \infty$. Thus, we have

$$\|x_{n_k} - Vx_{n_k}\| \le \|x_{n_k} - V_{\tau_{n_k}}x_{n_k}\| + \|V_{\tau_{n_k}}x_{n_k} - Vx_{n_k}\| \to 0. \tag{23}$$

Using Lemma 3, we get $\omega_w(x_{n_k}) \subset Fix(V)$. Since both U and T are averaged, it follows from Proposition 1 (ii) that

$$\omega_w(x_{n_k}) \subset Fix(U), \quad \omega_w(x_{n_k}) \subset Fix(I - \tau A^*(I-T)A).$$

Then, by Lemma 5, we obtain $\omega_w(x_{n_k}) \subset S$ immediately. Meanwhile, we have

$$\limsup_{k\to\infty}\langle h(x^*) - x^*, x_{n_k} - x^*\rangle = \langle h(x^*) - x^*, \tilde{x} - x^*\rangle, \ \ \forall \tilde{x} \in S. \tag{24}$$

In addition, since x^* is the unique solution of the variational inequality problem (13), we have

$$\limsup_{k\to\infty}\langle h(x^*) - x^*, x_{n_k} - x^*\rangle \le 0$$

together with (20) and hence $\limsup_{k\to\infty} \delta_{n_k} \leq 0$. This completes the proof. □

Next, we consider the bounded perturbation of (14) generated by the following iterative process:

$$\begin{cases} y_n = x_n + \beta_n \nu_n, \\ x_{n+1} = \alpha_n h(y_n + e(y_n)) + (1-\alpha_n)U(I - \tau_n A^*(I-T)Ay_n + e(y_n)). \end{cases} \tag{25}$$

Theorem 2. *Assume that the sequences $\{\beta_n\}$ and $\{\nu_n\}$ satisfy the condition (6). Let H_1,H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator with $L = \|A^*A\|$, where A^* is the adjoint of A. Suppose that $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two averaged mappings with the coefficients γ_1 and γ_2, respectively. Assume that problem (4) is consistent (i.e., $S \neq \emptyset$). Let $h : H_1 \to H_1$ be a ρ-contraction with $0 \leq \rho < 1$. For any $x_0 \in H_1$, define the sequence $\{x_n\}$ by (25). If the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*
(ii) $0 < \liminf_{n\to\infty} \tau_n \leq \limsup_{n\to\infty} \tau_n < \frac{1}{\gamma_2 L}$*;*
(iii) $\sum_{n=0}^{\infty} \|e(y_n)\| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to x^*, where x^* is a solution of the problem (4), which is also the unique solution of the variational inequality problem (13).

Proof. Now, put

$$\begin{aligned}\tilde{e}_n =& \alpha_n(h(y_n + e(y_n)) - h(x_n)) \\ &+ (1-\alpha_n)(U(y_n - \tau_n A^*(I-T)Ay_n + e(y_n)) - U(x_n - \tau_n A^*(I-T)Ax_n)).\end{aligned}$$

Then, Equation (25) can be rewritten as follows:

$$x_{n+1} = \alpha_n h(x_n) + (1-\alpha_n)U(I - \tau_n A^*(I-T)A)(x_n) + \tilde{e}_n, \tag{26}$$

In fact, by Proposition 1 (iii) and the nonexpansiveness of T, it is not hard to show that $A^*(I-T)A$ is $2L-$Lipschitz. Thus, we have

$$\begin{aligned}\|\tilde{e}_n\| &\leq \alpha_n\|h(y_n + e(y_n)) - h(x_n)\| \\ &\quad + (1-\alpha_n)\|y_n - x_n - \tau_n(A^*(I-T)Ay_n + e(y_n) - A^*(I-T)Ax_n)\| \\ &\leq \alpha_n\rho\|y_n - x_n + e(y_n)\| \\ &\quad + (1-\alpha_n)(\|y_n - x_n\| + 2\tau_n L\|y_n - x_n\| + \|e(y_n)\|) \\ &\leq (\alpha_n\rho + (1-\alpha_n)(1+2\tau_n L))\beta_n\|\nu_n\| + (\alpha_n\rho + (1-\alpha_n))\|e(y_n)\|.\end{aligned} \tag{27}$$

