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Some Implicativities for Groupoids and *BCK*-Algebras

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Received: 26 August 2019; Accepted: 10 October 2019; Published: 15 October 2019

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Abstract: In this paper, we generalize the notion of an implicativity discussed in BCK-algebras, and apply it to some groupoids and BCK-algebras. We obtain some relations among those axioms in the theory of groupoids.

Keywords: groupoid; d-algebra; BCK-algebra; (weakly) (i-)implicative; condition (L_i)

MSC: 06F35; 20N02

1. Introduction

As a generalization of BCK-algebras, the notion of d-algebras was introduced by Neggers and Kim [1]. They discussed some relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs. Several properties on d-algebras, e.g., d-ideals, deformations, and companion d-algebras, were studied [2–4]. Recently, some notions of the graph theory were applied to the theory of groupoids [5].

The notion of an implicativity has a very important role in the study of BCK-algebras. An implicative BCK-algebra has some connections with distributive lattices, Boolean algebras, and semi-Brouwerian algebras.

In this paper, we generalize the notion of the implicativity, which is a useful tool for investigation of BCK-algebras by using the notion of a word in general algebraic structures, the most simple mathematical structure, i.e., in the theory of a groupoid. Moreover, we generalized the notion of the implicativity by using $Bin(X)$-product "$\square$", and obtain the notion of a weakly i-implicativity, and obtain several properties in BCK-algebras and other algebraic structures.

2. Preliminaries

A groupoid $(X,*)$ is said to be a *left-zero-semigroup* if $x * y := x$ for all $x, y \in X$. Similarly, a groupoid $(X,*)$ is said to be a *right-zero-semigroup* if $x * y := y$ for all $x, y \in X$ [6]. A groupoid $(X,*,0)$ with constant 0 is said to be a d-algebra [1] if it satisfies the following conditions:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

For brevity, we call X a d-algebra. In a d-algebra X, we define a binary relation "$\leq$" by $x \leq y$ if and only if $x * y = 0$. A d-algebra $(X,*,0)$ is said to be an *edge* if $x * 0 = x$ for all $x \in X$. Example 1 below is an edge d-algebra. For general references on d-algebras we refer to [2–4].

A *BCK-algebra* [7] is a d-algebra X satisfying the following additional axioms:

(IV) $((x * y) * (x * z)) * (z * y) = 0$,
(V) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

Theorem 1 ([7]). *If $(X, *, 0)$ is a BCK-algebra, then*

$$(x * y) * z = (x * z) * y$$

for all $x, y, z \in X$.

Example 1. *Let $X := \{0, a, b, c, d, 1\}$ be a set with the following table:*

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	a	0	0	a	0	0
b	b	a	0	b	a	0
c	c	c	b	0	0	0
d	d	c	b	a	0	0
1	1	d	b	a	a	0

Then, $(X, *, 0)$ is an edge d-algebra which is not a BCK-algebra, since $(c * b) * d = b * d = a \neq 0 = 0 * b = (c * d) * b$. For general references on BCK-algebras, we refer to [7–9].

Let $(X, \leq)$ be a partially ordered set with minimal element 0, and let $(X, *)$ be its associated groupoid, i.e., $*$ is a binary operation on X defined by

$$x * y := \begin{cases} 0 & \text{if } x \leq y, \\ x & \text{otherwise.} \end{cases}$$

Then, $(X, *, 0)$ is a BCK-algebra, and we call it a *standard BCK-algebra*.

A BCK-algebra $(X, *, 0)$ is said to be *implicative* if $x = x * (y * x)$; *commutative* if $x * (x * y) = y * (y * x)$; *positive implicative* if $(x * y) * (y * z) = (x * y) * z$ for all $x, y \in X$ [7]. It is well known that a BCK-algebra is implicative if and only if it is both commutative and positive implicative. A group X is said to be *Boolean* if every element of X is its own inverse.

