## Article

# A Note on Minimal Translation Graphs in Euclidean Space 

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#### Abstract

In this note, we give a characterization of a class of minimal translation graphs generated by planar curves. Precisely, we prove that a hypersurface that can be written as the sum of $n$ planar curves is either a hyperplane or a cylinder on the generalized Scherk surface. This result can be considered as a generalization of the results on minimal translation hypersurfaces due to Dillen-Verstraelen-Zafindratafa in 1991 and minimal translation surfaces due to Liu-Yu in 2013.


Keywords: translation hypersurfaces; minimal hypersurfaces; Scherk surface

## 1. Introduction

The study of minimal hypersurfaces has a very long history. Many interesting and important results on minimal hypersurfaces in various ambient spaces have appeared in the past several centuries. From the view of differential geometry, one of the most interesting problems concerning the study of minimal hypersurfaces is to construct concrete examples of minimal hypersurfaces.

A hypersurface $M^{n} \subset \mathbb{R}^{n+1}$ is called a translation hypersurface if $M^{n}$ is a graph of a function $F\left(x_{1}, \cdots, x_{n}\right)=f_{1}\left(x_{1}\right)+\cdots+f_{n}\left(x_{n}\right)$, where each $f_{i}$ is a smooth function of one real variable for $i=1,2, \cdots, n$.

Dillen, Verstraelen, and Zafindratafa [1] in 1991 proved the following interesting result.
Theorem 1. Let $M^{n}$ be a minimal translation hypersurface in $\mathbb{R}^{n+1}$. Then, $M^{n}$ is either a hyperplane or $M^{n}=\sum \times \mathbb{R}^{n-2}$, where $\sum$ is Scherk's minimal translation surface in $\mathbb{R}^{3}$.

Scherk's minimal translation surface takes the following parameterization:

$$
F\left(x_{1}, x_{2}\right)=\frac{1}{a} \ln \left|\frac{\cos \left(a x_{1}\right)}{\cos \left(a x_{2}\right)}\right|,
$$

where $a$ is a nonzero constant. We recall that Scherk [2] in 1835 proved that, besides the plane, the only minimal translation surface is Scherk's surface.

Chen et al. [3] generalized Dillen et al.'s result to translation hypersurfaces with constant mean curvature, and Seo [4] studied the translation hypersurfaces with constant Gauss-Kronecker curvature in Euclidean space $\mathbb{R}^{n+1}$. Moreover, Lima-Santos-Sousa [5] obtained the complete classification of translation hypersurfaces of $\mathbb{R}^{n+1}$ with constant scalar curvature and discovered new examples of translation hypersurfaces.

Recently, Liu and Yu in [6] introduced a class of translation surfaces given by a graph of a function as follows:

$$
F\left(x_{1}, x_{2}\right)=f\left(x_{1}\right)+g\left(a x_{1}+x_{2}\right)
$$

for some nonzero constant $a$ and differentiable functions $f, g$. Such a surfaces is called an affine translation surface. Note that in the case $a=0$, the affine translation surfaces become the classical translation surfaces. In fact, this new class of translation surfaces is also generated by planar curves. The readers may refer to [7]. Moreover, Liu and Yu proved the following theorem.

Theorem 2. Besides the plane, the only minimal affine translation surface in Euclidean three-space $\mathbb{E}^{3}$ is the surface given by:

$$
\begin{equation*}
F\left(x_{1}, x_{2}\right)=\frac{1}{c} \ln \left|\frac{\cos \left(c \sqrt{1+a^{2}} x_{1}\right)}{\cos \left[c\left(a x_{1}+x_{2}\right)\right]}\right| \tag{1}
\end{equation*}
$$

where $a, c$ are constants and ac $\neq 0$.
Since this surface (1) is similar to the classical Scherk surface, we call (1) a generalized Scherk surface or an affine Scherk surface (see [8,9]).

A translation surface $M \in \mathbb{R}^{3}$ could be expressed as a general form $\phi(s, t)=\alpha(s)+\beta(t)$, where $\alpha$ : $I \subset \mathbb{R} \rightarrow \mathbb{R}^{3}$ and $\beta: J \subset \mathbb{R} \rightarrow \mathbb{R}^{3}$ are two regular curves with $\alpha^{\prime}(s) \times \beta^{\prime}(t) \neq 0$, which are called generators of $M$. In 1998, Dillen et al. [7] proved that if $M$ is minimal and $\alpha$ is a planar curve, then $\beta$ must be a planar curve; furthermore, $M$ is a plane or a generalized Scherk surface. In the case that minimal translation surfaces whose generators are both non-planar curves, very recently, López and Perdomo [10] completely solved the problem of characterizing all minimal translation surfaces of $\mathbb{R}^{3}$ in terms of the curvature and torsion of the generating curves.

In the two-dimensional situation, the study of geometric quantities on translation surfaces has a rich literature. Liu [11] provided a complete classification of translation surfaces with constant mean curvature in Euclidean three-space and the Lorentz-Minkowski three-space. Inoguchi, Lopez, and Munteanu [12,13] made important contributions to minimal translation surfaces in ambient spaces $\mathrm{Nil}_{3}$ and $\mathrm{Sol}_{3}$. For more recent results and progress on translation surfaces or translation hypersurfaces, see [14-19].

Motivated by the works mentioned above, in this paper, we consider a class of translation hypersurfaces in a Euclidean space $\mathbb{R}^{n+1}$ defined as follows.

Definition 1. We say that a hypersurface $M^{n}$ of the Euclidean space $\mathbb{R}^{n+1}$ is a translation graph if it is the graph of a function given by:

$$
F\left(x_{1}, \cdots, x_{n}\right)=f_{1}\left(x_{1}\right)+\cdots+f_{n-1}\left(x_{n-1}\right)+f_{n}(u)
$$

where $u=\sum_{i=1}^{n} c_{i} x_{i}, c_{i}$ are constants, $c_{n} \neq 0$, and each $f_{i}$ is a smooth function of one real variable for $i=1,2, \cdots, n$.