From the condition (iii) and condition (6), we have $\sum_{n=0}^{\infty} \|\tilde{e}_n\| < \infty$. Consequently, using Theorem 1, it follows that the algorithm (14) is bounded perturbation resilient. This completes the proof. □

4. Numerical Results

In this section, we consider the following numerical examples to present the effectiveness, realization and convergence of Theorems 1 and 2:

Example 1. *Let $H_1 = H_2 = \mathbb{R}^2$. Suppose $h(x) = \frac{1}{10}x$ and*

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Take $U = P_C$ and $T = P_Q$, where C and Q are defined as follows:

$$C = \{x \in \mathbb{R}^2 : \|x\|_2 \leq 4\}$$

and

$$Q = \{y \in \mathbb{R}^2 : 6 \leq y(i) \leq 12, \ \ i = 1, 2\},$$

where $y(i)$ denotes the ith element of y.

We can compute the solution set $S = \{x : 3 \leq x(1) \leq 6, 2 \leq x(2) \leq 4, \|x\|_2 \leq 4\}$.

Take the experiment parameters $\tau_n = \frac{1}{L}$ and $\alpha_n = \frac{1}{n+1}$ in the following iterative algorithms and the stopping criteria is $\|x_{n+1} - x_n\| < error$. According to the iterative process of Theorem 1, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = \frac{1}{n+1} * \frac{1}{10} x_n + (1 - \frac{1}{n+1}) U(x_n - \tau_n A^T (I - T) A x_n). \tag{28}$$

As $n \to \infty$, we have $x_n \to x^*$. Then, taking the random initial guess x_0 and using MATLAB software (MATLAB R2012a, MathWorks, Natick, MA, USA), we obtain the numerical experiment results in Table 1.

Table 1. $x_0 = rand(2, 1)$, results without bounded perturbation.

τ	n	Time (s)	x_n	Error
0.1111	52	0.02657	$[2.9995\ 1.9998]^T$	10^{-5}
0.1111	135	0.07538	$[2.9998\ 2.0000]^T$	10^{-6}
0.1111	216	0.15873	$[3.0000\ 2.0000]^T$	10^{-7}

Next, we consider the algorithm with bounded perturbation resilience. Choose the the bounded sequence $\{v_n\}$ and the summarable nonnegative real sequence $\{\beta_n\}$ as follows:

$$v_n = \begin{cases} -\dfrac{d_n}{\|d_n\|}, & \text{if } 0 \neq d_n \in \partial I_C(x_n), \\ 0, & \text{if } 0 = d_n \in \partial I_C(x_n), \end{cases} \tag{29}$$

and

$$\beta_n = c^n$$

for some $c \in (0, 1)$, where the indicator function

$$I_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C, \end{cases}$$

and

$$\partial I_C(x) = N_C(x) = \begin{cases} \{u \in H : \langle u, x - y \rangle \geq 0, \forall y \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C, \end{cases}$$

is the normal cone to C. The point d_n is taken from $N_C(x_n)$. Setting $c = 0.5$, the numerical results can be seen in Table 2.

Table 2. $x_0 = rand(2,1)$, results with bounded perturbation.

τ	n	Time (s)	x_n	Error
0.1111	45	0.02342	$[2.9992\ 1.9998]^T$	10^{-5}
0.1111	98	0.05317	$[2.9998\ 2.0000]^T$	10^{-6}
0.1111	153	0.10256	$[3.0000\ 2.0000]^T$	10^{-7}

As we have seen above, the accuracy of the solution is improved with the decrease of the stop criteria. In addition, the sequence $\{x_n\}$ converges to the point $(3,2)$, which is a solution of the numerical example. Of course, it is also the unique solution of the variational inequality $\langle (I-h)x^*, x-x^*\rangle \geq 0$.

In addition, we contrast the approximate value of solution x^* of Example 1 under the same parameter conditions, the same iterative numbers and the same initial value. The numerical results are reported in Tables 3 and 4, where $\{x_n^{(1)}\}$ and $\{x_n^{(2)}\}$ denote the iterative sequences generated by the algorithm (14) in this paper and Theorem 3.2 in Ref. [8], respectively.

Table 3. $x_0 = 2 * rand(2,1)$,results without bounded perturbation.