The notion of Smarandache algebras emerged and has been applied to several algebraic structures [10–12]. Two algebras $(X, *)$ and $(X, \circ)$ are said to be *Smarandache disjoint* [13,14] if we add some axioms of an algebra $(X, *)$ to an algebra $(X, \circ)$, then the algebra $(X, \circ)$ becomes a trivial algebra, i.e., $|X| = 1$; or if we add some axioms of an algebra $(X, \circ)$ to an algebra $(X, *)$, then the algebra $(X, \circ)$ becomes a trivial algebra, i.e., $|X| = 1$. Note that if we add an axiom (A) of an algebra $(X, *)$ to another algebra $(X, \circ)$, then we replace the binary operation "$\circ$" in (A) by the binary operation "$*$".

Let $Bin(X)$ be the collection of all groupoids $(X, *)$ defined on X. For any elements $(X, *)$ and $(X, \bullet)$ in $Bin(X)$, we define a binary operation "$\Box$" on $Bin(X)$ by

$$(X, *) \Box (X, \bullet) = (X, \Box), \tag{1}$$

where

$$x \Box y = (x * y) \bullet (y * x) \tag{2}$$

for any $x, y \in X$. Using the notion, Kim and Neggers proved the following theorem.

Theorem 2 ([6]). *$(Bin(X), \Box)$ is a semigroup, i.e., the operation "$\Box$" as defined in general is associative. Furthermore, the left zero semigroup is an identity for this operation.*

3. (Weakly) Implicativity in Groupoids

By using the notion of words, we generalize the notion of an implicativity in groupoids. A groupoid (or a *BCK*-algebra) $(X, *)$ is said to be *implicative* if

$$x * (y * x) = x$$

for all $x, y \in X$.

Proposition 1. *If $(X, *)$ is a left-zero semigroup (respectively, a right-zero semigroup), i.e., $x * y = x$ (respectively, $x * y = y$) for all $x, y \in X$, then $(X, *)$ is implicative.*

Proof. If $(X, *)$ is a left-zero semigroup, then $x * y = x$ for all $x, y \in X$. It follows that $x * (y * x) = x * y = x$, which proves that $(X, *)$ is implicative. Similarly, if $(X, *)$ is a right-zero semigroup, then it is also implicative. □

Proposition 2. *The class of implicative groupoids and the class of groups are Smarandache disjoint.*

Proof. Assume $(X, \bullet, e)$ is both a group and an implicative groupoid. Then, $e = e \bullet (x \bullet e) = x \bullet e = x$ for all $x \in X$. This shows that $X = \{e\}$. □

Notice that the class of implicative groupoids is equationally defined and thus that it is a variety, i.e., it is closed under subgroups, epimorphic images, and direct products.

A groupoid $(X, *)$ is said to be *weakly implicative* if there exists a word $w(x)$ such that, for all $x, y \in X$,

$$x * (y * x) = w(x).$$

Note that $w(x)$ is an expression of "x", e.g., $x * (x * x), x * x, ((x * x) * x) * x, \cdots$, and a zero element "0", e.g., $x * (0 * x), (0 * x) * (x * 0), \cdots$, if necessary.

Proposition 3. *Let $(X, *, 0)$ be a weakly implicative groupoid with $w(x) = x * (0 * x)$. If $(X, *, 0)$ is a BCK-algebra, then it is an implicative BCK-algebra.*

Proof. Let $(X, *, 0)$ be a weakly implicative groupoid with $w(x) := x * (0 * x)$. Since $(X, *, 0)$ is a *BCK*-algebra, we obtain $x * (y * x) = w(x) = x * (0 * x) = x * 0 = x$ for all $x, y \in X$. Hence, $(X, *, 0)$ is an implicative *BCK*-algebra. □

Corollary 1. *Let $(X, *, 0)$ be an edge d-algebra. If $(X, *, 0)$ is a weakly implicative with $w(x) = x * (0 * x)$, then is an implicative edge d-algebra.*

Proof. If $(X, *, 0)$ is an edge d-algebra, then $0 * x = 0$ and $x * 0 = x$ for all $x \in X$. By Proposition 3, $(X, *, 0)$ is an implicative edge d-algebra. □

Let $(X, *)$ be a groupoid. Define a binary operation "$\bullet$" on X by

$$x \bullet y := y * x$$

for all $x, y \in X$. We call $(X, \bullet)$ an *oppositie groupoid* of a groupoid $(X, *)$.

