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Further Extension of the Generalized Hurwitz-Lerch Zeta Function of Two Variables

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Abstract: The main aim of this paper is to provide a new generalization of Hurwitz-Lerch Zeta function of two variables. We also investigate several interesting properties such as integral representations, summation formula, and a connection with the generalized hypergeometric function. To strengthen the main results we also consider some important special cases.

Keywords: gamma function; beta function; hypergeometric function; generalized hurwitz-lerch zeta function

MSC: 11M06; 11M35; 33B15; 33C60

1. Introduction

The generalized hypergeometric function $F(-)$ [1] defined by

$$\begin{aligned} F\left(\beta_1, \cdots, \beta_p; \delta_1, \cdots, \delta_q; z\right) &= {}_p F_q\left(\beta_1, \cdots, \beta_p; \delta_1, \cdots, \delta_q; z\right) \\ &= \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\beta_i)_n \, z^n}{\prod_{j=1}^{q} (\beta_j)_n \, n!}, \end{aligned} \tag{1}$$

where $p, q \in \mathbb{Z}^+; b_j \neq 0, -1, -2, \cdots$.

The Appell hypergeometric function F_1 of two variables [2] is defined by

$$\begin{aligned} F_1[a, b, b'; c; z, t] &= \sum_{k,l=0}^{\infty} \frac{(a)_{k+l} (b)_k (b')_l}{(c)_{k+l}} \frac{z^k}{k!} \frac{t^l}{l!}, \\ &= \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \, {}_2F_1\left[\begin{matrix} a+k, \, b'; \\ c+k; \end{matrix} \; t\right] \frac{z^k}{k!}, \\ &\left(max\left\{\Re(z), \Re(t)\right\} \leq 1 \quad \text{and} \quad \Re(a) > 0\right). \end{aligned} \tag{2}$$

The confluent forms of Humbert functions are [2]:

$$\Phi_1[a, b; c; z, t] = \sum_{k,l=0}^{\infty} \frac{(a)_{k+l} (b)_k}{(c)_{k+l}} \frac{z^k}{k!} \frac{t^l}{l!}, \quad \left(|z| < 1, |t| < \infty\right), \tag{3}$$

$$\Phi_2[b, b'; c; z, t] = \sum_{k,l=0}^{\infty} \frac{(b)_k (b')_l}{(c)_{k+l}} \frac{z^k}{k!} \frac{t^l}{l!}, \quad \left(|z| < \infty, |t| < \infty\right), \tag{4}$$

and

$$\Phi_3[b; c; z, t] = \sum_{k,l=0}^{\infty} \frac{(b)_k}{(c)_{k+l}} \frac{z^k}{k!} \frac{t^l}{l!}, \quad \left(|z| < \infty, |t| < \infty\right). \tag{5}$$

The Appell's type generalized functions M_i by considering product of two ${}_3F_2$ functions is given in [3]. From these expansions, we recall one of the generalized Appell's type functions of two variables M_4 and is defined by

$$M_4\left(\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi';x,y\right)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\mu)_{k+l}(\eta)_k(\eta')_l(\delta)_k(\delta')_l}{(\nu)_{k+l}(\xi)_k(\xi')_l}\frac{x^k}{k!}\frac{y^l}{l!}. \tag{6}$$

If we set $\mu=\nu,\delta=\xi,\delta'=\xi'$ in (6) then

$$M_4[\mu,\eta,\eta',\delta,\delta';\mu,\delta,\delta';x,y]=(1-x)^{-\eta}\,(1-y)^{-\eta'}. \tag{7}$$

The Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ is defined by (see [4,5]):

$$\Phi(z,s,a)=\sum_{k=0}^{\infty}\frac{z^k}{(k+a)^s}, \tag{8}$$

$$\left(a\in\mathbb{C}\setminus\mathbb{Z}_0^-;s\in\mathbb{C}\right)\text{ when }\quad |z|<1;\Re(s)>1\text{when}\quad |z|=1.$$

For more details about the properties and particular cases found in [1,4,5]. Various type of generalizations, extensions, and properties of the Hurwitz-Lerch Zeta function can be found in [6–13].