We remark that this definition of a translation graph is an extension of the definitions due to Dillen et al. [1] for a hypersurface and Liu-Yu [6] for a surface in a Euclidean space.

In this note, we study the minimality of translation graphs and give a characterization of such hypersurfaces in $\mathbb{R}^{n+1}$. Precisely, we obtain the following theorem.

Theorem 3. Let $M^{n}$ be a minimal translation graph in $\mathbb{R}^{n+1}$. Then, $M^{n}$ is either a hyperplane or $M^{n}=$ $\sum \times \mathbb{R}^{n-2}$, where $\sum$ is a generalized Scherk's minimal translation surface in $\mathbb{R}^{3}$.

Remark 1. It is easy to see that Theorem 3 is a natural generalization of Dillen et al. and Liu-Yu's results in Theorems 1 and 2, respectively.

Analogous problems were considered in [20,21]. In addition, Saglam, Soytürk, and Sabuncuoǧlu gave a connection of minimal translation surfaces with geodesic planes, and Hasanis and Lóez gave
the classification of minimal translation surfaces in Euclidean space. The present paper is close to the research of Belova, Mikeš, and Strambach [22-25], where similar methods for investigations of geodesics and almost geodesics were used.

## 2. The Proof of Theorem 3

We first recall some basic background material for a graph of a hypersurface in a Euclidean space. Let $M^{n}$ be a hypersurface immersed in $\mathbb{R}^{n+1}$ given by:

$$
L\left(x_{1}, \ldots, x_{n}\right)=\left(x_{1}, \ldots, x_{n}, F\left(x_{1}, \ldots, x_{n}\right)\right)
$$

Denote the partial derivatives $\frac{\partial F}{\partial x_{i}}, \frac{\partial^{2} F}{\partial x_{i} x_{j}}, \ldots$, etc., by $F_{i}, F_{i j}, \ldots$, etc. Put $W=\sqrt{1+\sum_{i=1}^{n} F_{i}^{2}}$. It is easy to check that the unit normal $\xi$ is given by:

$$
\xi=\frac{1}{W}\left(-F_{1}, \ldots,-F_{n}, 1\right)
$$

and the coefficient $g_{i j}=g\left(\frac{\partial}{\partial x_{i}}, \frac{\partial}{\partial x_{j}}\right)$ of the metric tensor is given by:

$$
g_{i j}=\delta_{i j}+F_{i} F_{j},
$$

where $\delta_{i j}$ is the Kronecker symbol. Moreover, the matrix of the second fundamental form $h$ is given by relation:

$$
h_{i j}=\frac{F_{i j}}{W} .
$$

Hence, the components of the matrix of the shape operator $A$ are:

$$
a_{i}^{j}=\sum_{k} h_{i k} g^{k j}=\frac{1}{W}\left(F_{i j}-\sum_{k} \frac{F_{i k} F_{k} F_{j}}{W^{2}}\right) .
$$

Then, the mean curvature $H$ is given by:

$$
\begin{equation*}
H=\frac{1}{n W}\left(\sum_{i} F_{i i}-\frac{1}{W^{2}} \sum_{i, j} F_{i} F_{j} F_{i j}\right) \tag{2}
\end{equation*}
$$

From (2), the following result could be deduced directly.
Proposition 1. Let $M^{n}$ be a graph immersed in $\mathbb{R}^{n+1}$ given by:

$$
L\left(x_{1}, \ldots, x_{n}\right)=\left(x_{1}, \ldots, x_{n}, F\left(x_{1}, \ldots, x_{n}\right)\right)
$$

Then, the graph $M^{n}$ is minimal if and only if:

$$
\begin{equation*}
\sum_{i} F_{i i}+\sum_{\substack{i, j=1 \\ i \neq j}}^{n}\left(F_{i}^{2} F_{j j}-F_{i} F_{j} F_{i j}\right)=0 \tag{3}
\end{equation*}
$$

We remark that the above result is necessary and crucial for the study of a minimal hypersurface as a graph in a Euclidean space, especially for a translation hypersurface. Hence, in the following part, we use the basic characterization for the minimal graph's Equation (3) in Proposition 1 to give a detailed proof of Theorem 3.

Let $M^{n}$ be a translation hypersurface immersed in $\mathbb{R}^{n+1}$ given by:

$$
L\left(x_{1}, \ldots, x_{n}\right)=\left(x_{1}, \ldots, x_{n}, F\left(x_{1}, \ldots, x_{n}\right)\right)
$$

with:

$$
F\left(x_{1}, \ldots, x_{n}\right)=f_{1}\left(x_{1}\right)+\cdots+f_{n-1}\left(x_{n-1}\right)+f_{n}(u)
$$

where $u=\sum_{i=1}^{n} c_{i} x_{i}, c_{i}$ are constants with $c_{n} \neq 0$, and each $f_{i}$ is a smooth function for $i=1, \cdots, n-1$. It is easy to check that:

$$
\begin{array}{lc}
F_{i}=f_{i}^{\prime}+c_{i} f_{n}^{\prime}, & F_{n}=c_{n} f_{n}^{\prime} \\
F_{i i}=f_{i}^{\prime \prime}+c_{i}^{2} f_{n}^{\prime \prime}, & F_{n n}=c_{n}^{2} f_{n}^{\prime \prime} \\
F_{i j}=c_{i} c_{j} f_{n}^{\prime \prime}, & F_{i n}=c_{i} c_{n} f_{n}^{\prime \prime} \tag{6}
\end{array}
$$

for $1 \leq i \leq n-1$. Since $M^{n}$ is minimal, by Proposition 1, we substitute (4)-(6) into (3) and have:

