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An Iterative Algorithm for Solving Generalized Variational Inequalities and Fixed Points Problems

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Abstract: In this paper, a generalized variational inequality and fixed points problem is presented. An iterative algorithm is introduced for finding a solution of the generalized variational inequalities and fixed point of two quasi-pseudocontractive operators under a nonlinear transformation. Strong convergence of the suggested algorithm is demonstrated.

Keywords: split problems; variational inequalities; pseudocontractive operators; fixed points

MSC: 49J53; 90C25

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, respectively. Let $\emptyset \neq \mathcal{C} \subset \mathcal{H}$ be a closed convex set. For the given two nonlinear operators $\mathcal{A} \colon \mathcal{C} \to \mathcal{H}$ and $\varphi \colon \mathcal{C} \to \mathcal{C}$, the generalized variational inequality (GVI) aims to find an element $x^\dagger \in \mathcal{C}$ such that

$$\langle \mathcal{A}x^\dagger, \varphi(y) - \varphi(x^\dagger) \rangle \geq 0, \ \forall y \in \mathcal{C}. \tag{1}$$

We use $GVI(\mathcal{A}, \varphi, \mathcal{C})$ to denote the solution set of Equation (1).

If $\varphi \equiv \mathcal{I}$, then GVI (1) can be reduced to find an element $x^\dagger \in \mathcal{C}$ such that

$$\langle \mathcal{A}x^\dagger, y - x^\dagger \rangle \geq 0, \ \forall y \in \mathcal{C}. \tag{2}$$

We use $VI(\mathcal{A}, \mathcal{C})$ to denote the solution set of Equation (2).

Variational inequalities were introduced by Stampacchia [1] and provide a useful tool for researching a large variety of interesting problems arising in physics, economics, finance, elasticity, optimization, network analysis, medical images, water resources, and structural analysis [2–8]. For some related work, please refer to References [9–27].

Iterative computing fixed points of nonlinear operators is nowadays an active research field [28–35]. The interest in pseudocontractive operators is due mainly to their usefulness as an additional assumption to Lipschitz type conditions in proving convergence of fixed point iterative procedures and their connection with the important class of nonlinear monotone (accretive) operators.

Recall that an operator $S\colon \mathcal{C} \to \mathcal{C}$ is called pseudocontractive if

$$\|Su^\dagger - Sv^\dagger\|^2 \le \|u^\dagger - v^\dagger\|^2 + \|(\mathcal{I} - S)u^\dagger - (\mathcal{I} - S)v^\dagger\|^2, \tag{3}$$

for all $u^\dagger, v^\dagger \in \mathcal{C}$.

Iterative algorithms for finding the fixed points of pseudocontractive operators have been studied by many mathematicians, see, for example, References [36–40]. In this article, we focus on a general class of quasi-pseudocontractive operators. Recall that a mapping $S\colon \mathcal{C} \to \mathcal{C}$ is called quasi-pseudocontractive if

$$\|Su^\dagger - v^\dagger\|^2 \le \|u^\dagger - v^\dagger\|^2 + \|Su^\dagger - v^\dagger\|^2, \tag{4}$$

for all $u^\dagger \in \mathcal{C}$ and $v^\dagger \in \mathrm{Fix}(S)$, where $\mathrm{Fix}(S)$ stands for the set of fixed points of S, i.e., $\mathrm{Fix}(S) = \{z^\dagger : z^\dagger = Sz^\dagger\}$.

Now it is well-known that the quasi-pseudocontractive operators include the directed operators and the demicontractive operators as special cases [19]. In this paper, we consider the following generalized variational inequalities and fixed points problems of finding an element $\tilde{x}$ such that

$$\tilde{x} \in GVI(\mathcal{A}, \varphi, \mathcal{C}) \text{ and } \varphi(\tilde{x}) \in \mathrm{Fix}(S) \cap \mathrm{Fix}(T), \tag{5}$$

where S and T are two quasi-pseudocontractive operators.

In order to solve Equation (5), we introduce a new iterative algorithm. Under some mild restrictions, we will demonstrate the strong convergence analysis of the presented algorithm.

2. Notation and Lemmas

Let $\mathcal{H}$ be a real Hilbert space. Let $\emptyset \ne \mathcal{C} \subset \mathcal{H}$ be a closed convex set. Recall that an operator $S\colon \mathcal{C} \to \mathcal{C}$ is called L-Lipschitz if $\|Su^\dagger - Sv^\dagger\| \le L\|u^\dagger - v^\dagger\|$ for all $u^\dagger, v^\dagger \in \mathcal{C}$, where $L > 0$ is a constant.

Definition 1. *An operator $\mathcal{A}\colon \mathcal{C} \to \mathcal{H}$ is said to be*

- *Monotone if $\langle u^\ddagger - v^\ddagger, \mathcal{A}u^\ddagger - \mathcal{A}v^\ddagger \rangle \ge 0$, $\forall u^\ddagger, v^\ddagger \in \mathcal{C}$.*
- *Strongly monotone if $\langle u^\ddagger - v^\ddagger, \mathcal{A}u^\ddagger - \mathcal{A}v^\ddagger \rangle \ge \delta\|u^\ddagger - v^\ddagger\|^2$, $\forall u^\ddagger, v^\ddagger \in \mathcal{C}$, where $\delta > 0$ is a constant.*
- *α-inverse strongly monotone if $\langle u^\ddagger - v^\ddagger, \mathcal{A}u^\ddagger - \mathcal{A}v^\ddagger \rangle \ge \alpha\|\mathcal{A}u^\ddagger - \mathcal{A}v^\ddagger\|^2$, $\forall u^\ddagger, v^\ddagger \in \mathcal{C}$, where $\alpha > 0$ is a constant.*
- *α-inverse strongly φ-monotone if $\langle \mathcal{A}u^\ddagger - \mathcal{A}v^\ddagger, \varphi(u^\ddagger) - \varphi(v^\ddagger) \rangle \ge \alpha\|\mathcal{A}u^\ddagger - \mathcal{A}v^\ddagger\|^2$, $\forall u^\ddagger, v^\ddagger \in \mathcal{C}$, where $\varphi\colon \mathcal{C} \to \mathcal{C}$ is a nonlinear operator and $\alpha > 0$ is a constant.*

An operator $R\colon \mathcal{H} \rightrightarrows 2^{\mathcal{H}}$ is called monotone on $\mathcal{H}$ if and only if $\langle u^\dagger - v^\dagger, x - y \rangle \ge 0$ for all $x, y \in \mathrm{dom}(R)$, $u^\dagger \in Rx$, and $v^\dagger \in Ry$. A monotone operator R on $\mathcal{H}$ is called maximal monotone if the graph of R is a maximal monotone set.

We use $\mathrm{proj}_{\mathcal{C}}$ to denote the nearest point projection from $\mathcal{H}$ onto $\mathcal{C}$, that is, for $\forall u^\dagger \in \mathcal{H}$, $\|u^\dagger - \mathrm{proj}_{\mathcal{C}}[u^\dagger]\| \le \|u - u^\dagger\|$, for all $u \in \mathcal{C}$. Now it is known that the operator $\mathrm{proj}_{\mathcal{C}}\colon \mathcal{H} \to \mathcal{C}$ is firmly nonexpansive, that is,

$$\|\mathrm{proj}_{\mathcal{C}}[u^\dagger] - \mathrm{proj}_{\mathcal{C}}[v^\dagger]\|^2 \le \langle \mathrm{proj}_{\mathcal{C}}[u^\dagger] - \mathrm{proj}_{\mathcal{C}}[v^\dagger], u^\dagger - v^\dagger \rangle, \ \forall u^\dagger, v^\dagger \in \mathcal{H}.$$

Consequently,

$$\langle u^\dagger - \mathrm{proj}_{\mathcal{C}}[u^\dagger], x^\dagger - \mathrm{proj}_{\mathcal{C}}[u^\dagger] \rangle \le 0, \ \forall u^\dagger \in \mathcal{H}, x^\dagger \in \mathcal{C}. \tag{6}$$

Recall that an operator S is said to be demiclosed if $w_n \rightharpoonup \tilde{u}$ weakly and $Sw_n \to u$ strongly, implies $S(\tilde{u}) = u$. We collect several lemmas for our main results in the next section.

