Article

Certain Unified Integrals Associated with Product of M-Series and Incomplete *H*-functions

Manish Kumar Bansal [1] (iD), **Devendra Kumar** [2], **Ilyas Khan** [3,*], **Jagdev Singh** [4]
and Kottakkaran Sooppy Nisar [5] (iD)

[1]   Department of Applied Sciences, Government Engineering College, Banswara 327001, Rajasthan, India;
      bansalmanish443@gmail.com
[2]   Department of Mathematics, University of Rajasthan, Jaipur 302004, Rajasthan, India;
      devendra.maths@gmail.com
[3]   Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City 72915, Vietnam
[4]   Department of Mathematics, JECRC University, Jaipur 303905, Rajasthan, India;
      jagdevsinghrathore@gmail.com
[5]   Department of Mathematics, Faculty of Arts and Sciences, Prince Sattam Bin Abdulaziz University,
      Wadi Al-dawasir 11991, Saudi Arabia; n.sooppy@psau.edu.sa or ksnisar1@gmail.com
[*]   Correspondence: ilyaskhan@tdtu.edu.vn

![check for updates]

Abstract: In this paper, we established some interesting integrals associated with the product of M-series and incomplete *H*-functions, which are expressed in terms of incomplete *H*-functions. Next, we give some special cases by specializing the parameters of M-series and incomplete *H*-functions (for example, Fox's *H*-Function, Incomplete Fox Wright functions, Fox Wright functions and Incomplete generalized hypergeometric functions) and also listed few known results. The results obtained in this work are general in nature and very useful in science, engineering and finance.

Keywords: M-series; incomplete *H*-functions; gamma function; improper integral

MSC: 33C60, 33C99

1. Introduction, Definitions and Preliminaries

The integral formula containing several generalized special functions (GSF) have been explored by numerous authors [1–5]. Many integral formulas involving GSF have been proposed and play a pivotal role in solving scientific and engineering problems. In fact, GSF are connected with different kinds of problems in various fields of mathematical sciences. These relations of GSF with different field of research have motivated many scientists to investigate the field of integrals and connected GSF. Several unified integral formulas established by many authors involving a various kind of special functions (see, for example, [6–8]). The key aim of this work is to develop Oberhettinger's integral formulas containing the product of M-series and incomplete *H*-functions. The Oberhettinger's integral formulas established in the present work are very useful to obtain the Mellin transform of various simpler special functions. The Mellin transforms of special functions find their applications in mathematical statistics, number theory, and the theory of asymptotic expansions. The main findings of the present work are very useful in solving the problems arising in digital signals, image processing, finance and ship target recognition by sonar

system and radar signals [9–12]. We recall here the frequently used *incomplete Gamma functions* $\Gamma(\Im, y)$ and $\gamma(\Im, y)$ defined by:

$$\gamma(\Im, y) := \int_{0}^{y} t^{\Im-1} e^{-t} dt \qquad (\mathfrak{R}(\Im) > 0; y \geqq 0) \tag{1}$$

and

$$\Gamma(\Im, y) := \int_{y}^{\infty} t^{\Im-1} e^{-t} dt \qquad (y \geqq 0; \mathfrak{R}(\Im) > 0 \quad \text{when} \quad y = 0), \tag{2}$$

respectively, satisfy the decomposition formula given by:

$$\Gamma(\Im, y) + \gamma(\Im, y) = \Gamma(\Im) \qquad (\mathfrak{R}(\Im) > 0). \tag{3}$$

The condition that we have used on the parameter y in and anywhere else in the current paper is unrestrained of $\mathfrak{R}(z)$ $(z \in \mathbb{C})$.