$$
\begin{align*}
\sum_{i=1}^{n-1} f_{i}^{\prime \prime} & +\left(\sum_{i=1}^{n} c_{i}^{2}+c_{n}^{2} \sum_{i=1}^{n-1} f_{i}^{\prime 2}\right) f_{n}^{\prime \prime}+c_{n}^{2}\left(\sum_{i=1}^{n-1} f_{i}^{\prime \prime}\right) f_{n}^{\prime 2} \\
& +\sum_{\substack{i, j=1 \\
i \neq j}}^{n-1}\left(f_{i}^{\prime}+c_{i} f_{n}^{\prime}\right)^{2} f_{j}^{\prime \prime}+\frac{1}{2} \sum_{\substack{i, j=1 \\
i \neq j}}^{n-1}\left(c_{i} f_{j}^{\prime}-c_{j} f_{i}^{\prime}\right)^{2} f_{n}^{\prime \prime}=0 . \tag{7}
\end{align*}
$$

Since $c_{n} \neq 0$, differentiating (7) with respect to $x_{n}$ gives:

$$
\begin{align*}
& {\left[\sum_{i=1}^{n} c_{i}^{2}+c_{n}^{2} \sum_{i=1}^{n-1} f_{i}^{\prime 2}+\frac{1}{2} \sum_{\substack{i, j=1 \\
i \neq j}}^{n-1}\left(c_{i} f_{j}^{\prime}-c_{j} f_{i}^{\prime}\right)^{2}\right] f_{n}^{\prime \prime \prime}} \\
& \quad+2\left[\sum_{\substack{i=1}}^{n-1}\left(\sum_{\substack{j=1 \\
j \neq i}}^{n} c_{j}^{2}\right) f_{i}^{\prime \prime}\right] f_{n}^{\prime} f_{n}^{\prime \prime}+2 \sum_{\substack{i, j=1 \\
i \neq j}}^{n-1} c_{i} f_{i}^{\prime} f_{j}^{\prime \prime} f_{n}^{\prime \prime}=0 \tag{8}
\end{align*}
$$

Based on (8), we divide into the following two cases for further discussions.
Case A. $f_{n}^{\prime \prime \prime}=0$. In this case, after suitable translation, there exists a constant $m$ such that $f_{n}=m u^{2}$. Hence, (8) becomes:

$$
\begin{equation*}
2 m^{2} u \sum_{i=1}^{n-1}\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} c_{j}^{2}\right) f_{i}^{\prime \prime}+m \sum_{\substack{i, j=1 \\ i \neq j}}^{n-1} c_{i} f_{i}^{\prime} f_{j}^{\prime \prime}=0 \tag{9}
\end{equation*}
$$

If $m=0$, then $f_{n}=0$, and the hypersurface reduces to a classical translation hypersurface studied before by Dillen et al. in [1]. Hence, according to Dillen et al.'s result, we get Theorem 3 immediately.

If $m \neq 0$, we will derive a contradiction. In fact, it follows from (9) that:

$$
\begin{equation*}
\sum_{i=1}^{n-1}\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} c_{j}^{2}\right) f_{i}^{\prime \prime}=0, \quad \sum_{\substack{i, j=1 \\ i \neq j}}^{n-1} c_{i} f_{i}^{\prime} f_{j}^{\prime \prime}=0 \tag{10}
\end{equation*}
$$

It follows from the first equation of (10) that each $f_{i}^{\prime \prime}$ is constant for $i=1, \cdots, n-1$. Then, there exist constants $a_{i}$ such that $f_{i}\left(x_{i}\right)=a_{i} x_{i}^{2}$. In this case, (10) becomes:

$$
\begin{equation*}
\sum_{i=1}^{n-1}\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} c_{j}^{2}\right) a_{i}=0, \quad a_{i} c_{i} \sum_{\substack{j=1 \\ j \neq i}}^{n-1} a_{j}=0, \quad \text { where } i=1, \cdots, n-1 \tag{11}
\end{equation*}
$$

Substituting $f_{n}=m u^{2}$ and $f_{i}=a_{i} x_{i}^{2}$ for $i=1, \cdots, n-1$ into (7), we have:

$$
\begin{align*}
& 4 m^{2}\left[\sum_{i=1}^{n-1}\left(\sum_{\substack{j=1 \\
j \neq i}}^{n} c_{j}^{2}\right) a_{i}\right] u^{2}+8 m\left(\sum_{\substack{i, j=1 \\
i \neq j}}^{n-1} a_{i} c_{i} a_{j} x_{i}\right) u-4 m \sum_{\substack{i, j=1 \\
i \neq j}}^{n-1} a_{i} a_{j} c_{i} c_{j} x_{i} x_{j} \\
& \quad+4 \sum_{i=1}^{n-1} a_{i}^{2}\left(m \sum_{\substack{j=1 \\
j \neq i}}^{n} c_{j}^{2}+\sum_{\substack{j=1 \\
j \neq i}}^{n-1} a_{j}\right) x_{i}^{2}+\sum_{i=1}^{n-1} a_{i}+m \sum_{i=1}^{n} c_{i}^{2}=0 . \tag{12}
\end{align*}
$$

Taking into account (11), (12) becomes:

$$
\begin{equation*}
4 m \sum_{\substack{i, j=1 \\ i \neq j}}^{n-1} a_{i} a_{j} c_{i} c_{j} x_{i} x_{j}=4 \sum_{i=1}^{n-1} a_{i}^{2}\left(m \sum_{\substack{j=1 \\ j \neq i}}^{n} c_{j}^{2}+\sum_{\substack{j=1 \\ j \neq i}}^{n-1} a_{j}\right) x_{i}^{2}+\sum_{i=1}^{n-1} a_{i}+m \sum_{i=1}^{n} c_{i}^{2} \tag{13}
\end{equation*}
$$