Lemma 1 ([41]). *Let $\mathcal{H}$ be a real Hilbert space. Let $\emptyset \neq \mathcal{C} \subset \mathcal{H}$ be a closed convex set. Let $T\colon \mathcal{C} \to \mathcal{C}$ be an L-Lipschitz quasi-pseudocontractive operator. Then, we have*

$$\|(1-\sigma)x^\dagger + \sigma T((1-\xi)x^\dagger + \xi T x^\dagger) - y^\dagger\|^2 \leq \|x^\dagger - y^\dagger\|^2 + \sigma(\sigma - \xi)\|T((1-\xi)x^\dagger + \xi T x^\dagger) - x^\dagger\|^2,$$

for all $x^\dagger \in \mathcal{C}$ and $y^\dagger \in \mathrm{Fix}(T)$ when $0 < \sigma < \xi < \frac{1}{\sqrt{1+L^2}+1}$.

Lemma 2 ([41]). *Let $\mathcal{H}$ be a real Hilbert space. Let $\emptyset \neq \mathcal{C} \subset \mathcal{H}$ be a closed convex set. If the operator $T\colon \mathcal{C} \to \mathcal{C}$ is L-Lipschitz with $L \geq 1$, then we have*

$$\mathrm{Fix}(((1-\delta)\mathcal{I} + \delta T)T) = \mathrm{Fix}(T((1-\delta)\mathcal{I} + \delta T)) = \mathrm{Fix}(T),$$

where $\delta \in (0, \frac{1}{L})$.

Lemma 3 ([41]). *Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. If the operator $T\colon \mathcal{C} \to \mathcal{C}$ is L-Lipschitz with $L \geq 1$ and $\mathcal{I} - T$ is demiclosed at 0, then the composition operator $\mathcal{I} - T((1-\delta)\mathcal{I} + \delta T)$ is also demiclosed at 0 provided $\delta \in (0, \frac{1}{L})$.*

Lemma 4 ([42]). *Suppose $\{\varpi_n\} \subset [0, \infty)$, $\{\nu_n\} \subset (0, 1)$, and $\{\varrho_n\}$ are three real number sequences satisfying*

(i) $\varpi_{n+1} \leq (1-\nu_n)\varpi_n + \varrho_n, \forall n \geq 1;$
(ii) $\sum_{n=1}^{\infty} \nu_n = \infty;$
(iii) $\limsup_{n\to\infty} \frac{\varrho_n}{\nu_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\varrho_n| < \infty.$

Then, $\lim_{n\to\infty} \varpi_n = 0$.

Lemma 5 ([43]). *Let $\{w_n\}$ be a sequence of real numbers. Assume there exists at least a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \leq w_{n_k+1}$ for all $k \geq 0$. For every $n \geq N_0$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{i \leq n : w_{n_i} < w_{n_i+1}\}.$$

Then, $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq N_0$, we have $\max\{w_{\tau(n)}, w_n\} \leq w_{\tau(n)+1}$.

3. Main Results

Let $\mathcal{H}$ be a real Hilbert space. Let $\emptyset \neq \mathcal{C} \subset \mathcal{H}$ be a closed convex set. Let $\phi\colon \mathcal{C} \to \mathcal{H}$ be an L-Lipschitz operator. Let $\varphi\colon \mathcal{C} \to \mathcal{C}$ be a δ-strongly monotone and weakly continuous operator such that its rang $R(\varphi) = \mathcal{C}$. Let the operator $\mathcal{A}\colon \mathcal{C} \to \mathcal{H}$ be α-inverse strongly φ-monotone. Let $S\colon \mathcal{C} \to \mathcal{C}$ be an L_1-Lipschitzian quasi-pseudocontractive operator with $L_1 > 1$ and $T\colon \mathcal{C} \to \mathcal{C}$ be an L_2-Lipschitzian quasi-pseudocontractive operator with $L_2 > 1$. Denote the solution set of Equation (5) by Ω, that is, $\Omega = GVI(\mathcal{A}, \varphi, \mathcal{C}) \bigcap \varphi^{-1}(\mathrm{Fix}(S) \cap \mathrm{Fix}(T))$. In what follows, assume $\Omega \neq \emptyset$. Next, we firstly suggest the following algorithm for solving the problem in Equation (5).

For initial guess $x_0 \in \mathcal{C}$, define the sequence $\{x_n\}$ by the following form

$$\begin{cases} u_n = \mathrm{proj}_{\mathcal{C}}[\alpha_n \nu \phi(x_n) + (1-\alpha_n)(\varphi(x_n) - \varsigma_n \mathcal{A} x_n)], \\ y_n = (1-\sigma_n)u_n + \sigma_n T((1-\delta_n)u_n + \delta_n T u_n), \\ z_n = (1-\zeta_n)y_n + \zeta_n S((1-\eta_n)y_n + \eta_n S y_n), \\ \varphi(x_{n+1}) = \vartheta_n \varphi(x_n) + (1-\vartheta_n)z_n, \ n \geq 0, \end{cases} \tag{7}$$

where $\nu > 0$ is a constant, $\{\alpha_n\}$, $\{\sigma_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$, and $\{\vartheta_n\}$ are six sequences in $(0, 1)$ and $\{\varsigma_n\}$ is a sequence in $(0, \infty)$.

Theorem 1. *Suppose $I - S$ and $I - T$ are demiclosed at 0. Assume the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$;
(ii) $0 < a_1 < \sigma_n < c_1 < \delta_n < b_1 < \frac{1}{\sqrt{1+L_2{}^2}+1}$ *and* $0 < a_2 < \zeta_n < c_2 < \eta_n < b_2 < \frac{1}{\sqrt{1+L_1^2}+1}$;
(iii) $0 < \liminf_{n\to\infty} \vartheta_n \leq \limsup_{n\to\infty} \vartheta_n < 1$;
(iv) $L\nu < \delta < 2\alpha$ *and* $0 < \liminf_{n\to\infty} \varsigma_n \leq \limsup_{n\to\infty} \varsigma_n < 2\alpha$.