Recently, Pathan and Daman [14] give another generalization of the form

$$\begin{aligned}&\Phi_{\alpha,\beta;\gamma,\lambda,\mu;\nu}\left(z,t,s,a\right):=\sum_{k,l=0}^{\infty}\frac{(\alpha)_k(\beta)_k(\lambda)_l(\mu)_l}{(\gamma)_k(\nu)_l k!l!}\frac{z^kt^l}{(k+l+a)^s},\\&\gamma,\nu,a\neq\{0,-1,-2,\cdots,\},s\in\mathbb{C};\\&\quad\Re\left(s+\gamma+\nu-\alpha-\beta-\mu-\lambda\right)>0\quad\text{when}\quad|z|=1\quad\text{and}\quad|t|=1.\end{aligned} \tag{9}$$

Very recently, Choi and Parmar [15] introduced two variable generalization by

$$\begin{aligned}&\Phi_{\mu,\eta,\eta';\nu}\left(z,t,s,a\right):=\sum_{k,l=0}^{\infty}\frac{(\mu)_{k+l}(\eta)_k(\eta')_l}{(\nu)_{k+l}k!l!}\frac{z^kt^l}{(k+l+a)^s},\\&\left(\mu,\eta,\eta'\in\mathbb{C};\quad a,\nu\in\mathbb{C}\setminus\mathbb{Z}_0^-;\quad s,z,t\in\mathbb{C}\quad\text{when}\quad|z|<1\quad\text{and}\quad|t|<1;\right.\\&\quad\left.\text{and}\quad\Re\left(s+\nu-\mu-\eta-\eta'\right)>0\quad\text{when}\quad|z|=1\quad\text{and}\quad|t|=1\right).\end{aligned} \tag{10}$$

In this paper, we further extended the Hurwitz-Lerch Zeta function of two variables and is defined by

$$\begin{aligned}&\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}\left(z,t,s,a\right):=\textstyle\sum_{k,l=0}^{\infty}\frac{(\mu)_{k+l}(\eta)_k(\eta')_l(\delta)_k(\delta')_l}{(\nu)_{k+l}(\xi)_k(\xi')_l k!l!}\frac{z^kt^l}{(k+l+a)^s},\\&\left(\mu,\eta,\eta',\delta,\delta'\in\mathbb{C};a,\nu,\xi,\xi'\in\mathbb{C}\setminus\mathbb{Z}_0^-;s,z,t\in\mathbb{C}\quad\text{when}\quad|z|<1\quad\text{and}\quad|t|<1;\right.\\&\quad\left.\text{and}\quad\Re\left(s+\nu+\xi+\xi'-\mu-\eta-\eta'-\delta-\delta'\right)>0\quad\text{when}\,|z|=1\text{and}\,|t|=1\right).\end{aligned} \tag{11}$$

Special cases:

Case 1. If $\delta=\xi,\delta'=\xi'$, then (11) reduces to (3) of [15] which is given in (10).

Case 2. If $\mu=\nu$ and $\delta=\xi,\delta'=\xi'$ in (11), then we get the generalized Hurwitz-Lerch Zeta function of [14]:

$$\begin{aligned}&\Phi_{\eta,\eta'}\left(z,t,s,a\right):=\sum_{k,l=0}^{\infty}\frac{(\eta)_k(\eta')_l}{k!l!}\frac{z^kt^l}{(k+l+a)^s},\\&\left(\eta,\eta'\in\mathbb{C};\quad a\in\mathbb{C}\setminus\mathbb{Z}_0^-;\quad s\in\mathbb{C}\quad\text{when}\quad|z|<1\quad\text{and}\quad|t|<1;\right.\\&\quad\left.\text{and}\quad\Re\left(s-\eta-\eta'\right)>0\quad\text{when}\quad|z|=1\quad\text{and}\quad|t|=1\right).\end{aligned} \tag{12}$$

The limiting cases of (11) are as follows:

Case 3. If $\eta' \to \infty$ then we have

$$\begin{aligned}
&\Phi_{\mu,\eta,\delta,\delta';\nu,\xi,\xi'}(z,t,s,a) := \lim_{\eta' \to \infty}\left\{\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z,t/\eta',s,a)\right\} \\
&= \sum_{k,l=0}^{\infty}\frac{(\mu)_{k+l}(\eta)_k(\delta)_k(\delta')_l}{(\nu)_{k+l}(\xi)_k(\xi')_l k!l!}\frac{z^k t^l}{(k+l+a)^s}, \qquad (13) \\
&\quad (\mu,\eta,\delta,\delta' \in \mathbb{C}; a,\nu,\xi,\xi' \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s,z,t \in \mathbb{C} \quad \text{when} \quad |z|<1 \quad \text{and} \quad |t|<1; \\
&\text{and} \quad \Re\left(s+\nu+\xi+\xi'-\mu-\eta-\delta-\delta'\right)>0 \quad \text{when} \quad |z|=1 \text{and}\, |t|=1\right).
\end{aligned}$$