Note that (13) is a quadratic polynomial with $x_{1}, \cdots, x_{n-1}$. By the arbitrariness of $x_{i}$, we have:

$$
\begin{equation*}
a_{i}^{2}\left(m \sum_{\substack{j=1 \\ j \neq i}}^{n} c_{j}^{2}+\sum_{\substack{j=1 \\ j \neq i}}^{n-1} a_{j}\right)=0 \tag{14}
\end{equation*}
$$

for every $i=1, \cdots, n-1$,

$$
\begin{equation*}
\sum_{i=1}^{n-1} a_{i}+m \sum_{i=1}^{n} c_{i}^{2}=0 \tag{15}
\end{equation*}
$$

and:

$$
\begin{equation*}
a_{i} a_{j} c_{i} c_{j}=0 \tag{16}
\end{equation*}
$$

for every $i, j=1, \cdots, n-1$ and $i \neq j$. It follows from (14) and (15) that:

$$
\begin{equation*}
a_{i}^{3}=-m\left(c_{i} a_{i}\right)^{2} \tag{17}
\end{equation*}
$$

for every $i=1, \cdots, n-1$. From (16), we can see that at most one $a_{k} c_{k}$ is not zero. Without loss of generality, we assume $a_{k_{0}} c_{k_{0}} \neq 0$ and all $a_{k} c_{k}=0$ for $k \neq k_{0}$. From (17), we have $a_{k_{0}} \neq 0$ and $a_{k}=0$ for $k \neq k_{0}$. This contradicts the first equation of (11). Thus, all $a_{k}=0$ for $k=1, \cdots, n-1$ and $f_{i}^{\prime \prime}\left(x_{i}\right)=0$. Then, (7) becomes:

$$
\left(\sum_{i=1}^{n} c_{i}^{2}\right) 2 m=0
$$

which is a contradiction.
Case B. $f_{n}^{\prime \prime \prime} \neq 0$. Dividing by $f_{n}^{\prime \prime \prime}$ on both sides of (8), we have:

$$
\begin{align*}
& {\left[\sum_{i=1}^{n} c_{i}^{2}+c_{n}^{2} \sum_{i=1}^{n-1} f_{i}^{\prime 2}+\frac{1}{2} \sum_{\substack{i, j=1 \\
i \neq j}}^{n-1}\left(c_{i} f_{j}^{\prime}-c_{j} f_{i}^{\prime}\right)^{2}\right]} \\
& \quad+2 c_{n}\left[\sum_{i=1}^{n-1}\left(\sum_{\substack{j=1 \\
j \neq i}}^{n} c_{j}^{2}\right) f_{i}^{\prime \prime \prime}\right] \frac{f_{n}^{\prime} f_{n}^{\prime \prime}}{f_{n}^{\prime \prime \prime}}+2 c_{n} \sum_{\substack{i, j=1 \\
i \neq j}}^{n-1} c_{i} f_{i}^{\prime} f_{j}^{\prime \prime} \frac{f_{n}^{\prime \prime}}{f_{n}^{\prime \prime \prime}}=0 . \tag{18}
\end{align*}
$$

Differentiating (18) with respect to $u$, we have:

$$
\begin{equation*}
\left[\sum_{i=1}^{n-1}\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} c_{j}^{2}\right) f_{i}^{\prime \prime}\right]\left(\frac{f_{n}^{\prime} f_{n}^{\prime \prime}}{f_{n}^{\prime \prime \prime}}\right)_{u}+\sum_{\substack{i, j=1 \\ i \neq j}}^{n-1} c_{i} f_{i}^{\prime} f_{j}^{\prime \prime}\left(\frac{f_{n}^{\prime \prime}}{f_{n}^{\prime \prime \prime}}\right)_{u}=0 \tag{19}
\end{equation*}
$$

Next, we will give some claims.
Claim 1. $\left(\frac{f_{n}^{\prime \prime}}{f_{n}^{\prime \prime \prime}}\right)_{u} \neq 0$.
In fact, if $\left(\frac{f_{n}^{\prime \prime}}{f_{n}^{\prime \prime \prime}}\right)_{u}=0$, there exist constants $a, b$ such that:

$$
\begin{equation*}
f_{n}=a f_{n}^{\prime}+b u \tag{20}
\end{equation*}
$$

The assumption $f_{n}^{\prime \prime \prime} \neq 0$ implies that $a \neq 0$. Solving: (20) gives

$$
f_{n}(u)=k e^{\frac{u}{a}}+b u+a b
$$

where $k$ is nonzero constant. In this case,

$$
\left(\frac{f_{n}^{\prime} f_{n}^{\prime \prime}}{f_{n}^{\prime \prime \prime}}\right)_{u}=\frac{k}{a} e^{\frac{u}{a}} \neq 0 .
$$

Then, from (19), we have:

$$
\begin{equation*}
\sum_{i=1}^{n-1}\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} c_{j}^{2}\right) f_{i}^{\prime \prime}=0 \tag{21}
\end{equation*}
$$

Therefore, all $f_{i}^{\prime \prime}$ are constants for $i=1, \cdots, n-1$. Thus, there exist constants $a_{i}$ such that $f_{i}\left(x_{i}\right)=a_{i} x_{i}^{2}$, and (21) becomes:

$$
\sum_{i=1}^{n-1}\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} c_{j}^{2}\right) a_{i}=0
$$