Then, the iterative sequence $\{x_n\}$ defined by Equation (7) strongly converges to $\tilde{x} \in \Omega$ which solves the variational inequality

$$\langle \nu\phi(\tilde{x}) - \varphi(\tilde{x}), \varphi(x^\dagger) - \varphi(\tilde{x})\rangle \leq 0, \quad \forall x^\dagger \in \Omega. \tag{8}$$

Proof. Since φ is δ-strongly monotone, we deduce

$$\|\varphi(x) - \varphi(y)\| \geq \delta\|x - y\|, \forall x, y \in \mathcal{C}. \tag{9}$$

Note that VI (8) has a unique solution which is denoted by $\tilde{x}$. Thus, $\tilde{x} \in GVI(\mathcal{A}, \varphi, \mathcal{C})$ and $\varphi(\tilde{x}) \in \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$. By virtue of Equation (6), we get $\varphi(\tilde{x}) = \mathrm{proj}_{\mathcal{C}}[\varphi(\tilde{x}) - \varsigma_n \mathcal{A}\tilde{x}]$ for all $n \geq 0$. Note that $\mathcal{A}$ is α-inverse strongly φ-monotone. By Definition 1, we have

$$\begin{aligned}\|(\varphi(x) - \varsigma\mathcal{A}x) - (\varphi(\tilde{x}) - \varsigma\mathcal{A}\tilde{x})\|^2 &= \|\varphi(x) - \varphi(\tilde{x})\|^2 - 2\varsigma\langle \mathcal{A}x - \mathcal{A}\tilde{x}, \varphi(x) - \varphi(\tilde{x})\rangle \\ &\quad + \varsigma^2\|\mathcal{A}x - \mathcal{A}\tilde{x}\|^2 \\ &\leq \|\varphi(x) - \varphi(\tilde{x})\|^2 - 2\varsigma\alpha\|\mathcal{A}x - \mathcal{A}\tilde{x}\|^2 + \varsigma^2\|\mathcal{A}x - \mathcal{A}\tilde{x}\|^2 \\ &\leq \|\varphi(x) - \varphi(\tilde{x})\|^2 + \varsigma(\varsigma - 2\alpha)\|\mathcal{A}x - \mathcal{A}\tilde{x}\|^2.\end{aligned} \tag{10}$$

According to Equation (10), we get

$$\begin{aligned}\|(\varphi(x_n) - \varsigma_n\mathcal{A}x_n) - (\varphi(\tilde{x}) - \varsigma_n\mathcal{A}\tilde{x})\|^2 &\leq \|\varphi(x_n) - \varphi(\tilde{x})\|^2 + \varsigma_n(\varsigma_n - 2\alpha)\|\mathcal{A}x_n - \mathcal{A}\tilde{x}\|^2 \\ &\leq \|\varphi(x_n) - \varphi(\tilde{x})\|^2,\end{aligned} \tag{11}$$

and

$$\begin{aligned}\|\varphi(x_{n+1}) - \varsigma_{n+1}\mathcal{A}x_{n+1} - (\varphi(x_n) - \varsigma_{n+1}\mathcal{A}x_n)\|^2 &\leq \|\varphi(x_{n+1}) - \varphi(x_n)\|^2 \\ &\quad + \varsigma_{n+1}(\varsigma_{n+1} - 2\alpha)\|\mathcal{A}x_{n+1} - \mathcal{A}x_n\|^2.\end{aligned} \tag{12}$$

From Equations (7), (9), and (11), we have

$$\begin{aligned}\|u_n - \varphi(\tilde{x})\| &= \|\mathrm{proj}_{\mathcal{C}}[\alpha_n\nu\phi(x_n) + (1-\alpha_n)(\varphi(x_n) - \varsigma_n\mathcal{A}x_n)] - \mathrm{proj}_{\mathcal{C}}[\varphi(\tilde{x}) - \varsigma_n\mathcal{A}\tilde{x}]\| \\ &\leq \|\alpha_n(\nu\phi(x_n) - \varphi(\tilde{x}) + \varsigma_n\mathcal{A}\tilde{x}) + (1-\alpha_n)((\varphi(x_n) - \varsigma_n\mathcal{A}x_n) - (\varphi(\tilde{x}) - \varsigma_n\mathcal{A}\tilde{x}))\| \\ &\leq \alpha_n\|\nu\phi(x_n) - \nu\phi(\tilde{x})\| + \alpha_n\|\nu\phi(\tilde{x}) - \varphi(\tilde{x}) + \varsigma_n\mathcal{A}\tilde{x}\| \\ &\quad + (1-\alpha_n)\|(\varphi(x_n) - \varsigma_n\mathcal{A}x_n) - (\varphi(\tilde{x}) - \varsigma_n\mathcal{A}\tilde{x})\| \\ &\leq \alpha_n\nu L\|x_n - \tilde{x}\| + \alpha_n\|\nu\phi(\tilde{x}) - \varphi(\tilde{x}) + \varsigma_n\mathcal{A}\tilde{x}\| + (1-\alpha_n)\|\varphi(x_n) - \varphi(\tilde{x})\| \\ &\leq \alpha_n\nu L/\delta\|\varphi(x_n) - \varphi(\tilde{x})\| + \alpha_n\|\nu\phi(\tilde{x}) - \varphi(\tilde{x}) + \varsigma_n\mathcal{A}\tilde{x}\| \\ &\quad + (1-\alpha_n)\|\varphi(x_n) - \varphi(\tilde{x})\| \\ &= [1 - (1 - \nu L/\delta)\alpha_n]\|\varphi(x_n) - \varphi(\tilde{x})\| + \alpha_n\|\nu\phi(\tilde{x}) - \varphi(\tilde{x}) + \varsigma_n\mathcal{A}\tilde{x}\| \\ &\leq [1 - (1 - \nu L/\delta)\alpha_n]\|\varphi(x_n) - \varphi(\tilde{x})\| + \alpha_n(\|\nu\phi(\tilde{x}) - \varphi(\tilde{x})\| + 2\alpha\|\mathcal{A}\tilde{x}\|).\end{aligned} \tag{13}$$

By Equations (11) and (13), we obtain

$$\begin{aligned}\|u_n-\varphi(\tilde{x})\|^2 &\le \|\alpha_n(\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x})+(1-\alpha_n)((\varphi(x_n)-\varsigma_n\mathcal{A}x_n)-(\varphi(\tilde{x})-\varsigma_n\mathcal{A}\tilde{x}))\|^2\\ &\le \alpha_n\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2+(1-\alpha_n)\|(\varphi(x_n)-\varsigma_n\mathcal{A}x_n)-(\varphi(\tilde{x})-\varsigma_n\mathcal{A}\tilde{x}))\|^2\\ &\le \alpha_n\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2+(1-\alpha_n)[\|\varphi(x_n)-\varphi(\tilde{x})\|^2\\ &\quad +\varsigma_n(\varsigma_n-2\alpha)\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|^2].\end{aligned} \tag{14}$$

In view of Lemma 1, we deduce

$$\begin{aligned}\|y_n-\varphi(\tilde{x})\|^2 &= \|(1-\sigma_n)u_n+\sigma_nT((1-\delta_n)u_n+\delta_nTu_n)-\varphi(\tilde{x})\|^2\\ &\le \|u_n-\varphi(\tilde{x})\|^2+\sigma_n(\sigma_n-\delta_n)\|T((1-\delta_n)u_n+\delta_nTu_n)-u_n\|^2\\ &\le \|u_n-\varphi(\tilde{x})\|^2,\end{aligned} \tag{15}$$

and

$$\begin{aligned}\|z_n-\varphi(\tilde{x})\|^2 &= \|(1-\zeta_n)y_n+\zeta_nS((1-\eta_n)y_n+\eta_nSy_n)-\varphi(\tilde{x})\|^2\\ &\le \|y_n-\varphi(\tilde{x})\|^2+\zeta_n(\zeta_n-\eta_n)\|S((1-\eta_n)y_n+\eta_nSy_n)-y_n\|^2\\ &\le \|y_n-\varphi(\tilde{x})\|^2.\end{aligned} \tag{16}$$