Theorem 3. *The opposite groupoid of a BCK-algebra is weakly implicative.*

Proof. Let $(X, *, 0)$ be a *BCK*-algebra and let $w(x) := 0$ for all $x \in X$. Then, $x \bullet (y \bullet x) = (x * y) * x = (x * x) * y = 0 * y = 0 = w(x)$. Hence, $(X, \bullet)$ is weakly implicative. □

Proposition 4. *There is no nontrivial implicative opposite groupoid derived from a BCK-algebra.*

Proof. Let $(X, *, 0)$ be a BCK-algebra and let $|X| \geq 2$. Assume that $(X, \bullet)$ is implicative. Then, $x = x \bullet (y \bullet x) = (x * y) * x = (x * x) * y = 0 * y = 0$ for all $x \in X$, i.e., $X = \{0\}$, a contradiction. $\square$

Theorem 4. *The class of weakly implicative groupoids and the class of groups are Smarandache disjoint.*

Proof. Assume $(X, \cdot, e)$ is both a group and a weakly groupoid. Then, there exists a word $w(x)$ such that $x \cdot (y \cdot x) = w(x)$ for all $x, y \in X$. It follows that $e \cdot (x \cdot e) = w(e)$ for all $x \in X$. Since $x = e \cdot (x \cdot e)$, we obtain $x = w(e)$, a constant. Hence, $X = \{w(e)\}$, i.e., $|X| = 1$, a contradiction. $\square$

4. Levels of Implicativities

Let $(X, *)$ be a groupoid and let $x, y \in X$. We define binary operations "$\square_i$" on X by $x \square_1 y := (x * y) * (y * x) = x \square y$ and $x \square_{i+1} y := (x \square_i y) * (y \square_i x)$ for all $x, y \in X$, where $i = 1, 2, 3, \cdots$. Let $w(x)$ be a word of x. We define the following levels of implicativities as follows:

Level 0: (i) $x * (y * x) = w(x)$ (*weakly 0-implicative*); (ii) $x * (y * x) = x$ (*implicative*).
Level 1: (i) $x * (y \square_1 x) = w(x)$ (*weakly 1-implicative*); (ii) $x * (y \square_1 x) = x$ (*1-implicative*).
Level i: (i) $x * (y \square_i x) = w(x)$ (*weakly i-implicative*); (ii) $x * (y \square_i x) = x$ (*i-implicative*).

Theorem 5. *Let $(X, \cdot, e)$ be a group with $|X| \geq 2$. Then, X is weakly 1-implicative if and only if X is a Boolean group.*

Proof. Let $(X, \cdot, e)$ be a weakly 1-implicative groupoid. Then, $x \cdot (y \square_1 x) = w(x)$ for all $x, y \in X$. It follows that $x \cdot ((y \cdot x) \cdot (x \cdot y)) = w(x)$. If we let $x := e$, then $e \cdot ((y \cdot e) \cdot (e \cdot y)) = w(e)$, and hence $y^2 = w(e)$ for all $y \in X$. If we let $y := e$, then $w(e) = e^2 = e$. Hence $y^2 = w(e) = e$ for all $y \in X$. Hence, $(X, \cdot, e)$ is a Boolean group.

Assume $(X, \cdot, e)$ is a Boolean group. Then, $x^2 = e$ for all $x \in X$. It follows that, for any $x, y \in X$,

$$\begin{aligned} x \cdot (y \square_1 x) &= x \cdot ((y \cdot x) \cdot (x \cdot y)) \\ &= xyx^2y \\ &= x \\ &= w(x). \end{aligned}$$

Hence, $(X, \cdot, e)$ is a weakly 1-implicative groupoid. $\square$

Theorem 6. *Let $(X, \cdot, e)$ be a group. If $(X, \cdot, e)$ is a weakly i-implicative groupoid, then it is i-implicative.*