τ	n	$x_n^{(1)}$	$x_n^{(2)}$	Error
0.2179	52	$[2.9884\ 1.9993]^T$	$[2.9850\ 1.9940]^T$	10^{-5}
0.2179	132	$[3.0000\ 1.9995]^T$	$[2.9942\ 1.9988]^T$	10^{-6}
0.2179	208	$[3.0000\ 1.9999]^T$	$[2.9995\ 1.9996]^T$	10^{-7}

Table 4. $x_0 = 2 * rand(2,1)$, results with bounded perturbation.

τ	n	$x_n^{(1)}$	$x_n^{(2)}$	Error
0.2179	32	$[2.9990\ 1.9997]^T$	$[2.9920\ 1.9987]^T$	10^{-5}
0.2179	56	$[2.9997\ 1.9999]^T$	$[2.9942\ 1.9988]^T$	10^{-6}
0.2179	115	$[3.0000\ 2.0000]^T$	$[2.9995\ 1.9998]^T$	10^{-7}

Example 2. *Let $H_1 = H_2 = \mathbb{R}^3$. Suppose $h(x) = \frac{1}{3}x$ and*

$$A = \begin{pmatrix} 1 & 0 & -8 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T : y = (y(1), y(2), y(3))^T \mapsto (y(1), y(2), \frac{y(3) + sin\, y(3)}{2})^T. \tag{30}$$

It is obvious that T is $\frac{1}{2}-av$ and the set of fixed points $Fix(T) = \{y|\ (y(1), y(2), 0)^T\}$ is nonempty. Let $U = P_C$ and $C = \{x \in \mathbb{R}^3 |\ \|x\| \leq 1\}$. Then, we use the iterative algorithm of Theorem 1 to approximate a point $x^* \in C$ such that $Ax^* \in Fix(T)$.

Take the experiment parameters $\tau_n = \frac{1.9*n}{(n+1)L}$ and $\alpha_n = \frac{1}{n+1}$ in the following iterative algorithms. Let $F(x) = \frac{1}{2}\|(I-T)Ax\|^2 + I_C(x)$ and the stopping criteria is $F(x) < error$.

Then, taking the random initial guess x_0 and using MATLAB software, we obtain the numerical experiment results in Table 5.

Table 5. $x_0 = 10 * rand(3,1)$, results without bounded perturbation.

τ	n	Time (s)	x_n	Error
0.0255	120	0.0155	$[0.0085\ 0.0167\ 0.0037]^T * 10^{-2}$	10^{-5}
0.0256	235	0.0461	$[0.0078\ 0.0013\ 0.0013]^T * 10^{-3}$	10^{-6}
0.0257	518	0.1881	$[0.0029\ 0.0015\ 0.0002]^T * 10^{-4}$	10^{-7}

Next, we consider the bounded perturbation. The definitions of v_n and β_n are similar to the Example 1. Setting $c = 0.8$, the numerical results can be seen in Table 4.

As we have seen in Tables 5 and 6, the sequence $\{x_n\}$ approximates to the point $(0,0,0)^T$, which is a solution of the numerical example. Of course, it is also the unique solution of the variational inequality $\langle (I-h)x^*, x-x^* \rangle \geq 0$.

Table 6. $x_0 = 10 * rand(3,1)$, results with bounded perturbation.

τ	n	Time (s)	x_n	Error
0.0246	22	0.0036	$[0.0108\ 0.0139\ 0.0015]^T * 10^{-2}$	10^{-5}
0.0249	32	0.0040	$[0.0085\ 0.0014\ 0.0014]^T * 10^{-3}$	10^{-6}
0.0250	36	0.0058	$[0.0028\ 0.0025\ 0.0034]^T * 10^{-4}$	10^{-7}

5. Conclusions

The SCFPP is an inverse problem that consists in finding a point in a fixed point set such that its image under a bounded linear operator belongs to another fixed point set. Many iterative algorithms have been developed to solve these kinds of problems. In this paper, we have introduced a viscosity iterative sequence and obtained the strong convergence. We prove the main result using the weaker conditions than many existing similar methods—for example, Xu's algorithm [37] for the SFP. More specifically, his algorithm generates a sequence $\{x_n\}$ via the following recursions:

$$x_{n+1} = \alpha_n u + (1-\alpha_n) P_C(x_n - \tau_n A^*(I-P_Q)Ax_n),$$

where u is a a fixed element and $\{\alpha_n\} \subset [0,1]$ satisfies the assumptions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(ii) either $\sum_{n=0}^{\infty} \|\alpha_{n+1} - \alpha_n\| < \infty$ or $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$.