Combining Equations (10), (14), (15) with (16), we obtain

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi(\tilde{x})\| &\le \vartheta_n\|\varphi(x_n)-\varphi(\tilde{x})\|+(1-\vartheta_n)\|z_n-\varphi(\tilde{x})\|\\ &\le \vartheta_n\|\varphi(x_n)-\varphi(\tilde{x})\|+(1-\vartheta_n)\|u_n-\varphi(\tilde{x})\|\\ &\le \vartheta_n\|\varphi(x_n)-\varphi(\tilde{x})\|+(1-\vartheta_n)[1-(1-\nu L/\delta)\alpha_n]\|\varphi(x_n)-\varphi(\tilde{x})\|\\ &\quad +(1-\vartheta_n)\alpha_n(\|\nu\phi(\tilde{x})-\varphi(\tilde{x})\|+2\alpha\|\mathcal{A}\tilde{x}\|)\\ &= [1-(1-\nu L/\delta)(1-\vartheta_n)\alpha_n]\|\varphi(x_n)-\varphi(\tilde{x})\|\\ &\quad +(1-\nu L/\delta)(1-\vartheta_n)\alpha_n\frac{\|\nu\phi(\tilde{x})-\varphi(\tilde{x})\|+2\alpha\|\mathcal{A}\tilde{x}\|}{1-\nu L/\delta}.\end{aligned} \tag{17}$$

An induction to derive

$$\|\varphi(x_n)-\varphi(\tilde{x})\| \le \max\left\{\|\varphi(x_0)-\varphi(\tilde{x})\|, \frac{\|\nu\phi(\tilde{x})-\varphi(\tilde{x})\|+2\alpha\|\mathcal{A}\tilde{x}\|}{1-\nu L/\delta}\right\}.$$

It follows that

$$\|x_n-\tilde{x}\| \le \frac{1}{\delta}\|\varphi(x_n)-\varphi(\tilde{x})\| \le \frac{1}{\delta}\max\left\{\|\varphi(x_0)-\varphi(\tilde{x})\|, \frac{\|\nu\phi(\tilde{x})-\varphi(\tilde{x})\|+2\alpha\|\mathcal{A}\tilde{x}\|}{1-\nu L/\delta}\right\}.$$

Hence, $\{\varphi(x_n)\}$ and $\{x_n\}$ are all bounded.

By Equation (7), we get

$$\varphi(x_{n+1})-\varphi(x_n)=(1-\vartheta_n)(z_n-\varphi(x_n)),\ n\ge 0. \tag{18}$$

It follows that

$$\langle\varphi(x_n)-\varphi(\tilde{x}),\varphi(x_{n+1})-\varphi(x_n)\rangle=(1-\vartheta_n)\langle\varphi(x_n)-\varphi(\tilde{x}),z_n-\varphi(x_n)\rangle. \tag{19}$$

Observe that

$$2\langle\varphi(x_n)-\varphi(\tilde{x}),\varphi(x_{n+1})-\varphi(x_n)\rangle = \|\varphi(x_{n+1})-\varphi(\tilde{x})\|^2-\|\varphi(x_n)-\varphi(\tilde{x})\|^2 \\ -\|\varphi(x_{n+1})-\varphi(x_n)\|^2, \tag{20}$$

and

$$2\langle z_n-\varphi(x_n),\varphi(x_n)-\varphi(\tilde{x})\rangle = \|z_n-\varphi(\tilde{x})\|^2-\|\varphi(x_n)-\varphi(\tilde{x})\|^2-\|z_n-\varphi(x_n)\|^2. \tag{21}$$

By virtue of Equations (19)–(21), we deduce

$$\|\varphi(x_{n+1})-\varphi(\tilde{x})\|^2 = (1-\vartheta_n)[\|z_n-\varphi(\tilde{x})\|^2-\|\varphi(x_n)-\varphi(\tilde{x})\|^2-\|z_n-\varphi(x_n)\|^2] \\ +\|\varphi(x_n)-\varphi(\tilde{x})\|^2+\|\varphi(x_{n+1})-\varphi(x_n)\|^2. \tag{22}$$

Combining Equations (18) with (22), we have

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi(\tilde{x})\|^2 &= (1-\vartheta_n)[\|z_n-\varphi(\tilde{x})\|^2-\|\varphi(x_n)-\varphi(\tilde{x})\|^2-\|z_n-\varphi(x_n)\|^2] \\ &\quad +(1-\vartheta_n)^2\|z_n-\varphi(x_n)\|^2+\|\varphi(x_n)-\varphi(\tilde{x})\|^2 \\ &= (1-\vartheta_n)[\|z_n-\varphi(\tilde{x})\|^2-\|\varphi(x_n)-\varphi(\tilde{x})\|^2]-\vartheta_n(1-\vartheta_n)\|z_n-\varphi(x_n)\|^2 \\ &\le (1-\vartheta_n)[\|u_n-\varphi(\tilde{x})\|^2-\|\varphi(x_n)-\varphi(\tilde{x})\|^2]-\vartheta_n(1-\vartheta_n)\|z_n-\varphi(x_n)\|^2.\end{aligned} \tag{23}$$

Returning to Equation (13), we get

$$\|u_n-\varphi(\tilde{x})\|^2 \le [1-(1-\nu L/\delta)\alpha_n]\|\varphi(x_n)-\varphi(\tilde{x})\|^2 \\ +(1-\nu L/\delta)\alpha_n\left(\frac{\|\nu\phi(\tilde{x})-\varphi(\tilde{x})\|+2\alpha\|\mathcal{A}\tilde{x}\|}{(1-\nu L/\delta)}\right)^2. \tag{24}$$

There exists two possible cases. CASE 1. There exists $m>0$ such that $\{\|\varphi(x_n)-\varphi(\tilde{x})\|\}$ is decreasing when $n\ge m$. Thus, $\lim_{n\to\infty}\|\varphi(x_n)-\varphi(\tilde{x})\|$ exists. From Equations (23) and (24), we have

$$\begin{aligned}\vartheta_n(1-\vartheta_n)\|z_n-\varphi(x_n)\|^2 &\le \|\varphi(x_n)-\varphi(\tilde{x})\|^2-\|\varphi(x_{n+1})-\varphi(\tilde{x})\|^2 \\ &\quad +(1-\vartheta_n)[\|u_n-\varphi(\tilde{x})\|^2-\|\varphi(x_n)-\varphi(\tilde{x})\|^2] \\ &\le \|\varphi(x_n)-\varphi(\tilde{x})\|^2-\|\varphi(x_{n+1})-\varphi(\tilde{x})\|^2 \\ &\quad +(1-\nu L/\delta)\alpha_n\left(\frac{\|\nu\phi(\tilde{x})-\varphi(\tilde{x})\|+2\alpha\|\mathcal{A}\tilde{x}\|}{(1-\nu L/\delta)}\right)^2 \\ &\to 0.\end{aligned}$$

This together with assumptions (i) and (iii) implies that

$$\lim_{n\to\infty}\|z_n-\varphi(x_n)\|=0. \tag{25}$$

Furthermore, it follows from Equation (18) that

$$\lim_{n\to\infty}\|\varphi(x_{n+1})-\varphi(x_n)\|=0. \tag{26}$$

By Equation (14), we have

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi(\tilde{x})\|^2 &= \|\vartheta_n(\varphi(x_n)-\varphi(\tilde{x}))+(1-\vartheta_n)(z_n-\varphi(\tilde{x}))\|^2 \\ &\leq \vartheta_n\|\varphi(x_n)-\varphi(\tilde{x})\|^2+(1-\vartheta_n)\|z_n-\varphi(\tilde{x})\|^2 \\ &\leq \vartheta_n\|\varphi(x_n)-\varphi(\tilde{x})\|^2+(1-\vartheta_n)\|u_n-\varphi(\tilde{x})\|^2 \\ &\leq (1-\vartheta_n)\alpha_n\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2+\vartheta_n\|\varphi(x_n)-\varphi(\tilde{x})\|^2 \\ &\quad+(1-\vartheta_n)(1-\alpha_n)\|\varphi(x_n)-\varphi(\tilde{x})\|^2 \\ &\quad+(1-\vartheta_n)(1-\alpha_n)\varsigma_n(\varsigma_n-2\alpha)\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|^2] \\ &\leq \|\varphi(x_n)-\varphi(\tilde{x})\|^2+(1-\vartheta_n)\alpha_n\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2 \\ &\quad+(1-\vartheta_n)(1-\alpha_n)\varsigma_n(\varsigma_n-2\alpha)\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|^2.\end{aligned} \tag{27}$$