In this case, (18) becomes:

$$
\sum_{i=1}^{n} c_{i}^{2}+c_{n}^{2} \sum_{i=1}^{n-1}\left(2 a_{i} x_{i}\right)^{2}+\frac{1}{2} \sum_{\substack{i, j=1 \\ i \neq j}}^{n-1}\left(2 c_{i} a_{j} x_{j}-2 c_{j} a_{i} x_{i}\right)^{2}+8 a c_{n} \sum_{\substack{i, j=1 \\ i \neq j}}^{n-1} c_{i} a_{i} a_{j} x_{i}=0
$$

which is impossible.
Claim 2. $\sum_{i=1}^{n-1}\left(\sum_{\substack{j=1 \\ j=1}} c_{j}^{2}\right) f_{i}^{\prime \prime} \neq 0$.
We assume $\sum_{i=1}^{n-1}\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} c_{j}^{2}\right) f_{i}^{\prime \prime}=0$. Then, all $f_{i}^{\prime \prime}$ are constants for $i=1, \cdots, n-1$. Hence, there exist constants $a_{i}$ such that $f_{i}\left(x_{i}\right)=a_{i} x_{i}^{2}$ and $\sum_{i=1}^{n-1}\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} c_{j}^{2}\right) a_{i}=0$. It follows from (19) that $\sum_{\substack{i, j=1 \\ i \neq j}}^{n-1} c_{i} f_{i}^{\prime} f_{j}^{\prime \prime}=0$. Then, (18) becomes:

$$
\sum_{i=1}^{n} c_{i}^{2}+c_{n}^{2} \sum_{i=1}^{n-1}\left(2 a_{i} x_{i}\right)^{2}+\frac{1}{2} \sum_{\substack{i, j=1 \\ i \neq j}}^{n-1}\left(2 c_{i} a_{j} x_{j}-2 c_{j} a_{i} x_{i}\right)^{2}=0
$$

This is a contradiction. Thus, we have $\sum_{i=1}^{n-1}\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} c_{j}^{2}\right) f_{i}^{\prime \prime} \neq 0$.

According to Claim 1 and Claim 2, it follows from (19) that there exists a constant $m$ such that:

$$
\begin{equation*}
\frac{\sum_{i, j=1}^{n \neq j}}{n-1} c_{i} f_{i}^{\prime} f_{j}^{\prime \prime}{ }_{\sum_{i=1}^{n-1}\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} c_{j}^{2}\right) f_{i}^{\prime \prime}}=-\frac{\left(\frac{f_{n}^{\prime} f_{n}^{\prime \prime}}{f_{n}^{\prime \prime \prime}}\right)_{u}}{\left(\frac{f_{n}^{\prime \prime}}{f_{n}^{\prime \prime \prime}}\right)_{u}}=m \tag{22}
\end{equation*}
$$

Hence,

$$
\left(\frac{f_{n}^{\prime} f_{n}^{\prime \prime}}{f_{n}^{\prime \prime \prime}}\right)_{u}=-m\left(\frac{f_{n}^{\prime \prime}}{f_{n}^{\prime \prime \prime}}\right)_{u}
$$

By integration, there exists a constant $c$ such that:

$$
\begin{equation*}
\frac{f_{n}^{\prime} f_{n}^{\prime \prime}}{f_{n}^{\prime \prime \prime}}=-m \frac{f_{n}^{\prime \prime}}{f_{n}^{\prime \prime \prime}}+c \tag{23}
\end{equation*}
$$

that is

$$
f_{n}^{\prime} f_{n}^{\prime \prime}=-m f_{n}^{\prime \prime}+c f_{n}^{\prime \prime \prime}
$$

By integration, we get:

$$
\begin{equation*}
f_{n}^{\prime 2}+2 m f_{n}^{\prime}=2 c f_{n}^{\prime \prime}+c_{0} \tag{24}
\end{equation*}
$$

for some constant $c_{0}$. Moreover, solving the ODE (24), after a translation, gives:

$$
f_{n}=\left\{\begin{array}{l}
-m u-2 c \ln \cos \left(\frac{\sqrt{-\left(m^{2}+c_{0}\right)}}{2 c} u\right), \quad \text { if } m^{2}+c_{0}<0 \\
-m u-2 c \ln \sinh \left(\frac{\sqrt{m^{2}+c_{0}}}{2 c} u\right)-2 c \ln 2, \quad \text { if } m^{2}+c_{0}>0 \\
-m u-2 c \ln |u|, \quad \text { if } m^{2}+c_{0}=0
\end{array}\right.
$$

On the other hand, from (22), we have:

$$
\begin{equation*}
\sum_{\substack{i, j=1 \\ i \neq j}}^{n-1} c_{i} f_{i}^{\prime} f_{j}^{\prime \prime}=m \sum_{i=1}^{n-1}\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} c_{j}^{2}\right) f_{i}^{\prime \prime} \tag{25}
\end{equation*}
$$

Since $\sum_{i=1}^{n-1}\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} c_{j}^{2}\right) f_{i}^{\prime \prime} \neq 0$ in Claim 2, it is impossible that all $f_{i}^{\prime \prime}$ are zeros for $i=1, \cdots, n-1$. We assume that $f_{i_{0}}^{\prime \prime} \neq 0$. Then, differentiating (25) with respect to $x_{i_{0}}$, we have:

$$
\begin{equation*}
\left[m\left(\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}\right)-\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n-1}\left(c_{i} f_{i}^{\prime}\right)\right] \frac{f_{i_{0}}^{\prime \prime \prime}}{f_{i_{0}}^{\prime \prime}}=c_{i_{0}} \sum_{\substack{i=1 \\ i \neq i_{0}}}^{n-1} f_{i}^{\prime \prime} \tag{26}
\end{equation*}
$$

Now, we make another claim.
Claim 3. $m\left(\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}\right)-\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n-1}\left(c_{i} f_{i}^{\prime}\right)=0$.
Assume that $m\left(\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}\right)-\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n-1}\left(c_{i} f_{i}^{\prime}\right) \neq 0$. We will get a contradiction again. It follows from (26) that there exists a constant $a$ such that:

$$
\begin{equation*}
\frac{f_{i_{0}}^{\prime \prime \prime}}{c_{i_{0}} f_{i_{0}}^{\prime \prime}}=\frac{\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n-1} f_{i}^{\prime \prime}}{m\left(\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}\right)-\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n-1}\left(c_{i} f_{i}^{\prime}\right)}=a . \tag{27}
\end{equation*}
$$