Srivastava et al. [13] defined incomplete generalized hypergeometric functions ${}_p\Gamma_q$ and ${}_p\gamma_q$ by the Mellin–Barnes type integrals in terms of incomplete Gamma functions $\Gamma(s, y)$ and $\gamma(s, y)$ as follows:

$$
{}_p\Gamma_q \left[\begin{array}{c} (e_1, y) \quad ,e_2, \cdots ,e_p; \\ \\ f_1, \cdots ,f_q; \end{array} z \right] := \frac{\prod\limits_{j=1}^{q} \Gamma(f_j)}{\prod\limits_{j=1}^{p} \Gamma(e_j)} \sum_{l=0}^{\infty} \frac{\Gamma(e_1 + l, y) \prod\limits_{j=2}^{p} \Gamma(e_j + l)}{\prod\limits_{j=1}^{q} \Gamma(f_j + l)} \frac{z^l}{l!}
$$

$$
= \frac{1}{2\pi i} \frac{\prod\limits_{j=1}^{q} \Gamma(f_j)}{\prod\limits_{j=1}^{p} \Gamma(e_j)} \int_{\mathfrak{L}} \frac{\Gamma(e_1 + s, y) \prod\limits_{j=2}^{p} \Gamma(e_j + s)}{\prod\limits_{j=1}^{q} \Gamma(f_j + s)} \Gamma(-s)(-z)^s ds \tag{4}
$$

$$(|arg(-z)| < \pi),$$

and

$$
{}_p\gamma_q \left[\begin{array}{c} (e_1, y) \quad ,e_2, \cdots ,e_p; \\ \\ f_1, \cdots ,f_q; \end{array} z \right] := \frac{\prod\limits_{j=1}^{q} \Gamma(f_j)}{\prod\limits_{j=1}^{p} \Gamma(e_j)} \sum_{l=0}^{\infty} \frac{\gamma(e_1 + l, y) \prod\limits_{j=2}^{p} \Gamma(e_j + l)}{\prod\limits_{j=1}^{q} \Gamma(f_j + l)} \frac{z^l}{l!}
$$

$$
= \frac{1}{2\pi i} \frac{\prod\limits_{j=1}^{q} \Gamma(f_j)}{\prod\limits_{j=1}^{p} \Gamma(e_j)} \int_{\mathfrak{L}} \frac{\gamma(e_1 + s, y) \prod\limits_{j=2}^{p} \Gamma(e_j + s)}{\prod\limits_{j=1}^{q} \Gamma(f_j + s)} \Gamma(-s)(-z)^s ds \tag{5}
$$

$$(|arg(-z)| < \pi),$$

where, $\mathfrak{L}$ is the Mellin–Barnes type contour, having $\tau - i\infty$ as the starting point and $\tau + i\infty$ $(\tau \in \mathfrak{R})$ as the end point with the usual indentations to separate a set of poles from another of the integrand in each and every case.

The incomplete H-functions $\gamma_{p,q}^{m,n}(z)$ and $\Gamma_{p,q}^{m,n}(z)$ have introduced and investigated by Srivastava et al. [14] (Equations (2.1)–(2.4)) in the following manner:

$$\Gamma_{p,q}^{m,n}(z) = \Gamma_{p,q}^{m,n}\left[z \left| \begin{array}{c} (e_1, E_1, y), (e_j, E_j)_{2,p} \\ \\ (f_j, F_j)_{1,q} \end{array} \right. \right] = \Gamma_{p,q}^{m,n}\left[z \left| \begin{array}{c} (e_1, E_1, y), (e_2, E_2), \cdots, (e_p, E_p) \\ \\ (f_1, F_1), (f_2, F_2), \cdots, (f_q, F_q) \end{array} \right. \right] \tag{6}$$

$$:= \frac{1}{2\pi i} \int_{\mathfrak{L}} f(s, y) \, z^{-s} ds,$$

where,

$$f(s, y) = \frac{\Gamma(1 - e_1 - E_1 s, y) \prod\limits_{j=1}^{m} \Gamma(f_j + F_j s) \prod\limits_{j=2}^{n} \Gamma(1 - e_j - E_j s)}{\prod\limits_{j=m+1}^{q} \Gamma(1 - f_j - F_j s) \prod\limits_{j=n+1}^{p} \Gamma(e_j + E_j s)}.$$

and

$$\gamma_{p,q}^{m,n}(z) = \gamma_{p,q}^{m,n}\left[z \left| \begin{array}{c} (e_1, E_1, y), (e_j, E_j)_{2,p} \\ \\ (f_j, F_j)_{1,q} \end{array} \right. \right] = \gamma_{p,q}^{m,n}\left[z \left| \begin{array}{c} (e_1, E_1, y), (e_2, E_2), \cdots, (e_p, E_p) \\ \\ (f_1, F_1), (f_2, F_2), \cdots, (f_q, F_q) \end{array} \right. \right] \tag{7}$$