Proof. Given $x \in X$, we have $e \square_1 x = (e \cdot x) \cdot (x \cdot e) = x^2$, $x \square_1 e = (x \cdot e) \cdot (e \cdot x) = x^2$, $e \square_2 x = (e \square_1 x) \cdot (x \square_1 e) = x^2 \cdot x^2 = x^4$, and $x \square_2 e = x^4$. Similarly, we obtain $e \square_i x = x^{2^i} = x \square_i e$. Since X is a group and $w(x)$ is a word on x, we have $w(e) = e$. This shows that $e = w(e) = e \cdot (y \square_i e) = e \cdot y^{2^i} = y^2$ for all $y \in X$. Hence, $w(x) = x \cdot (e \square_i x) = x \cdot x^{2^i} = x \cdot e^i = x$ for all $x \in X$, proving that $(X, \cdot, e)$ is i-implicative. $\square$

Proposition 5. *Let $(X, \cdot, e)$ be a group. If $x^{2^i} = e$ for any $x \in X$, then X is i-implicative.*

Proof. Given $x, y \in X$, we have $x \cdot (y \square_i x) = x \cdot x^{2^i} y^{2^i} = x$. Hence, X is i-implicative. $\square$

Theorem 7. *Let $(X, *, 0)$ be a BCK-algebra. If it is weakly i-implicative, then it is i-implicative.*

Proof. Suppose that $(X, *, 0)$ is weakly i-implicative. Then, there exists a mapping $H : X \times X \to X$ such that, for any $x, y \in X$, $x * (y \Box_i x) = H(x)$. Since $(X, *, 0)$ is a BCK-algebra, we obtain $0 \Box_1 x = (0 * x) * (x * 0) = 0, 0 \Box_2 x = (0 \Box_1 x) * (x \Box_1 0) = 0$. In this fashion, we obtain $0 \Box_i x = 0$. Thus, $H(x) = x * (0 \Box_i x) = x * 0 = x$, which proves that $x * (y \Box_i x) = H(x) = x * (0 \Box_i x) = x$. Hence, $(X, *, 0)$ is i-implicative. □

Theorem 8. *Let $(X, *)$ be both a weakly 0-implicative groupoid and an 1-implicative groupoid. If $(X, \Box) := (X, *) \Box (X, *)$, then $(X, \Box)$ is weakly 0-implicative.*

Proof. Since $(X, \Box) = (X, *) \Box (X, *)$, we have $x \Box (y \Box x) = (x * (y \Box x)) * ((y \Box x) * x)$ for any $x, y \in X$. It follows from $(X, *)$ is 1-implicative that $x = x * (y \Box_1 x) = x * (y \Box x)$ for all $x, y \in X$. Let $z := y \Box x$. Since $(X, *)$ is weakly 0-implicative, we have $x * (z * x) = w(x)$ for some word $w(x)$. It follows that

$$\begin{aligned} x \Box (y \Box x) &= (x * (y \Box x)) * ((y \Box x) * x) \\ &= x * ((y \Box x) * x) \\ &= x * (z * x) \\ &= w(x), \end{aligned}$$

which proves that $(X, \Box)$ is weakly 0-implicative. □

Corollary 2. *Let $(X, *)$ be both an implicative groupoid and a 1-implicative groupoid. If $(X, \Box) := (X, *) \Box (X, *)$, then $(X, \Box)$ is implicative.*

Proof. Let $w(x) := x$ in Theorem 8. □

Let $(X, *)$ be a groupoid and let $(X, \Box) := (X, *) \Box (X, *)$. If we assume that $x \Box y := x * y$ for any $x, y \in X$, then $x \Box_1 y = x \Box y = x * y$ and hence $x \Box_2 y = (x \Box_1 y) * (y \Box_1 x) = (x * y) * (y * x) = x \Box_1 y = x \Box y = x * y$. In this fashion, we obtain $x \Box_i y = x * y$ for all $i = 1, 2, \cdots$.