Hence,

$$\begin{aligned}&(1-\vartheta_n)(1-\alpha_n)\varsigma_n(2\alpha-\varsigma_n)\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|^2 \\ &\leq \|\varphi(x_n)-\varphi(\tilde{x})\|^2-\|\varphi(x_{n+1})-\varphi(\tilde{x})\|^2+(1-\vartheta_n)\alpha_n\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_nB\tilde{x}\|^2 \\ &\leq (\|\varphi(x_n)-\varphi(\tilde{x})\|+\|\varphi(x_{n+1})-\varphi(\tilde{x})\|)\|\varphi(x_{n+1})-\varphi(x_n)\| \\ &\quad+(1-\vartheta_n)\alpha_n\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2 \\ &\to 0 \text{ (by } (i) \text{ and Equation (26)).}\end{aligned}$$

This, together with assumption (iv), implies that

$$\lim_{n\to\infty}\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|=0. \tag{28}$$

Set $v_n=\varphi(x_n)-\varsigma_n\mathcal{A}x_n-(\varphi(\tilde{x})-\varsigma_n\mathcal{A}\tilde{x})$ for all n. Applying Equation (6), we get

$$\begin{aligned}\|u_n-\varphi(\tilde{x})\|^2 &= \|\mathrm{proj}_C[\alpha_n\nu\phi(x_n)+(1-\alpha_n)(\varphi(x_n)-\varsigma_nBx_n)]-\mathrm{proj}_C[\varphi(\tilde{x})-\varsigma_n\mathcal{A}\tilde{x}]\|^2 \\ &\leq \langle\alpha_n(\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x})+(1-\alpha_n)v_n,u_n-\varphi(\tilde{x})\rangle \\ &= \frac{1}{2}\Big\{\|\alpha_n(\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x})+(1-\alpha_n)v_n\|^2+\|u_n-\varphi(\tilde{x})\|^2 \\ &\quad-\|\alpha_n(\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x})+(1-\alpha_n)v_n-u_n+\varphi(\tilde{x})\|^2\Big\} \\ &\leq \frac{1}{2}\Big\{\alpha_n\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2+(1-\alpha_n)\|\varphi(x_n)-\varphi(\tilde{x})\|^2 \\ &\quad+\|u_n-\varphi(\tilde{x})\|^2-\|\alpha_n(\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-v_n)+\varphi(x_n) \\ &\quad-u_n-\varsigma_n(\mathcal{A}x_n-\mathcal{A}\tilde{x})\|^2\Big\} \\ &= \frac{1}{2}\Big\{\alpha_n\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2+(1-\alpha_n)\|\varphi(x_n)-\varphi(\tilde{x})\|^2 \\ &\quad+\|u_n-\varphi(\tilde{x})\|^2-\|\varphi(x_n)-u_n\|^2-\alpha_n^2\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-v_n\|^2 \\ &\quad-\varsigma_n^2\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|+2\varsigma_n\alpha_n\langle\mathcal{A}x_n-\mathcal{A}\tilde{x},\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-v_n\rangle \\ &\quad+2\varsigma_n\langle\varphi(x_n)-u_n,\mathcal{A}x_n-\mathcal{A}\tilde{x}\rangle \\ &\quad-2\alpha_n\langle\varphi(x_n)-u_n,\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-v_n\rangle\Big\}.\end{aligned} \tag{29}$$

It follows that

$$\begin{aligned}\|u_n-\varphi(\tilde{x})\|^2 \le{}& \alpha_n\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2+(1-\alpha_n)\|\varphi(x_n)-\varphi(\tilde{x})\|^2\\&-\|\varphi(x_n)-u_n\|^2+2\varsigma_n\|\varphi(x_n)-u_n\|\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|\\&+2\alpha_n\|\varphi(x_n)-u_n\|\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-v_n\|\\&+2\varsigma_n\alpha_n\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-v_n\|.\end{aligned} \tag{30}$$

In the light of Equations (27) and (30), we have

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi(\tilde{x})\|^2 \le{}& \vartheta_n\|\varphi(x_n)-\varphi(\tilde{x})\|^2+(1-\vartheta_n)\|u_n-\varphi(\tilde{x})\|^2\\ \le{}& \vartheta_n\|\varphi(x_n)-\varphi(\tilde{x})\|^2+(1-\vartheta_n)\alpha_n\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2\\&+(1-\alpha_n)(1-\vartheta_n)\|\varphi(x_n)-\varphi(\tilde{x})\|^2-(1-\vartheta_n)\|\varphi(x_n)-u_n\|^2\\&+2\varsigma_n(1-\vartheta_n)\alpha_n\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-v_n\|\\&+2\varsigma_n(1-\vartheta_n)\|\varphi(x_n)-u_n\|\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|\\&+2(1-\vartheta_n)\alpha_n\|\varphi(x_n)-u_n\|\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-v_n\|\\ \le{}& \|\varphi(x_n)-\varphi(\tilde{x})\|^2+\alpha_n\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2\\&+2\varsigma_n\alpha_n\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-v_n\|\\&+2\varsigma_n\|\varphi(x_n)-u_n\|\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|-(1-\vartheta_n)\|\varphi(x_n)-u_n\|^2\\&+2\alpha_n\|\varphi(x_n)-u_n\|\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-v_n\|.\end{aligned}$$

Then,

$$\begin{aligned}(1-\vartheta_n)\|\varphi(x_n)-u_n\|^2 \le{}& (\|\varphi(x_n)-\varphi(\tilde{x})\|+\|\varphi(x_{n+1})-\varphi(\tilde{x})\|)\|\varphi(x_{n+1})-\varphi(x_n)\|\\&+\alpha_n\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2\\&+2\varsigma_n\alpha_n\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-v_n\|\\&+2\varsigma_n\|\varphi(x_n)-u_n\|\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|\\&+2\alpha_n\|\varphi(x_n)-u_n\|\|\nu\phi(x_n)-\varphi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-v_n\|.\end{aligned}$$

According to (iii), Equations (26) and (28), we easily deduce

$$\lim_{n\to\infty}\|\varphi(x_n)-u_n\|=0. \tag{31}$$

In view of Equations (15) and (16), we get

$$\begin{aligned}&\sigma_n(\delta_n-\sigma_n)\|T((1-\delta_n)u_n+\delta_nTu_n)-u_n\|^2+\zeta_n(\eta_n-\zeta_n)\|S((1-\eta_n)y_n+\eta_nSy_n)-y_n\|^2\\&\le\|u_n-\varphi(\tilde{x})\|^2-\|z_n-\varphi(\tilde{x})\|^2\\&\le\|u_n-z_n\|(\|u_n-\varphi(\tilde{x})\|+\|z_n-\varphi(\tilde{x})\|).\end{aligned} \tag{32}$$