By (27), there exist constants $b_{i_{0}}$ and $d_{i_{0}}$ such that:

$$
\begin{equation*}
f_{i_{0}}=b_{i_{0}} e^{a c_{i_{0}} x_{i_{0}}}-\frac{d_{i_{0}}}{a c_{i_{0}}} x_{i_{0}} \tag{28}
\end{equation*}
$$

From (27), we also have:

$$
f_{i}^{\prime \prime}+a c_{i} f_{i}^{\prime}=a m \sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}
$$

for every $i=1, \cdots, i_{0}-1, i_{0}+1, \cdots, n-1$. Solving this equation, we have:

$$
\begin{equation*}
f_{i}=b_{i} e^{-a c_{i} x_{i}}+\frac{m\left(\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n-1} c_{i}^{2}\right)}{c_{i}} x_{i} \tag{29}
\end{equation*}
$$

where $b_{i}$ are constants for $i=1, \cdots, i_{0}-1, i_{0}+1, \cdots, n-1$. On the other hand, differentiating (25) with respect to $x_{k}$ for $k=1, \cdots, i_{0}-1, i_{0}+1, \cdots, n-1$, we have:

$$
\begin{equation*}
f_{k}^{\prime \prime \prime}\left[\sum_{\substack{i=1 \\ i \neq k}}^{n-1} c_{i} f_{i}^{\prime}-m \sum_{\substack{i=1 \\ i \neq k}}^{n} c_{i}^{2}\right]+c_{k} f_{k}^{\prime \prime} \sum_{\substack{i=1 \\ i \neq k}}^{n-1} f_{i}^{\prime \prime}=0 \tag{30}
\end{equation*}
$$

Substituting (28) and (29) into (30), we have:

$$
\begin{equation*}
2 a c_{k}^{3} b_{k} \sum_{\substack{i=1 \\ i \neq k, i \neq i_{0}}}^{n-1} c_{i}^{2} b_{i} e^{-a\left(c_{i} x_{i}+c_{k} x_{k}\right)}=c_{k}^{3} b_{k} B_{k} e^{-a c_{k} x_{k}} \tag{31}
\end{equation*}
$$

for $k=1, \cdots, i_{0}-1, i_{0}+1, \cdots, n-1$, where:

$$
B_{k}=(n-3) m \sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}-m \sum_{\substack{i=1 \\ i \neq k}}^{n} c_{i}^{2}-\frac{d_{i_{0}}}{a}
$$

From (31) and the arbitrariness of $x_{k}$, we have $c_{k}^{3} b_{k} c_{i}^{2} b_{i}=0$, which is equivalent to $c_{k} b_{k} c_{i} b_{i}=0$ for $i, k=1, \cdots, i_{0}-1, i_{0}+1, \cdots, n-1$ and $i \neq k$. It follows that at most one $c_{k} b_{k}$ is not zero. We assume that all $c_{k} b_{k}=0$ for $k=1, \cdots, i_{0}-1, i_{0}+1, \cdots, n-1$ and $k \neq j_{0}$. In this case:

$$
\begin{equation*}
f_{j_{0}}^{\prime}=-a b_{j_{0}} c_{j_{0}} e^{-a c_{j_{0}} x_{j_{0}}}+\frac{m \sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}}{c_{j_{0}}} \tag{32}
\end{equation*}
$$

and:

$$
\begin{equation*}
f_{k}^{\prime}=\frac{m \sum_{\substack{k=1 \\ k \neq i_{0}}}^{n} c_{k}^{2}}{c_{k}} \tag{33}
\end{equation*}
$$

for $k=1, \cdots, i_{0}-1, i_{0}+1, \cdots, n-1$ and $k \neq j_{0}$. Substituting (28), (32), and (33) into (25), we have:

$$
(n-3) m\left(\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}\right) a^{2} b_{i_{0}} c_{i_{0}}^{2} e^{a c_{i_{0}} x_{i_{0}}}+a^{2} b_{j_{0}} c_{j_{0}}^{2} C e^{-a c_{j_{0}} x_{j_{0}}}=0
$$

where:

$$
C=(n-3) m \sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}-m \sum_{\substack{i=1 \\ i \neq j_{0}}}^{n} c_{i}^{2}-\frac{d_{i_{0}}}{a}
$$

It follows that $n=3$, and (18) becomes:

$$
\begin{align*}
& {\left[\sum_{i=1}^{3} c_{i}^{2}+c_{3}^{2} \sum_{i=1}^{2} f_{i}^{\prime 2}+\left(c_{1} f_{2}^{\prime}-c_{2} f_{1}^{\prime}\right)^{2}\right]} \\
& \quad+2 c_{3}\left[\sum_{\substack{i=1}}^{2}\left(\sum_{\substack{j=1 \\
j \neq i}}^{3} c_{j}^{2}\right) f_{i}^{\prime \prime}\right] \frac{f_{3}^{\prime} f_{3}^{\prime \prime}}{f_{3}^{\prime \prime \prime}}+2 c_{3} \sum_{\substack{i, j=1 \\
i \neq j}}^{2} c_{i} f_{i}^{\prime} f_{j}^{\prime \prime} \frac{f_{3}^{\prime \prime}}{f_{3}^{\prime \prime \prime}}=0 \tag{34}
\end{align*}
$$