Theorem 9. *Every implicative BCK-algebra $(X, *, 0)$ is an i-implicative BCK-algebra where $i = 1, 2, \cdots$.*

Proof. Let $(X, *, 0)$ be an implicative BCK-algebra. Then, $x * (y * x) = x$ for any $x, y \in X$. It follows from Theorem 1 that

$$\begin{aligned} y \Box x &= (y * x) * (x * y) \\ &= (y * (x * y)) * x \\ &= y * x, \end{aligned}$$

i.e., $y \Box x = y * x$. This shows that $x * (y \Box_i x) = x * (y \Box x) = x * (y * x) = x$ for any $i = 1, 2, \cdots$. Hence, $(X, *, 0)$ is an i-implicative BCK-algebra. □

5. Weakly Implicative Groupoids with P(L_i)

A groupoid $(X, *, 0)$ is said to have a *condition* (L_i) if it satisfies the following condition, for any $x, y \in X$,

$$x \Box_{i+1} y = x \Box_i y, (L_i);$$

and a groupoid $(X, *, 0)$ is said to have a *condition* (L_0) if it satisfies the following condition, for any $x, y \in X$,

$$x \Box_1 y = x \Box_0 y, (L_0),$$

i.e., $(x * y) * (y * x) = x * y$. Assume that a groupoid $(X, *)$ has the condition (L_i). Then, $x\Box_{i+2}y = (x\Box_{i+1}y) * (y\Box_{i+1}x) = (x\Box_i y) * (y\Box_i x) = x\Box_{i+1}y$ for any $x, y \in X$. Similarly, $x\Box_{i+3}y = x\Box_{i+2}y = x\Box_{i+1}y$. In this fashion, we have $x\Box_{i+k}y = x\Box_{i+k-1}y$ for any $k = 1, 2, \cdots$. Hence, $(X, *)$ satisfies the condition (L_{i+k}).

Proposition 6. *If a groupoid $(X, *)$ is a weakly i-implicative groupoid with (L_i), then it is a weakly $(i + k)$-implicative groupoid.*

Proof. Let $(X, *)$ be a weakly i-implicative groupoid with (L_i). Then, $x * (y\Box_i x) = w(x)$ and $y\Box_{i+k}x = y\Box_i x$ for any $x, y \in X$, where $k = 1, 2, \cdots$. It follows that $x * (y\Box_{i+k}x) = x * (y\Box_i x) = w(x)$ for any $k = 1, 2, \cdots$. This proves that $(X, *)$ is a weakly $(i + k)$-implicative groupoid. $\Box$

Theorem 10. *Any standard BCK-algebra has the condition (L_0).*

Proof. Let $(X, *, 0)$ be a standard BCK-algebra. Given $x, y \in X$, we have 3 cases: (i) $x * y = 0$; (ii) $y * x = 0$; (iii) $x * y \neq 0, y * x \neq 0$. Case (i). If $x * y = 0$, then $x\Box y = (x * y) * (y * x) = 0 * (y * x) = 0 = x * y$. Case (ii). If $y * x = 0$, then $x\Box y = (x * y) * (y * x) = (x * y) * 0 = x * y$. Case (iii). If $x * y \neq 0, y * x \neq 0$, then $x * y = x$ and $y * x = y$. It follows that $x\Box y = (x * y) * (y * x) = x * y$. Hence, $x\Box_1 y = x\Box_0 y = x * y$. $\Box$

Note that nonstandard BCK-algebras need not have the condition (L_0). Consider the following example.

Example 2. *Let $X := \{0, 1, 2, 3\}$ be a set with the following table:*

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then, $(X, *, 0)$ is a BCK-algebra ([7], p. 245). Since $2 * 3 = 1$ and $(2 * 3) * (3 * 2) = 1 * 3 = 0$, we have $2\Box 3 \neq 2 * 3$, i.e., $(X, *, 0)$ does not satisfy the condition (L_0).

A groupoid $(X, *)$ is said to have a *condition* (α) if $X \times X = A \cup B \cup C$, where

$$A = \{(x, y) | y * x = 0\},$$
$$B = \{(x, y) | x * y = 0\},$$
$$C = \{(x, y) | x * y = x, y * x = y\}.$$

Theorem 11. *Let $(X, *, 0)$ be a groupoid with a condition (α). If $(X, *, 0)$ satisfies the following conditions: (i) $0 * x = x$; (ii) $x * 0 = x$; (iii) $x * x = 0$; (iv) $y * x = 0$ implies $x * y \in \{0, x\}$, then $(x * (x * y)) * y = 0$ for all $x, y \in X$.*