$$:= \frac{1}{2\pi i} \int_{\mathfrak{L}} F(s, y) \, z^{-s} ds,$$

where,

$$F(s, y) = \frac{\gamma(1 - e_1 - E_1 s, y) \prod\limits_{j=1}^{m} \Gamma(f_j + F_j s) \prod\limits_{j=2}^{n} \Gamma(1 - e_j - E_j s)}{\prod\limits_{j=m+1}^{q} \Gamma(1 - f_j - F_j s) \prod\limits_{j=n+1}^{p} \Gamma(e_j + E_j s)}.$$

The incomplete H-functions $\Gamma_{p,q}^{m,n}(z)$ and $\gamma_{p,q}^{m,n}(z)$ in (6) and (7) respectively, exist for all $y \geqq 0$ under the same set of conditions and the same set of contour stated in the articles presented by Kilbas et al. [15], Mathai and Saxena [16] and Mathai et al. [17].

Aforementioned functions possess numerous special cases out of which we choose to enumerate a few:

(i) Taking $y = 0$ in (6), incomplete H-function $\Gamma_{p,q}^{m,n}(z)$ reduce to the commonly used Fox's H-function [18] as follows:

$$\Gamma_{p,q}^{m,n}\left[z \left| \begin{array}{c} (e_1, E_1, 0), (e_2, E_2), \cdots, (e_p, E_p) \\ \\ (f_1, F_1), (f_2, F_2), \cdots, (f_q, F_q) \end{array} \right. \right] = H_{p,q}^{m,n}\left[z \left| \begin{array}{c} (e_1, E_1), (e_2, E_2), \cdots, (e_p, E_p) \\ \\ (f_1, F_1), (f_2, F_2), \cdots, (f_q, F_q) \end{array} \right. \right]. \tag{8}$$

(ii) Setting $m = 1, n = p$ and replacing q by $q + 1$ and taking suitable parameters, the functions (6) and (7) reduces to incomplete Fox-Wright functions ${}_p\Psi_q^{(\Gamma)}$ and ${}_p\Psi_q^{(\gamma)}$ (see for details, [14] [P. 132, Equation (6.3) and (6.4)]):

$$
\Gamma_{p,q+1}^{1,p} \left[-z \left| \begin{array}{c} (1-e_1,E_1,y),(1-e_j,E_j)_{2,p} \\ \\ (0,1),(1-f_j,F_j)_{1,q} \end{array} \right. \right] = {}_p\Psi_q^{(\Gamma)} \left[\begin{array}{cc} (e_1,E_1,y),(e_j,E_j)_{2,p}; & \\ & z \\ (f_j,F_j)_{1,q}; & \end{array} \right]. \tag{9}
$$

and

$$
\gamma_{p,q+1}^{1,p} \left[-z \left| \begin{array}{c} (1-e_1,E_1,y),(1-e_j,E_j)_{2,p} \\ \\ (0,1),(1-f_j,F_j)_{1,q} \end{array} \right. \right] = {}_p\Psi_q^{(\gamma)} \left[\begin{array}{cc} (e_1,E_1,y),(e_j,E_j)_{2,p}; & \\ & z \\ (f_j,F_j)_{1,q}; & \end{array} \right]. \tag{10}
$$

(iii) Further, taking $y = 0$ in (9) incomplete Fox–Wright function ${}_p\Psi_q^{(\Gamma)}$ reduces to the well known Fox–Wright function ${}_p\Psi_q$ (see for details, [18] ([p. 39, Equation (2.6.11)]))

$$
{}_p\Psi_q^{(\Gamma)} \left[\begin{array}{cc} (e_1,E_1,0),(e_j,E_j)_{2,p}; & \\ & z \\ (f_j,F_j)_{1,q}; & \end{array} \right] = {}_p\Psi_q \left[\begin{array}{cc} (e_1,E_1),(e_j,E_j)_{2,p}; & \\ & z \\ (f_j,F_j)_{1,q}; & \end{array} \right]. \tag{11}
$$