From (23) and (25), (34) becomes:

$$
\begin{equation*}
c_{1}^{2}+c_{2}^{2}+c_{3}^{2}+c_{3}^{2}\left(f_{1}^{\prime 2}+f_{2}^{\prime 2}\right)+\left(c_{1} f_{2}^{\prime}-c_{2} f_{1}^{\prime}\right)^{2}+2 c c_{3}\left[\left(c_{3}^{2}+c_{2}^{2}\right) f_{1}^{\prime \prime}+\left(c_{3}^{2}+c_{1}^{2}\right) f_{2}^{\prime \prime}\right]=0 \tag{35}
\end{equation*}
$$

Differentiating (35) with respect to $x_{1}$, we have:

$$
c_{3}^{2} f_{1}^{\prime} f_{1}^{\prime \prime}-c_{2}\left(c_{1} f_{2}^{\prime}-c_{2} f_{1}^{\prime}\right) f_{1}^{\prime \prime}+c c_{3}\left(c_{2}^{2}+c_{3}^{2}\right) f_{1}^{\prime \prime \prime}=0
$$

Rewriting (36), we have:

$$
\begin{equation*}
\left(c_{2}^{2}+c_{3}^{2}\right)\left(f_{1}^{\prime}+c c_{3} \frac{f_{1}^{\prime \prime \prime}}{f_{1}^{\prime \prime}}\right)=c_{1} c_{2} f_{2}^{\prime} \tag{36}
\end{equation*}
$$

Therefore, $f_{2}^{\prime}$ is constant, and $f_{2}^{\prime \prime}=0$, which is a contradiction. We complete the proof of Claim 3.
Now, we have proven that $m\left(\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}\right)-\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n-1}\left(c_{i} f_{i}^{\prime}\right)=0$, and $f_{i}^{\prime}$ are constants for $i=1, \cdots, i_{0}-$ $1, i_{0}+1, \cdots, n-1$. There exist constants $a_{i}$ such that $f_{i}^{\prime}\left(x_{i}\right)=a_{i}$ and:

$$
\begin{equation*}
m\left(\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}\right)=\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n-1}\left(c_{i} a_{i}\right) \tag{37}
\end{equation*}
$$

Then, (18) becomes:

$$
\begin{align*}
& \sum_{i=1}^{n} c_{i}^{2}+c_{n}^{2} \sum_{\substack{i=1 \\
i \neq i_{0}}}^{n-1} a_{i}^{2}+f_{i_{0}}^{\prime 2} \sum_{\substack{i=1 \\
i \neq i_{0}}}^{n} c_{i}^{2}+\sum_{\substack{j=1 \\
j \neq i_{0}}}^{n-1}\left(a_{j}^{2} \sum_{\substack{i=1 \\
i \neq j}}^{n-1} c_{i}^{2}\right) \\
&-2 c_{i_{0}} f_{i_{0}}^{\prime} \sum_{\substack{i=1 \\
i \neq i_{0}}}^{n-1} c_{i} a_{i}-\sum_{\substack{j=1 \\
j \neq i_{0}}}^{n-1}\left(c_{j} a_{j} \sum_{\substack{i=1 \\
i \neq j}}^{n-1} c_{i} a_{i}\right) \\
& \quad+2 c_{n} f_{i_{0}}^{\prime \prime}\left(\sum_{\substack{i=1 \\
i \neq i_{0}}}^{n} c_{i}^{2}\right) \frac{f_{n}^{\prime} f_{n}^{\prime \prime}}{f_{n}^{\prime \prime \prime}}+2 c_{n} f_{i_{0}}^{\prime \prime}\left(\sum_{\substack{i=1 \\
i \neq i_{0}}}^{n-1} c_{i} a_{i}\right) \frac{f_{n}^{\prime \prime}}{f_{n}^{\prime \prime \prime}}=0 . \tag{38}
\end{align*}
$$

From (23) and (37), (38) becomes:

$$
\begin{align*}
\sum_{i=1}^{n} c_{i}^{2} & +c_{n}^{2} \sum_{\substack{i=1 \\
i \neq i_{0}}}^{n-1} a_{i}^{2}+f_{i_{0}}^{\prime 2} \sum_{\substack{i=1 \\
i \neq 1}}^{n} c_{i}^{2}+\sum_{\substack{j=1 \\
j \neq i_{0}}}^{n-1}\left(a_{j}^{2} \sum_{\substack{i=1 \\
i \neq j}}^{n-1} c_{i}^{2}\right) \\
& -2 c_{i_{0}} f_{i_{1}^{\prime}}^{\prime} \sum_{\substack{i=1 \\
i \neq i_{0}}}^{n-1} c_{i} a_{i}-\sum_{\substack{j=1 \\
j \neq i_{0}}}^{n-1}\left(c_{j} a_{j} \sum_{\substack{i=1 \\
i \neq j, i \neq i_{0}}}^{n-1} c_{i} a_{i}\right) \\
& +2 c c_{n} f_{i_{0}}^{\prime \prime}\left(\sum_{\substack{i=1 \\
i \neq i_{0}}}^{n} c_{i}^{2}\right)=0 . \tag{39}
\end{align*}
$$

Rewriting (39), we have:

$$
\begin{equation*}
\left(\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}\right) f_{i_{0}}^{\prime 2}-2 c_{i_{0}}\left(\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n-1} c_{i} a_{i}\right) f_{i_{0}}^{\prime}+2 c c_{n}\left(\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}\right) f_{i_{0}}^{\prime \prime}+B=0, \tag{40}
\end{equation*}
$$

where:

$$
\begin{aligned}
B & =\sum_{i=1}^{n} c_{i}^{2}+c_{n}^{2} \sum_{\substack{i=1 \\
i \neq i_{0}}}^{n-1} a_{i}^{2}+\sum_{\substack{j=1 \\
j \neq i_{0}}}^{n-1}\left(a_{j}^{2} \sum_{\substack{i=1 \\
i \neq j}}^{n-1} c_{i}^{2}\right)-\sum_{\substack{j=1 \\
j \neq i_{0}}}^{n-1}\left(c_{j} a_{j} \sum_{\substack{i=1 \\
i \neq j}}^{n-1} c_{i} a_{i}\right) \\
& =\sum_{i=1}^{n} c_{i}^{2}+\left(c_{n}^{2}+c_{i_{0}}^{2}\right) \sum_{\substack{i=1 \\
i \neq i_{0}}}^{n-1} a_{i}^{2}+\frac{1}{2} \sum_{\substack{i, j=1 \\
i \neq j, i, j \neq i_{0}}}^{n-1}\left(a_{i} c_{j}-a_{j} c_{i}\right)^{2}>0 .
\end{aligned}
$$