Case 4. If $\mu \to \infty$ then we have

$$\begin{aligned}
&\Phi_{\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z,t,s,a) := \lim_{\mu \to \infty}\left\{\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z/\mu,t/\mu,s,a)\right\} \\
&= \sum_{k,l=0}^{\infty}\frac{(\eta)_k(\eta')_l(\delta)_k(\delta')_l}{(\nu)_{k+l}(\xi)_k(\xi')_l k!l!}\frac{z^k t^l}{(k+l+a)^s}, \qquad (14) \\
&\quad (\eta,\eta',\delta,\delta' \in \mathbb{C}; a,\nu,\xi,\xi' \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s,z,t \in \mathbb{C} \quad \text{when} \quad |z|<1 \quad \text{and} \quad |t|<1; \\
&\text{and} \quad \Re\left(s+\nu+\xi+\xi'-\eta-\delta-\delta'\right)>0 \quad \text{when} \quad |z|=1 \quad \text{and} \quad |t|=1\right).
\end{aligned}$$

Case 5. If $min(|\mu|,|\eta'|) \to \infty$ then we have

$$\begin{aligned}
&\Phi_{\eta,\delta,\delta';\nu,\xi,\xi'}(z,t,s,a) := \lim_{min(|\mu|,|\eta'|) \to \infty}\left\{\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}\left(\frac{z}{\mu},\frac{t}{(\mu\eta')},s,a\right)\right\} \\
&= \sum_{k,l=0}^{\infty}\frac{(\eta)_k(\delta)_k(\delta')_l}{(\nu)_{k+l}(\xi)_k(\xi')_l k!l!}\frac{z^k t^l}{(k+l+a)^s}, \qquad (15) \\
&\quad (\eta,\delta,\delta' \in \mathbb{C}; a,\nu,\xi,\xi' \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s,z,t \in \mathbb{C} \quad \text{when} \quad |z|<1 \quad \text{and} \quad |t|<1; \\
&\quad \text{and} \quad \Re\left(s+\nu+\xi+\xi'-\eta-\delta-\delta'\right)>0 \quad \text{when} \quad |z|=1 \quad \text{and} \quad |t|=1\right).
\end{aligned}$$

2. Integral Representations

Theorem 1. *The following integral representation of* (11) *holds true:*

$$\begin{aligned}
&\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z,t,s,a) \\
&= \frac{1}{\Gamma(s)}\int_0^{\infty} x^{s-1}e^{-ax}M_4\left(\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi';ze^{-x},te^{-x}\right)dx, \qquad (16) \\
&(\min\{\Re(s),\Re(a)\}>0 \quad \textit{when}\, |z|\le 1(z\ne 1), |t|\le 1(t\ne 1), \\
&\Re(s)>1, \textit{when}\ z=1, t=1\right).
\end{aligned}$$

Proof. Using the following Eulerian integral

$$\frac{1}{(k+l+a)^s} := \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}e^{-(k+l+a)t}dt \quad (\min\Re(s),\Re(a)>0, k,l \in \mathbb{N}_0) \qquad (17)$$

in (11), we get

$$\begin{aligned}
&\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z,t,s,a) \\
&= \sum_{k,l=0}^{\infty}\frac{(\mu)_{k+l}(\eta)_k(\eta')_l(\delta)_k(\delta')_l}{(\nu)_{k+l}(\xi)_k(\xi')_l}\frac{z^k t^l}{k!l!}\frac{1}{\Gamma(s)}\int_0^{\infty} x^{s-1}e^{-(k+l+a)x}dx.
\end{aligned}$$

Interchanging the order of integration and summation, which is verified by uniform convergence of the involved series under the given conditions, we have

$$\begin{aligned}&\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}\left(z,t,s,a\right)\\=&\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}e^{-ax}\sum_{k,l=0}^{\infty}\frac{(\mu)_{k+l}(\eta)_k(\eta')_l(\delta)_k(\delta')_l}{(\nu)_{k+l}(\xi)_k(\xi')_l}\frac{z^kt^l}{k!l!}\left(e^{-t}\right)^k\left(e^{-t}\right)^l dx.\end{aligned}\tag{18}$$