It follows from Equations (25), (31), and (32) that

$$\lim_{n\to\infty}\|T((1-\delta_n)u_n+\delta_nTu_n)-u_n\|=0, \tag{33}$$

and

$$\lim_{n\to\infty}\|S((1-\eta_n)y_n+\eta_nSy_n)-y_n\|=0. \tag{34}$$

Note that $z_n-y_n=\delta_n[S((1-\eta_n)y_n+\eta_nSy_n)-y_n]$. Therefore,

$$\lim_{n\to\infty}\|z_n-y_n\|=0. \tag{35}$$

Next, we prove $\limsup_{n\to\infty}\langle \nu\phi(\tilde{x}) - \varphi(\tilde{x}), u_n - \varphi(\tilde{x})\rangle \leq 0$. Let $\{u_{n_i}\}$ be a subsequence of $\{u_n\}$ such that

$$\begin{aligned}\limsup_{n\to\infty}\langle \nu\phi(\tilde{x}) - \varphi(\tilde{x}), u_n - \varphi(\tilde{x})\rangle &= \lim_{i\to\infty}\langle \nu\phi(\tilde{x}) - \varphi(\tilde{x}), u_{n_i} - \varphi(\tilde{x})\rangle \\ &= \lim_{i\to\infty}\langle \nu\phi(\tilde{x}) - \varphi(\tilde{x}), \varphi(x_{n_i}) - \varphi(\tilde{x})\rangle.\end{aligned} \tag{36}$$

Note that $\{x_{n_i}\}$ is bounded. We can choose a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \rightharpoonup z \in \mathcal{C}$ weakly. Assume $x_{n_i} \rightharpoonup z$ without loss of generality. This indicates that $\varphi(x_{n_i}) \rightharpoonup \varphi(z)$ due to the weak continuity of φ. Thus, $u_{n_i} \rightharpoonup \varphi(z)$ and $y_{n_i} \rightharpoonup \varphi(z)$.

Apply Lemmas 2 and 6 to Equations (33) and (34) to deduce $\varphi(z) \in \mathrm{Fix}(T)$ and $\varphi(z) \in \mathrm{Fix}(S)$, respectively. That is, $\varphi(z) \in \mathrm{Fix}(T) \cap \mathrm{Fix}(S)$. Next, we show $z \in GVI(\mathcal{A}, \varphi, \mathcal{C})$. Let

$$Rv = \begin{cases} \mathcal{A}v + N_{\mathcal{C}}(v), & v \in \mathcal{C}, \\ \emptyset, & v \notin \mathcal{C}. \end{cases}$$

According to Reference [32], we can deduce that R is maximal φ-monotone. Let $(v, w) \in G(R)$. Since $w - \mathcal{A}v \in N_{\mathcal{C}}(v)$ and $x_n \in \mathcal{C}$, we have $\langle \varphi(v) - \varphi(x_n), w - \mathcal{A}v\rangle \geq 0$. Noting that $u_n = proj_{\mathcal{C}}[\alpha_n \nu\phi(x_n) + (1-\alpha_n)(\varphi(x_n) - \varsigma_n)]$, we get

$$\langle \varphi(v) - u_n, u_n - [\alpha_n\nu\phi(x_n) + (1-\alpha_n)(\varphi(x_n) - \varsigma_n\mathcal{A}x_n)]\rangle \geq 0.$$

It follows that

$$\langle \varphi(v) - u_n, \frac{u_n - \varphi(x_n)}{\varsigma_n} + \mathcal{A}x_n - \frac{\alpha_n}{\varsigma_n}(\nu\phi(x_n) - \varphi(x_n) + \varsigma_n\mathcal{A}x_n)\rangle \geq 0.$$

Thus,

$$\begin{aligned}\langle \varphi(v) - \varphi(x_{n_i}), w\rangle &\geq \langle \varphi(v) - \varphi(x_{n_i}), \mathcal{A}v\rangle \\ &\geq \langle \varphi(v) - \varphi(x_{n_i}), \mathcal{A}v\rangle - \langle \varphi(v) - u_{n_i}, \frac{u_{n_i} - \varphi(x_{n_i})}{\varsigma_{n_i}}\rangle \\ &\quad - \langle \varphi(v) - u_{n_i}, \mathcal{A}x_{n_i}\rangle + \frac{\alpha_{n_i}}{\varsigma_{n_i}}\langle \varphi(v) - u_{n_i}, \nu\phi(x_{n_i}) - \varphi(x_{n_i}) + \varsigma_{n_i}\mathcal{A}x_{n_i}\rangle \\ &= \langle \varphi(v) - \varphi(x_{n_i}), \mathcal{A}v - \mathcal{A}x_{n_i}\rangle + \langle \varphi(v) - \varphi(x_{n_i}), \mathcal{A}x_{n_i}\rangle \\ &\quad - \langle \varphi(v) - u_{n_i}, \frac{u_{n_i} - \varphi(x_{n_i})}{\varsigma_{n_i}}\rangle - \langle \varphi(v) - u_{n_i}, \mathcal{A}x_{n_i}\rangle \\ &\quad + \frac{\alpha_{n_i}}{\varsigma_{n_i}}\langle \varphi(v) - u_{n_i}, \nu\phi(x_{n_i}) - \varphi(x_{n_i}) + \varsigma_{n_i}\mathcal{A}x_{n_i}\rangle \\ &\geq -\langle \varphi(v) - u_{n_i}, \frac{u_{n_i} - \varphi(x_{n_i})}{\varsigma_{n_i}}\rangle - \langle \varphi(x_{n_i}) - u_{n_i}, \mathcal{A}x_{n_i}\rangle \\ &\quad + \frac{\alpha_{n_i}}{\varsigma_{n_i}}\langle \varphi(v) - u_{n_i}, \nu\phi(x_{n_i}) - \varphi(x_{n_i}) + \varsigma_{n_i}\mathcal{A}x_{n_i}\rangle.\end{aligned} \tag{37}$$

By virtue of Equation (37), we derive that $\langle \varphi(v) - \varphi(z), w\rangle \geq 0$ due to $\|\varphi(x_{n_i}) - u_{n_i}\| \to 0$ and $\varphi(x_{n_i}) \rightharpoonup \varphi(z)$. By the maximal φ-monotonicity of R, $z \in R^{-1}0$. So, $z \in GVI(\mathcal{A}, \varphi, \mathcal{C})$. Therefore, $z \in \Omega$.

From Equation (36), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle \nu\phi(\tilde{x}) - \varphi(\tilde{x}), u_n - \varphi(\tilde{x})\rangle &= \lim_{i\to\infty}\langle \nu\phi(\tilde{x}) - \varphi(\tilde{x}), \varphi(x_{n_i}) - \varphi(\tilde{x})\rangle \\ &= \langle \nu\phi(\tilde{x}) - \varphi(\tilde{x}), \varphi(z) - \varphi(\tilde{x})\rangle \leq 0.\end{aligned} \tag{38}$$