The second condition is not necessary in our theorems. We also consider the bounded perturbation resilience of the proposed method and get theoretical convergence results. Finally, numerical experiments have been presented to illustrate the effectiveness of the proposed algorithms.

Author Contributions: All authors read and approved the final manuscript. Conceptualization, P.D.; Data Curation, P.D. and X.Z.; Formal Analysis, P.D. and J.Z.

Funding: This work was supported of the scientific research project of the Tianjin Municipal Education Commission (2018KJ253) and the Fundamental Research Funds for the Central Universities (3122017072).

Acknowledgments: The authors would like to thank the referee for valuable suggestions to improve the manuscript.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. Byrne, C. A united treatment of some iterative algorithms in signal processing and image reconstruction. *Inverse Probl.* **2004**, *20*, 103–120. [CrossRef]
2. Censor, Y.; Elfving, T.; Kopf, N.; Bortfeld, T. The multiple-sets split feasibility problem and its applications for inverse problems. *Inverse Probl.* **2005**, *21*, 2071–2084. [CrossRef]

3. Censor, Y.; Bortfeld, T.; Martin, B.; Trofimov, A. A unified approach for inversion problems in intensity modulated radiation therapy. *Phys. Med. Biol.* **2006**, *51*, 2353–2365. [CrossRef] [PubMed]
4. Censor, Y.; Elfving, T. A multiprojection algorithm using Bregman projections in product space. *Numer. Algorithms* **1994**, *8*, 221–239. [CrossRef]
5. Byrne, C. Iterative oblique projection onto convex sets and the split feasibility problem. *Inverse Probl.* **2002**, *18*, 441–453. [CrossRef]
6. Censor, Y.; Segal, A. The split common fixed point problem for directed operators. *J. Convex Anal.* **2009**, *16*, 587–600.
7. Moudafi, A. A note on the split common fixed-point problem for quasi-nonexpansive operators. *Nonlinear Anal.* **2011**, *74*, 4083–4087. [CrossRef]
8. Kraikaew, R.; Saejung, S. On split common fixed point problems. *J. Math. Anal. Appl.* **2014**, *415*, 513–524. [CrossRef]
9. Dong, Q.L.; He, S.N.; Zhao, J. Solving the split equality problem without prior knowledge of operator norms. *Optimization* **2015**, *64*, 1887–1906. [CrossRef]
10. He, S.N.; Tian, H.; Xu, H.K. The selective projection method for convex feasibility and split feasibility problems. *J. Nonlinear Convex Anal.* **2018**, *19*, 1199–1215.
11. Padcharoen, A.; Kumam, P.; Cho, Y.J. Split common fixed point problems for demicontractive operators. *Numer. Algorithms* **2018**. [CrossRef]
12. Zhao, J. Solving split equality fixed-point problem of quasi-nonexpansive mappings without prior knowledge of operators norms. *Optimization* **2015**, *64*, 2619–2630. [CrossRef]
13. Zhao, J.; Hou, D.F. A self-adaptive iterative algorithm for the split common fixed point problems. *Numer. Algorithms* **2018**. [CrossRef]
14. Byrne, C.; Censor, Y.; Gibali, A.; Reich, S. Weak and strong convergence of algorithms for the split common null point problem. *J. Nonlinear Convex Anal.* **2012**, *13*, 759–775.
15. Combettes, P.L. Solving monotone inclusions via compositions of nonexpansive averaged operators. *Optimization* **2004**, *53*, 475–504. [CrossRef]
16. Chuang, C.S.; Lin, I.J. New strong convergence theorems for split variational inclusion problems in Hilbert spaces. *J. Inequal. Appl.* **2015**, *176*, 1–20. [CrossRef]
17. Dong, Q.L.; Yuan, H.B.; Cho, Y.J.; Rassias, T.M. Modified inertial Mann algorithm and inertial CQ-algorithm for nonexpansive mappings. *Optim. Lett.* **2018**, *12*, 87–102. [CrossRef]
18. Censor, Y.; Motova, A.; Segal, A. Perturbed projections and subgradient projections for the multiple-sets split feasibility problem. *J. Math. Anal. Appl.* **2007**, *327*, 1244–1256. [CrossRef]
19. Censor, Y.; Davidi, R.; Herman, G.T. Perturbation resilience and superiorization of iterative algorithms. *Inverse Probl.* **2010**, *26*, 65008. [CrossRef]
20. Guo, Y.N.; Cui, W. Strong convergence and bounded perturbation resilience of a modified proximal gradient algorithm. *J. Inequal. Appl.* **2018**, *103*, 1–15. [CrossRef]
21. Jin, W.; Censor, Y.; Jiang, M. Bounded perturbation resilience of projected scaled gradient methods. *Comput. Optim. Appl.* **2016**, *63*, 365–392. [CrossRef]
22. Xu, H.K. Bounded perturbation resilience and superiorization techniques for the projected scaled gradient method. *Inverse Probl.* **2017**, *33*, 044008. [CrossRef]
23. Dong, Q.L.; Zhao, J.; He, S.N. Bounded perturbation resilience of the viscosity algorithm. *J. Inequal. Appl.* **2016**, *299*, 1–12. [CrossRef]
24. Censor, Y.; Chen, W.; Combettes, P.L.; Davidi, R.; Herman, G.T. On the effectiveness of projection methods for convex feasibility problem with linear inequality constrains. *Comput. Optim. Appl.* **2012**, *51*, 1065–1088. [CrossRef]
25. Davidi, R.; Herman, G.T.; Censor, Y. Perturbation-resilient block-iterative projection methods with application to image reconstruction from projections. *Int. Trans. Oper. Res.* **2009**, *16*, 505–524. [CrossRef] [PubMed]
26. Davidi, R.; Censor, Y.; Schulte, R.W.; Geneser, S.; Xing, L. Feasibility-seeking and superiorization algorithm applied to inverse treatment plannning in rediation therapy. *Contemp. Math.* **2015**, *636*, 83–92.
27. Moreau, J.J. Proprietes des applications 'prox'. *C. R. Acad. Sci. Paris Ser. A Math.* **1963**, *256*, 1069–1071.
28. Marino, G.; Scardamaglia, B.; Karapinar, E. Strong convergence theorem for strict pseudo-contractions in Hilbert spaces. *J. Inequal. Appl.* **2016**, *134*, 1–12. [CrossRef]