In view of (6), we arrived the desired result. □

Similarly, if we use (17) in the limiting cases (13), (14) and (15) then we obtain the following corollaries:

Corollary 1. *The following integral representations for $\Phi_{\mu,\eta,\delta,\delta';\nu,\xi,\xi'}\left(z,t,s,a\right),\Phi_{\eta,\eta',\delta,\delta';\nu,\xi,\xi'}\left(z,t,s,a\right)$ and $\Phi_{\eta,\delta,\delta';\nu,\xi,\xi'}\left(z,t,s,a\right)$ in (13), (14) and (15) holds true when $\delta=\xi,\delta'=\xi'$:*

$$\Phi_{\mu,\eta;\nu}\left(z,t,s,a\right)=\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}e^{-ax}\Phi_1\left(\mu,\eta;\nu;ze^{-x},te^{-x}\right)dx,\tag{19}$$

which is (14) of [15].

$$\Phi_{\eta,\eta';\nu}\left(z,t,s,a\right)=\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}e^{-ax}\Phi_2\left(\eta,\eta';\nu;ze^{-x},te^{-x}\right)dx,\tag{20}$$

which is (15) of [15] and

$$\Phi_{\eta;\nu}\left(z,t,s,a\right)=\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}e^{-ax}\Phi_3\left(\eta;\nu;ze^{-x},te^{-x}\right)dx.\tag{21}$$

$(\min\{\Re(s),\Re(a)\}>0$ *when* $|z|\le 1(z\neq 1),|t|\le 1(t\neq 1),\ \ \Re(s)>1,$ *when* $z=1,t=1)$, *which is (16) of [15].*

Corollary 2. *In view of (7), we have*

$$\begin{aligned}&\Phi_{\mu,\eta,\eta';\mu}\left(z,t,s,a\right)\\&=\frac{1}{\Gamma(s)}\int_0^\infty\frac{x^{s-1}e^{-ax}}{\left(1-ze^{-x}\right)^{\eta}\left(1-te^{-x}\right)^{-\eta'}}dx,\end{aligned}\tag{22}$$

$(\min\Re(s)>0,\Re(a)>0$ *when* $|z|\le 1(z\neq 1),|t|\le 1(t\neq 1),\ \ \Re(s)>1,$ *when* $z=1,t=1)$.

Remark 1. *If we take $t=0$ in (22), then it gives (19) of [15] and by setting $t=0,\eta=1$ then (22) reduces to (20) of [15]*

Theorem 2. *Each of the following integrals for $\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}\left(z,t,s,a\right)$ holds true*

$$\begin{aligned}&\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}\left(z,t,s,a\right)\\&=\frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)}\int_0^\infty\frac{y^{\mu-1}}{(1+y)^{\nu}}\Phi_{\eta,\eta',\xi,\delta;\delta',\xi'}\left(\frac{zy}{1+y},\frac{ty}{1+y},s,a\right),\end{aligned}\tag{23}$$

and

$$\begin{aligned}&\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}\left(z,t,s,a\right)\\&=\frac{\Gamma(\nu)}{\Gamma(s)\Gamma(\mu)\Gamma(\nu-\mu)}\int_0^\infty\int_0^\infty\frac{x^{s-1}e^{-ax}y^{\mu-1}}{(1+y)^{\nu}}\\&\times\sum_{k=0}^{\infty}\frac{(\eta)_k(\delta)_k}{k!(\xi)_k}\left(\frac{zye^{-x}}{1+y}\right)^k\sum_{l=0}^{\infty}\frac{(\eta')_l(\delta')_k}{l!(\xi')_k}\left(\frac{tye^{-x}}{1+y}\right)^l dxdy.\end{aligned}\tag{24}$$

Proof. Setting $a = \mu + k + l, b = \nu + k + l$ in the Eulerian beta function formula,

$$B\left(a, b-a\right) = \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)} = \int_0^\infty \frac{y^{a-1}}{(1+y)^b} dy, \Re(b) > \Re(a) > 0, \tag{25}$$

gives

$$\frac{\Gamma(\mu+k+l)\Gamma(\nu-\mu)}{\Gamma(\nu+k+l)} = \int_0^\infty \frac{y^{\mu+k+l-1}}{(1+y)^{\nu+k+l}} dy, \tag{26}$$

$$\Rightarrow \frac{(\mu)_{k+l}\Gamma(\mu)\Gamma(\nu-\mu)}{(\nu)_{k+l}\Gamma(\nu)} = \int_0^\infty \frac{y^{\mu+k+l-1}}{(1+y)^{\nu+k+l}} dy,$$
$$\Re(\nu) > \Re(\mu) > 0, k, l \in \mathbb{N}.$$

$$\Rightarrow \frac{(\mu)_{k+l}}{(\nu)_{k+l}} = \frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^\infty \frac{y^{\mu+k+l-1}}{(1+y)^{\nu+k+l}} dy. \tag{27}$$