Applying Equation (6), we obtain

$$\begin{aligned}\|u_n-\varphi(\tilde{x})\|^2 &= \|\mathrm{proj}_C[\alpha_n\nu\phi(x_n)+(1-\alpha_n)(\varphi(x_n)-\varsigma_n\mathcal{A}x_n)]-\mathrm{proj}_C[\varphi(\tilde{x})-(1-\alpha_n)\varsigma_n\mathcal{A}\tilde{x}]\|^2\\
&\le \langle\alpha_n(\nu\phi(x_n)-\varphi(\tilde{x}))+(1-\alpha_n)z_n,u_n-\varphi(\tilde{x})\rangle\\
&\le \alpha_n\nu\langle\phi(x_n)-\phi(\tilde{x}),u_n-\varphi(\tilde{x})\rangle+\alpha_n\langle\nu\phi(\tilde{x})-\varphi(\tilde{x}),u_n-\varphi(\tilde{x})\rangle\\
&\quad+(1-\alpha_n)\|\varphi(x_n)-\varsigma_n\mathcal{A}x_n-(\varphi(\tilde{x})-\varsigma_n\mathcal{A}\tilde{x})\|\|u_n-\varphi(\tilde{x})\|\\
&\le \alpha_nL\nu\|x_n-\tilde{x}\|\|u_n-\varphi(\tilde{x})\|+\alpha_n\langle\nu\phi(\tilde{x})-\varphi(\tilde{x}),u_n-\varphi(\tilde{x})\rangle\\
&\quad+(1-\alpha_n)\|\varphi(x_n)-\varphi(\tilde{x})\|\|u_n-\varphi(\tilde{x})\|\\
&\le \alpha_n(\nu L/\delta)\|\varphi(x_n)-\varphi(\tilde{x})\|\|u_n-\varphi(\tilde{x})\|+\alpha_n\langle\nu\phi(\tilde{x})-\varphi(\tilde{x}),u_n-\varphi(\tilde{x})\rangle\\
&\quad+(1-\alpha_n)\|\varphi(x_n)-\varphi(\tilde{x})\|\|u_n-\varphi(\tilde{x})\|\\
&= [1-(1-L\nu/\delta)\alpha_n]\|\varphi(x_n)-\varphi(\tilde{x})\|\|u_n-\varphi(\tilde{x})\|\\
&\quad+\alpha_n\langle\nu\phi(\tilde{x})-\varphi(\tilde{x}),u_n-\varphi(\tilde{x})\rangle\\
&\le \frac{1-(1-L\nu/\delta)\alpha_n}{2}\|\varphi(x_n)-\varphi(\tilde{x})\|^2+\frac{1}{2}\|u_n-\varphi(\tilde{x})\|^2\\
&\quad+\alpha_n\langle\nu\phi(\tilde{x})-\varphi(\tilde{x}),u_n-\varphi(\tilde{x})\rangle.\end{aligned}$$

It follows that

$$\|u_n-\varphi(\tilde{x})\|^2\le[1-(1-L\nu/\delta)\alpha_n]\|\varphi(x_n)-\varphi(\tilde{x})\|^2+2\alpha_n\langle\nu\phi(\tilde{x})-\varphi(\tilde{x}),u_n-\varphi(\tilde{x})\rangle.$$

Therefore,

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi(\tilde{x})\|^2 &\le \vartheta_n\|\varphi(x_n)-\varphi(\tilde{x})\|^2+(1-\vartheta_n)\|u_n-\varphi(\tilde{x})\|^2\\
&\le \vartheta_n\|\varphi(x_n)-\varphi(\tilde{x})\|^2+(1-\vartheta_n)[1-(1-\nu L/\delta)\alpha_n]\|\varphi(x_n)-\varphi(\tilde{x})\|^2\\
&\quad+2(1-\vartheta_n)\alpha_n\langle\nu\phi(\tilde{x})-\varphi(\tilde{x}),u_n-\varphi(\tilde{x})\rangle\\
&= [1-(1-\nu L/\delta)(1-\vartheta_n)\alpha_n]\|\varphi(x_n)-\varphi(\tilde{x})\|^2\\
&\quad+2(1-\vartheta_n)\alpha_n\langle\nu\phi(\tilde{x})-\varphi(\tilde{x}),u_n-\varphi(\tilde{x})\rangle\\
&= [1-(1-\nu L/\delta)(1-\vartheta_n)\alpha_n]\|\varphi(x_n)-\varphi(\tilde{x})\|^2\\
&\quad+(1-\nu L/\delta)(1-\vartheta_n)\alpha_n\Big(\frac{2}{1-\nu L/\delta}\langle\nu\phi(\tilde{x})-\varphi(\tilde{x}),u_n-\varphi(\tilde{x})\rangle\Big).\end{aligned}\tag{39}$$

We can therefore apply Lemma 4 to Equation (39) to conclude that $\varphi(x_n)\to\varphi(\tilde{x})$ and $x_n\to\tilde{x}$.

CASE 2. There exists n_0 such that $\|\varphi(x_{n_0})-\varphi(\tilde{x})\|\le\|\varphi(x_{n_0+1})-\varphi(\tilde{x})\|$. At this case, we set $\omega_n=\{\|\varphi(x_n)-\varphi(\tilde{x})\|\}$. Then, we have $\omega_{n_0}\le\omega_{n_0+1}$. For $n\ge n_0$, we define a sequence $\{\tau_n\}$ by

$$\tau(n)=\max\{l\in\mathbb{N}|n_0\le l\le n,\omega_l\le\omega_{l+1}\}.$$

We can show easily that $\tau(n)$ is a non-decreasing sequence such that

$$\lim_{n\to\infty}\tau(n)=\infty$$

and

$$\omega_{\tau(n)}\le\omega_{\tau(n)+1}.$$

According to techniques similar to Equations (36) and (39), we obtain

$$\limsup_{n\to\infty}\langle\nu\phi(\tilde{x})-\varphi(\tilde{x}),u_{\tau(n)}-\varphi(\tilde{x})\rangle\le 0\tag{40}$$

and

$$\omega_{\tau(n)+1}^2 \le [1-(1-\nu L/\delta)(1-\vartheta_{\tau(n)})\alpha_{\tau(n)}]\omega_{\tau(n)}^2 + (1-\nu L/\delta)(1-\vartheta_{\tau(n)})\alpha_{\tau(n)}\Big(\frac{2}{1-\nu L/\delta}\langle \nu\phi(\tilde{x})-\varphi(\tilde{x}), u_{\tau(n)}-\varphi(\tilde{x})\rangle\Big). \tag{41}$$

Since $\omega_{\tau(n)} \le \omega_{\tau(n)+1}$, we have from Equation (41) that

$$\omega_{\tau(n)}^2 \le \frac{2}{1-\nu L/\delta}\langle \nu\phi(\tilde{x})-\varphi(\tilde{x}), u_{\tau(n)}-\varphi(\tilde{x})\rangle. \tag{42}$$

Combining Equations (41) with (42), we have

$$\limsup_{n\to\infty} \omega_{\tau(n)} \le 0,$$

and thus

$$\lim_{n\to\infty} \omega_{\tau(n)} = 0. \tag{43}$$

By Equations (40) and (41), we also get

$$\limsup_{n\to\infty} \omega_{\tau(n)+1} \le \limsup_{n\to\infty} \omega_{\tau(n)}.$$

The last inequality together with Equation (43) imply that

$$\lim_{n\to\infty} \omega_{\tau(n)+1} = 0.$$

Applying Lemma 5 to get

$$0 \le \omega_n \le \max\{\omega_{\tau(n)}, \omega_{\tau(n)+1}\}.$$

Therefore, $\omega_n \to 0$, i.e., $x_n \to \tilde{x}$. The proof is completed. □

For initial guess $x_0 \in \mathcal{C}$, define a sequence $\{x_n\}$ by the following form

$$\begin{cases} u_n = \mathrm{proj}_{\mathcal{C}}[\alpha_n\nu\phi(x_n)+(1-\alpha_n)(\varphi(x_n)-\varsigma_n\mathcal{A}x_n)], \\ y_n = (1-\sigma_n)u_n + \sigma_n T((1-\delta_n)u_n+\delta_n Tu_n), \\ z_n = (1-\zeta_n)y_n + \zeta_n S((1-\eta_n)y_n+\eta_n Sy_n), \\ x_{n+1} = \vartheta_n x_n + (1-\vartheta_n)z_n, \ n \ge 0, \end{cases} \tag{44}$$

where $\nu > 0$ is a constant, $\{\alpha_n\}$, $\{\sigma_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$, and $\{\vartheta_n\}$ are six sequences in $(0,1)$ and $\{\varsigma_n\}$ is a sequence in $(0,\infty)$.