29. Xu, H.K. Viscosity approximation methods for nonexpansive mappings. *J. Math. Anal. Appl.* **2004**, *298*, 279–291. [CrossRef]
30. Xu, H.K. Averaged mappings and the gradient-projection algorithm. *J. Optim. Theory Appl.* **2011**, *150*, 360–378. [CrossRef]
31. Geobel, K.; Kirk, W.A. *Topics in Metric Fixed Point Theory*; Cambridge Studies in Advanced Mathematics 28; Cambridge University Press: Cambridge, UK, 1990.
32. He, S.N.; Yang, C.P. Solving the variational inequality problem defined on intersection of finite level sets. *Abstr. Appl. Anal.* **2013**, *2013*, 942315. [CrossRef]
33. Yang, C.P.; He, S.N. General alterative regularization methods for nonexpansive mappings in Hilbert spaces. *Fixed Point Theory Appl.* **2014**, *2014*, 203. [CrossRef]
34. Moudafi, A. Viscosity approximation methods for fixed point problems. *J. Math. Anal. Appl.* **2000**, *241*, 46–55. [CrossRef]
35. Cho, Y.J.; Qin, X. Viscosity approximation methods for a finite family of m-accretive mappings in reflexive Banach spaces. *Positivity* **2008**, *12*, 483–494. [CrossRef]
36. Qin, X.; Cho, Y.J.; Kang, S.M. Viscosity approximation methods for generalized equilibrium problems and fixed point problems with applications. *Nonlinear Anal.* **2010**, *72*, 99–112. [CrossRef]
37. Xu, H.K. A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem. *Inverse Probl.* **2006**, *22*, 2021–2034. [CrossRef]