Now substituting (27) in (11), we get

$$\begin{aligned} &\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}\left(z,t,s,a\right) \\ &= \sum_{k=0}^\infty \frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^\infty \frac{y^{\mu+k+l-1}}{(1+y)^{\nu+k+l}} \frac{(\eta)_k(\eta')_l(\delta)_k(\delta')_l}{(\xi)_k(\xi')_l k! l!} \frac{z^k t^l}{(k+l+a)^s} dy \end{aligned} \tag{28}$$

interchanging integration and summation gives

$$\begin{aligned} &\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}\left(z,t,s,a\right) \\ &= \frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^\infty \frac{y^{\mu-1}}{(1+y)^\nu} \sum_{k=0}^\infty \frac{(\eta)_k(\eta')_l(\delta)_k(\delta')_l}{(\xi)_k(\xi')_l k! l!} \left(\frac{zy}{1+y}\right)^k \left(\frac{ty}{1+y}\right)^l \frac{1}{(k+l+a)^s} dy. \end{aligned} \tag{29}$$

In view of (11) and (9) we arrived the desired result.

Now, we prove the second integral. From (18), $\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}\left(z,t,s,a\right)$ can be written as

$$\begin{aligned} &\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}\left(z,t,s,a\right) \\ &= \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} \sum_{k,l=0}^\infty \frac{(\mu)_{k+l}(\eta)_k(\eta')_l(\delta)_k(\delta')_l}{(\nu)_{k+l}(\xi)_k(\xi')_l} \frac{(ze^{-x})^k}{k!} \frac{(te^{-x})^l}{l!} dx, \end{aligned}$$

Now using (27), we get

$$\begin{aligned} \Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}\left(z,t,s,a\right) &= \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} \sum_{k,l=0}^\infty \frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^\infty \frac{y^{\mu+k+l-1}}{(1+y)^{\nu+k+l}} dy \\ &\times \frac{(\eta)_k(\eta')_l(\delta)_k(\delta')_l}{(\xi)_k(\xi')_l} \frac{(ze^{-x})^k}{k!} \frac{(te^{-x})^l}{l!} dx, \\ &= \frac{\Gamma(\nu)}{\Gamma(s)\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^\infty \int_0^\infty \frac{x^{s-1}e^{-ax}y^{\mu-1}}{(1+y)^\nu} \\ &\times \sum_{k=0}^\infty \frac{(\eta)_k(\delta)_k}{(\xi)_k k!} \left(\frac{zye^{-x}}{1+y}\right)^k \sum_{l=0}^\infty \frac{(\eta')_l(\delta')_l}{(\xi')_l l!} \left(\frac{tye^{-x}}{1+y}\right)^l dxdy. \end{aligned}$$

□

Corollary 3. *If $\delta = \delta' = 1$ and $\xi = \xi' = 1$, then we get the result (22) of [15] as*

$$\Phi_{\mu,\eta,\eta';\nu}(z,t,s,a) = \frac{\Gamma(\nu)}{\Gamma(s)\Gamma(\mu)\Gamma(\nu-\mu)} \times \int_0^\infty \int_0^\infty \frac{x^{s-1}e^{-ax}y^{\mu-1}}{(1+y)^\nu}\left(1-\frac{zye^{-x}}{1+y}\right)^{-\eta}\left(1-\frac{tye^{-x}}{1+y}\right)^{-\eta'} dxdy,$$
$$(\Re(\nu) > \Re(\mu) > 0; \min\{\Re(s), \Re(a)\} > 0).$$

Theorem 3. *The following summation formula hold true.*

$$\sum_{r=0}^{\infty} \frac{(s)_r}{r!} \Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z,t,s+r,a)\, x^r = \Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z,t,s,a-x), \quad (|x| < |a|; s \neq 1). \tag{30}$$