Corollary 1. *Suppose $I-S$ and $I-T$ are demiclosed at 0. Assume the following restrictions are satisfied:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$;
(ii) $0 < a_1 < \sigma_n < c_1 < \delta_n < b_1 < \frac{1}{\sqrt{1+L_2{}^2}+1}$ *and* $0 < a_2 < \zeta_n < c_2 < \eta_n < b_2 < \frac{1}{\sqrt{1+L_1^2}+1}$;
(iii) $0 < \liminf_{n\to\infty}\vartheta_n \le \limsup_{n\to\infty}\vartheta_n < 1$;
(iv) $\delta \in (\nu, 2\alpha)$ *and* $0 < \liminf_{n\to\infty}\varsigma_n \le \limsup_{n\to\infty}\varsigma_n < 2\alpha$.

Then, the sequence $\{x_n\}$ defined by Equation (44) strongly converges to $\tilde{x} \in VI(\mathcal{A},\mathcal{C}) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$ which solves the following variational inequality

$$\langle \nu\phi(\tilde{x})-\tilde{x}, x^\dagger-\tilde{x}\rangle \le 0, \quad \forall x^\dagger \in VI(\mathcal{A},\mathcal{C}) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T).$$

4. Examples and Applications

In this section, we provide some examples and applications of our suggested algorithms and theorems.

Let $\mathcal{H}$ be a real Hilbert space. Let $\emptyset \neq \mathcal{C} \subset \mathcal{H}$ be a closed convex set. Let $\phi\colon \mathcal{C} \to \mathcal{H}$ be an L-Lipschitz operator. Let $\varphi\colon \mathcal{C} \to \mathcal{C}$ be a δ-strongly monotone and weakly continuous operator such that its rang $R(\varphi) = \mathcal{C}$. Let the operator $\mathcal{A}\colon \mathcal{C} \to \mathcal{H}$ be α-inverse strongly φ-monotone. Let $T\colon \mathcal{C} \to \mathcal{C}$ be an L_3-Lipschitzian pseudocontractive operator with $L_3 > 1$. Set $\Omega^\dagger = GVI(\mathcal{A}, \varphi, \mathcal{C}) \bigcap \varphi^{-1}(\mathrm{Fix}(T)) \neq \emptyset$.

For the initial guess $x_0 \in \mathcal{C}$, define the sequence $\{x_n\}$ by the following form

$$\begin{cases} u_n = \mathrm{proj}_{\mathcal{C}}[\alpha_n \nu \phi(x_n) + (1-\alpha_n)(\varphi(x_n) - \varsigma_n \mathcal{A} x_n)], \\ y_n = (1-\sigma_n)u_n + \sigma_n T((1-\delta_n)u_n + \delta_n T u_n), \\ z_n = (1-\zeta_n)y_n + \zeta_n T((1-\eta_n)y_n + \eta_n T y_n), \\ \varphi(x_{n+1}) = \vartheta_n \varphi(x_n) + (1-\vartheta_n) z_n, \ n \geq 0, \end{cases} \tag{45}$$

where $\nu > 0$ is a constant, $\{\alpha_n\}$, $\{\sigma_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$, and $\{\vartheta_n\}$ are six sequences in $(0,1)$ and $\{\varsigma_n\}$ is a sequence in $(0,\infty)$.

Lemma 6 ([40]). *Let $\mathcal{H}$ be a real Hilbert space, $\mathcal{C}$ a closed convex subset of $\mathcal{H}$. Let $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be a continuous pseudocontractive operator. Then, $\mathcal{I} - \mathcal{T}$ is demi-closed at zero.*

Theorem 2. *Assume the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$;
(ii) $0 < a_1 < \sigma_n < c_1 < \delta_n < b_1 < \frac{1}{\sqrt{1+L_3{}^2}+1}$ *and* $0 < a_2 < \zeta_n < c_2 < \eta_n < b_2 < \frac{1}{\sqrt{1+L_3^2}+1}$;
(iii) $0 < \liminf_{n\to\infty} \vartheta_n \leq \limsup_{n\to\infty} \vartheta_n < 1$;
(iv) $L\nu < \delta < 2\alpha$ *and* $0 < \liminf_{n\to\infty} \varsigma_n \leq \limsup_{n\to\infty} \varsigma_n < 2\alpha$.

Then, the iterative sequence $\{x_n\}$ defined by Equation (45) strongly converges to $\tilde{x} \in \Omega^\dagger$ which solves the variational inequality

$$\langle \nu\phi(\tilde{x}) - \varphi(\tilde{x}), \varphi(x^\dagger) - \varphi(\tilde{x}) \rangle \leq 0, \quad \forall x^\dagger \in \Omega^\dagger.$$

Remark 1. *Algorithm (45) and Theorem 2 include the corresponding algorithm and theorem in Reference [18] as special cases, respectively.*

Let $S\colon \mathcal{C} \to \mathcal{C}$ be an L_1-Lipschitzian monotone operator with $L_1 > 1$ and $T\colon \mathcal{C} \to \mathcal{C}$ be an L_2-Lipschitzian monotone operator with $L_2 > 1$. Set $\Omega^\ddagger = GVI(\mathcal{A}, \varphi, \mathcal{C}) \bigcap \varphi^{-1}(S^{-1}(0) \cap T^{-1}(0)) \neq \emptyset$.

For initial guess $x_0 \in \mathcal{C}$, define the sequence $\{x_n\}$ by the following form

$$\begin{cases} u_n = \mathrm{proj}_{\mathcal{C}}[\alpha_n \nu \phi(x_n) + (1-\alpha_n)(\varphi(x_n) - \varsigma_n \mathcal{A} x_n)], \\ y_n = (1-\sigma_n)u_n + \sigma_n (I-T)[(1-\delta_n)u_n + \delta_n (I-T) u_n], \\ z_n = (1-\zeta_n)y_n + \zeta_n (I-S)[(1-\eta_n)y_n + \eta_n (I-S) y_n], \\ \varphi(x_{n+1}) = \vartheta_n \varphi(x_n) + (1-\vartheta_n) z_n, \ n \geq 0, \end{cases} \tag{46}$$

where $\nu > 0$ is a constant, $\{\alpha_n\}$, $\{\sigma_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$, and $\{\vartheta_n\}$ are six sequences in $(0,1)$ and $\{\varsigma_n\}$ is a sequence in $(0,\infty)$.

Theorem 3. *Assume the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$;
(ii) $0 < a_1 < \sigma_n < c_1 < \delta_n < b_1 < \frac{1}{\sqrt{1+L_2{}^2}+1}$ *and* $0 < a_2 < \zeta_n < c_2 < \eta_n < b_2 < \frac{1}{\sqrt{1+L_1^2}+1}$;
(iii) $0 < \liminf_{n\to\infty} \vartheta_n \leq \limsup_{n\to\infty} \vartheta_n < 1$;
(iv) $L\nu < \delta < 2\alpha$ *and* $0 < \liminf_{n\to\infty} \varsigma_n \leq \limsup_{n\to\infty} \varsigma_n < 2\alpha$.

Then, the iterative sequence $\{x_n\}$ *defined by Equation* (46) *strongly converges to* $\tilde{x} \in \Omega$ *which solves the variational inequality*

$$\langle \nu\phi(\tilde{x}) - \varphi(\tilde{x}), \varphi(x^\dagger) - \varphi(\tilde{x}) \rangle \leq 0, \quad \forall x^\dagger \in \Omega^\ddagger.$$

5. Conclusions

In this paper, we investigated a generalized variational inequality and fixed points problems. We presented an iterative algorithm for finding a solution of the generalized variational inequality and fixed point of two quasi-pseudocontractive operators under a nonlinear transformation. We demonstrated the strong convergence of the suggested algorithm under some mild conditions, noting that in our suggested iterative sequence (Equation (7)), the involved operator $\mathcal{A}$ requires some form of strong monotonicity. A natural question arises: how to weaken this assumption?

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