By (37), (40) becomes:

$$
f_{i_{0}}^{\prime 2}-2 m c_{i_{0}} f_{i_{0}}^{\prime}+2 c c_{n} f_{i_{0}}^{\prime \prime}+\frac{B}{\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} C_{i}^{2}}=0
$$

Solving this equation, we have:

$$
\begin{equation*}
f_{i_{0}}=2 c c_{n} \ln \cos \frac{A}{2 c c_{n}} x_{i_{0}}+m c_{i_{0}} x_{i_{0}} \tag{41}
\end{equation*}
$$

where:

$$
A=\sqrt{\frac{B}{\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}}-m^{2} c_{i_{0}}^{2}}
$$

On the other hand, since $f_{i}^{\prime}\left(x_{i}\right)=a_{i}$ for $i=1, \cdots, i_{0}-1, i_{0}+1, \cdots, n-1$, (7) becomes:

$$
\begin{align*}
& {\left[\sum_{i=1}^{n} c_{i}^{2}+c_{n}^{2} \sum_{\substack{i=1 \\
i \neq i_{0}}}^{n-1} a_{i}^{2}+f_{i_{0}}^{\prime 2} \sum_{\substack{i=1 \\
i \neq i_{0}}}^{n} c_{i}^{2}+\sum_{\substack{j=1 \\
j \neq i_{0}}}^{n-1}\left(a_{j}^{2} \sum_{\substack{i=1 \\
i \neq j}}^{n-1} c_{i}^{2}\right)\right.} \\
& \left.\quad-2 c_{i_{0}} f_{i_{0}}^{\prime} \sum_{\substack{i=1 \\
i \neq i_{0}}}^{n-1} c_{i} a_{i}-\sum_{\substack{j=1 \\
j \neq i_{0}}}^{n-1}\left(c_{j} a_{j} \sum_{\substack{i=1 \\
i \neq j}}^{n-1} c_{i} a_{i}\right)\right] f_{n}^{\prime \prime} \\
& \quad+\left[1+\sum_{\substack{i=1 \\
i \neq i_{0}}}^{n-1} a_{i}^{2}+\left(\sum_{\substack{i=1 \\
i \neq i_{0}}}^{n} c_{i}^{2}\right) f_{n}^{\prime 2}+2\left(\sum_{\substack{i=1 \\
i \neq i_{0}}}^{n-1} c_{i} a_{i}\right) f_{n}^{\prime}\right] f_{i_{0}}^{\prime \prime}=0 . \tag{42}
\end{align*}
$$

Combining (42) with (39), we have:

$$
\begin{equation*}
\left[1-2 c c_{n}\left(\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}\right) f_{n}^{\prime \prime}+\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n-1} a_{i}^{2}+\left(\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}\right) f_{n}^{\prime 2}+2\left(\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n-1} c_{i} a_{i}\right) f_{n}^{\prime}\right] f_{i_{0}}^{\prime \prime}=0 \tag{43}
\end{equation*}
$$

Since $f_{i_{0}}^{\prime \prime} \neq 0$, from (37) and (43), we have:

$$
f_{n}^{\prime 2}+2 m f_{n}^{\prime}=2 c c_{n} f_{n}^{\prime \prime}-\frac{1+\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n-1} a_{i}^{2}}{\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}}
$$

which together with (24) forces that:

$$
c_{0}=-\frac{1+\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n-1} a_{i}^{2}}{\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}}
$$

From (37), we have:

$$
m^{2}+c_{0}=\frac{-\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}-c_{n}^{2} \sum_{\substack{i=1 \\ i \neq i_{0}}}^{n-1} a_{i}^{2}-\left[\sum_{\substack{i=1 \\ i=i_{0}}}^{n-1} a_{i}^{2} \sum_{\substack{i=i_{0} \\ i \neq i_{0}}}^{n} c_{i}^{2}-\left(\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n-1} c_{i} a_{i}\right)^{2}\right]}{\left(\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}\right)^{2}}<0
$$

This is a contradiction, and hence, the second and third expressions of $f_{n}$ in (41) are impossible. After some translation, we have $f_{i}=0$ for $i=1, \cdots, i_{0}-1, i_{0}+1, \cdots, n-1$ and:

$$
f_{i_{0}}=2 c c_{n} \ln \cos \frac{x_{i_{0}}}{2 c c_{n}} \sqrt{\frac{\sum_{i=1}^{n} c_{i}^{2}}{\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}}}, \quad f_{n}=-2 c c_{n} \ln \cos \frac{u}{2 c c_{n}} \sqrt{\frac{1}{\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}}}
$$

Thus, the hypersurface is given by:

$$
F\left(x_{1}, \cdots, x_{n}\right)=2 c c_{n} \ln \frac{\cos \frac{x_{i_{0}}}{2 c c_{n}} \sqrt{\frac{\sum_{i=1}^{n} c_{i}^{2}}{\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n}}}}{\cos \frac{1}{2 c c_{n}} \sqrt{\frac{1}{\sum_{\substack{i=1 \\ i \neq i_{0}}}^{n} c_{i}^{2}}}\left(c_{1} x_{1}+\cdots+c_{n} x_{n}\right)} .
$$

At this moment, we complete the proof of Theorem 3.
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