Proof. Using (11), we have

$$\begin{aligned} &\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z,t,s,a-x) \\ &= \sum_{k,l=0}^{\infty} \frac{(\mu)_{k+l}(\eta)_k(\eta')_l(\delta)_k(\delta')_l}{(\nu)_{k+l}(\xi)_k(\xi)_l} \frac{z^k t^l}{k!l!(k+l+a-x)^s}, \\ &= \sum_{k,l=0}^{\infty} \frac{(\mu)_{k+l}(\eta)_k(\eta')_l(\delta)_k(\delta')_l}{(\nu)_{k+l}(\xi)_k(\xi)_l} \frac{z^k t^l}{k!l!(k+l+a)^s}\left(1-\frac{x}{k+l+a}\right)^{-s}, \end{aligned}$$

using binomial series, we get

$$= \sum_{r=0}^{\infty} \frac{(s)_r}{s!}\left\{\sum_{k,l=0}^{\infty} \frac{(\mu)_{k+l}(\eta)_k(\eta')_l(\delta)_k(\delta')_l}{(\nu)_{k+l}(\xi)_k(\xi)_l} \frac{z^k t^l}{k!l!(k+l+a)^{s+r}}\right\} x^r.$$

In view of definition (11), we reach the required result. □

3. A Connection with Generalized Hypergeometric Function

In this section, we establish the connection between (11) and generalized hypergeometric function.

Theorem 4. *For $a \neq \{-1,-2,\cdots\}$ and $z \neq 0$, the following explicit series representation holds true*

$$\begin{aligned} \Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z,t,s,a) &= \textstyle\sum_{k=0}^{\infty} \frac{(\mu)_k(\eta)_k(\delta)_k z^k}{(\nu)_k(\xi)_k(a+k)^s k!} \\ &\times F\left(\eta',\delta',1-\xi-k,-k;1-\eta-k,1-\delta-k,\xi';\tfrac{t}{z}\right), \end{aligned} \tag{31}$$

where $F(-)$ is the generalized hypergeometric function defined in (1).

Proof. Using (11) and the identity ([16] page 56, Equation (1))

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} A(l,k) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} A(l,k-l),$$

which implies that

$$\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z,t,s,a) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{(\mu)_k(\eta)_{k-l}(\eta')_l(\delta)_{k-l}(\delta')_l}{(\nu)_k(\xi)_{k-l}(\xi')_l(k-l)!l!} \frac{z^{k-l}t^l}{(k+a)^s}.$$

Now,

$$(k-l)! = \frac{(-1)^l k!}{(-k)_l}, 0 \le l \le k$$
$$(\eta)_{k-l} = \frac{(-1)^l (\eta)_k}{(1-\eta-k)_l}, 0 \le l \le k,$$

we get,

$$\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z,t,s,a) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{(\mu)_k(\eta)_k(\eta')_l(\delta)_k(\delta')_l(1-\xi-k)_l(-k)_l z^{k-l} t^l}{(\nu)_k(1-\eta-k)_l(1-\delta-k)_l(\xi)_k(\xi')_l(k)!l!} \frac{1}{(k+a)^s}.$$

Lastly, summing the l-series, we get the required result. □

Corollary 4. *If we set $\delta = \xi$ in Theorem 4, then we get (28) of [14] as*

$$\Phi_{\mu,\eta,\eta',\xi,\delta';\nu,\xi,\xi'}(z,t,s,a) = \sum_{k=0}^{\infty} \frac{(\mu)_k(\eta)_k z^k}{(\nu)_k(k+a)^s k!} F\left(\eta', \delta', -k, 1-\xi-k; \xi', 1-\eta-k, 1-\xi-k; \frac{t}{z}.\right) \quad (32)$$

Corollary 5. *If we set $\nu = \eta, \delta = \xi$ and $\delta' = \xi'$ in Theorem 4, then we get (29) of [14] as*

$$\Phi_{\mu,\eta,\eta',\xi,\xi';\eta,\xi,\xi'}(z,t,s,a) = \sum_{k=0}^{\infty} \frac{(\mu)_k z^k}{(a+k)^s k!} F\left(\eta', -k; 1-\eta-k; \frac{t}{z}\right). \quad (33)$$

4. Concluding Remarks

An extension of a generalized Hurwitz-Lerch Zeta function is defined and some of its properties are studied in this paper. An integral representation is established and a relation with Appell's type function is given. Finally, a connection with the hypergeometric function is also given. The results derived here are more general in nature by comparing the results of the papers [14,15] which help to derive some interesting special cases and are mentioned in Remark 1 and Corollaries 1–5.

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