Proof. Case (i). If $(x, y) \in A$, then $y * x = 0$. By (iv), we have $x * y \in \{0, x\}$. If $x * y = 0$, then $(x * (x * y)) * y = (x * 0) * y = x * y = 0$. If $x * y = x$, then $(x * (x * y)) * y = (x * x) * y = 0 * y = 0$. Case (ii). If $(x, y) \in B$, then $x * y = 0$ and hence $(x * (x * y)) * y = (x * 0) * y = x * y = 0$. Case (iii). If $(x, y) \in C$, then $x * y = x$ and $y * x = y$. It follows that $(x * (x * y)) * y = (x * x) * y = 0 * y = 0$. $\Box$

Theorem 12. *Let $(X, *, 0)$ be a groupoid with a condition (α). If $(X, *, 0)$ satisfies the following conditions: (i) $x * 0 = x$; (ii) $0 * (x * y) = y * x$ for all $x, y \in X$, then $(X, *, 0)$ satisfies the condition (L_0).*

Proof. Given $x, y \in X$, if $(x, y) \in A$, then $y * x = 0$ and hence $x \Box y = (x * y) * (y * x) = (x * y) * 0 = x * y$. If $(x, y) \in B$, then $x * y = 0$ and hence $x \Box y = (x * y) * (y * x) = 0 * (y * x) = x * y$. If $(x, y) \in C$, then $x * y = x, y * x = y$ and hence $x \Box y = (x * y) * (y * x) = x * y$, proving the theorem. $\Box$

Theorem 13. *Let K be a field and let $A, B, C \in K, |K| \geq 3$. Define a binary operation "$*$" on K by $x * y := A + Bx + Cy$ for all $x, y \in K$. If $(K, *)$ is an implicative groupoid, then $x * y$ is one of the following:*

(i) $x * y = x,$
(ii) $x * y = y,$
(iii) $x * y = A - y.$

Proof. Since (K, $*$) is an implicative groupoid, we have

$$\begin{aligned} x &= x * (y * x) \\ &= A + Bx + C(A + Bx + Cy) \\ &= A(1 + C) + (B + C^2)x + BCy \end{aligned}$$

for any $x, y \in$ K. It follows that $A(1 + C) = 0, B + C^2 = 1$, and $BC = 0$. Case 1. Assume $B = 0$. Since $B + C^2 = 1$, we obtain $C^2 = 1$, i.e., $C = \pm 1$. If $C = 1$, then $A = 0$, since $A(1 + C) = 0$. Hence, $x * y = y$. If $C = -1$, then A is arbitrary, since $A(1 + C) = 0$. Hence, $x * y = A - y$. Case 2. Assume $C = 0$. Since $A(1 + C) = 0, B + C^2 = 1$, we obtain $A = 0, B = 1$, i.e., $x * y = x$. $\Box$

Theorem 14. *Let K be a field and let $A, B, C \in K, |K| \geq 3$. Define a binary operation "$*$" on K by $x * y := A + Bx + Cy$ for all $x, y \in K$. If $(K, *)$ satisfies the condition (L_0), then $x * y$ is one of the following:*

(i) $x * y = A,$
(ii) $x * y = x,$
(iii) $x * y = \frac{1}{2}(x + y),$
(iv) $x * y = A - \frac{1}{2}(x - y).$

Proof. Since $x * y = A + Bx + Cy$ and $y * x = A + By + Cx$, we have

$$\begin{aligned} (x * y) * (y * x) &= (A + Bx + Cy) * (A + By + Cx) \\ &= A + B(A + Bx + Cy) + C(A + By + Cx) \\ &= A(1 + B + C) + (B^2 + C^2)x + 2BCy \\ &= x * y \\ &= A + Bx + Cy \end{aligned}$$

for any $x, y \in$ K. It follows that $A(1 + B + C) = A, B^2 + C^2 = B$ and $2BC = C$. This shows that $C = 0$ or $B = \frac{1}{2}$. Case 1. $C = 0$. Since $B^2 + C^2 = B$, we obtain that either $B = 0$ or $B = 1$. If $B = 0$, then $x * y = A$. If $B = 1$, then $A = A(1 + B + C) = 2A$, i.e., $A = 0$. Hence, $x * y = x$. Case 2. $B = \frac{1}{2}$. Since $B^2 + C^2 = B$, we obtain $C = \pm\frac{1}{2}$. If $C = \frac{1}{2}$, then $A = A(1 + B + C) = 2A$, i.e., $A = 0$. Hence, $x * y = \frac{1}{2}(x + y)$. If $C = -\frac{1}{2}$, then $A = A(1 + B + C) = A$, and hence A is arbitrary. Hence, $x * y = A - \frac{1}{2}(x - y)$. $\Box$

6. Conclusions

In this paper, we generalized the notion of an implicativity discussed mainly in BCK-algebras by using the notion of a word, and obtained several properties in groupoids and BCK-algebras. By using the notion of $Bin(X)$-product $\Box$, we generalized the notion of the implicativity in different directions, and obtained the notion of a weakly (i-)implicativity. We applied these notions to BCK-algebras and several groupoids, and investigated some relations among them. The notion of a weakly

(i-)implicativity can be applied to positive implicative BCK-algebras, e.g., $x * y = (x\Box_i y) * y$, and seek to find some relations with commutative BCK-algebras.

7. Future Research

Using the notions of the word and the $Bin(X)$-product, we will generalize the notions of the commutativity and the positive implicativity in BCK-algebras and groupoids, i.e., (weakly) i-commutative and (weakly) i-positive implicative BCK-algebras and groupoids. We will investigate some relations between (weakly) i-implicative BCK-algebras and (weakly) i-commutative and (weakly) i-positive implicative BCK-algebras. Moreover, we will generalize several equivalent conditions for positive implicative BCK-algebras, and investigate their relationships.

Author Contributions: Funding acquisition, I.H.H.; Investigation, H.S.K.; Resources, J.N.

Acknowledgments: The authors are deeply grateful to the referee for the valuable suggestions. This work was supported by Incheon National University Research Grant 2019–2020.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Neggers, J.; Kim, H.S. On d-algebras. *Math. Slovaca* **1999**, *49*, 19–26.
2. Allen, P.J.; Kim, H.S.; Neggers, J. On companion d-algebras. *Math. Slovaca* **2007**, *57*, 93–106. [CrossRef]
3. Allen, P.J.; Kim, H.S.; Neggers, J. Deformations of d/BCK-algebras. *Bull. Korean Math. Soc.* **2011**, *48*, 315–324. [CrossRef]
4. Neggers, J.; Jun, Y.B.; Kim, H.S. On d-ideals in d-algebras. *Math. Slovaca* **1999**, *49*, 243–251.
5. Kim, H.S.; Neggers, J.; Ahn, S.S. A method to identify simple graphs by special binary systems. *Symmetry* **2018**, *10*, 297. [CrossRef]
6. Kim, H.S.; Neggers, J. The semigroups of binary systems and some perspectives. *Bull. Korean Math. Soc.* **2008**, *45*, 651–661. [CrossRef]
7. Meng, J.; Jun, Y.B. *BCK-Algebras*; Kyungmoon Sa: Seoul, Korea, 1994.
8. Huang, Y. *BCI-Algebras*; Science Press: Beijing, China, 2006.
9. Iorgulescu, A. *Algebras of Logic as BCK-Algebras*; Editura ASE: Bucharest, Romania, 2008.
10. Kandasamy, W.B.V. *Bialgebraic Structures and Smarandache Bialgebraic Structures*; American Research Press: Rehoboth, DE, USA, 2003.
11. Kandasamy, W.B.V.; Smarandache, F. *N-Algebraic Structures and S-N-Algebraic Structures*; HEXIS: Phoenix, AZ, USA, 2005.
12. Kandasamy, W.B.V.; Smarandache, F. *Interval Groupoids*; Infrlearnquest: Ann Arbor, MI, USA, 2010.
13. Allen, P. J.; Kim, H. S.; Neggers, J. Smarandache disjoint in BCK/d-algebras. *Sci. Math. Jpn.* **2005**, *61*, 447–449.
14. Kim, H.S.; Neggers, J.; Ahn, S. S. On pre-commutative algebras. *Math. J.* **2019**, *7*, 336. [CrossRef]