(iv) Taking $E_j = F_k = 1$ $(j = 1, \cdots, p, k = 1, \cdots, q)$ in (9) and (10), the incomplete H-functions reduces to the incomplete generalized hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$ (see [13]):

$$
{}_p\Psi_q^{(\Gamma)} \left[\begin{array}{cc} (e_1,1,y),(e_j,1)_{2,p}; & \\ & z \\ (f_j,1)_{1,q}; & \end{array} \right] = {}_p\Gamma_q \left[\begin{array}{cc} (e_1,y),e_2,\cdots,e_p; & \\ & z \\ f_1,\cdots,f_q; & \end{array} \right]. \tag{12}
$$

and

$$
{}_p\Psi_q^{(\gamma)} \left[\begin{array}{cc} (e_1,1,y),(e_j,1)_{2,p}; & \\ & z \\ (f_j,1)_{1,q}; & \end{array} \right] = {}_p\gamma_q \left[\begin{array}{cc} (e_1,y),e_2,\cdots,e_p; & \\ & z \\ f_1,\cdots,f_q; & \end{array} \right]. \tag{13}
$$

The generalized M-series was introduce and investigate by Sharma et al. [19] as follows:

$$
{}_r\mathfrak{M}_s^{\rho,\sigma} \left[\begin{array}{c} \alpha_1, \cdots \alpha_r; \\ \\ \beta_1, \cdots \beta_s; \end{array} w \right] = {}_r\mathfrak{M}_s^{\rho,\sigma}(w) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_r)_k}{(\beta_1)_k \cdots (\beta_s)_k} \frac{w^k}{\Gamma(\rho k + \sigma)}; \qquad (\rho,\sigma,w \in \mathbf{C}, \Re(\rho) > 0). \tag{14}
$$

Here, we omit the details of convergence conditions and complete description due to all details are given in [19].

We are also required to recollect Oberttinger's integral [20] formula which is defined as follows:

$$
\int_0^{\infty} x^{\mu-1}(x + a + \sqrt{x^2 + 2ax})^{-\lambda} dx = 2\lambda a^{-\lambda} \left(\frac{a}{2} \right)^{\mu} \frac{\Gamma(2\mu)\Gamma(\lambda - \mu)}{\Gamma(1 + \lambda + \mu)}, \tag{15}
$$

provided that $0 < \Re(\mu) < \Re(\lambda)$.

2. Oberhettinger's Integral Formula

Numerous author time to time establish unified integral formulas involving several kind of special function (see, for example, [21–23], see also, [24–26]). So, in this section, we establish Oberhettinger's integral formulas of the incomplete *H*-functions (6) and (7). The Oberhettinger's integral formulas established in the present work are very useful to obtain the Mellin transform of various simpler special functions. The Mellin transforms of special functions find their applications mathematical statistics, number theory, and the theory of asymptotic expansions. Specially, the key findings of the present study are very useful in solving the problems arising in digital signals, image processing, finance and ship target recognition by sonar system and radar signals.

Theorem 1. *If $\lambda, \mu \in \mathbb{C}$ with $0 < \Re(\mu) < \Re(\lambda)$ and $x > 0$, then the following integral formula holds:*

$$
\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\, {}_r\mathfrak{M}_s^{\rho,\sigma}\left[\begin{array}{c} \alpha_1, \cdots \alpha_r; \\ \\ \beta_1, \cdots \beta_s; \end{array} \frac{z_1}{x+a+\sqrt{x^2+2ax}}\right]
$$

$$
\Gamma_{p,q}^{m,n}\left[\left.\frac{z_2}{x+a+\sqrt{x^2+2ax}}\;\right|\;\begin{array}{c}(e_1,E_1,y),(e_j,E_j)_{2,p}\\ \\ (f_j,F_j)_{1,q}\end{array}\right]dx = 2^{1-\mu}a^{\mu-\lambda}\Gamma(2\mu)\sum_{k=0}^\infty \frac{(\alpha_1)_k\cdots(\alpha_r)_k}{(\beta_1)_k\cdots(\beta_s)_k}\frac{\left(\frac{z_1}{a}\right)^k}{\Gamma(\rho k+\sigma)}
$$

$$
\Gamma_{p+2,q+2}^{m,n+2}\left[\left.\frac{z_2}{a}\;\right|\;\begin{array}{c}(e_1,E_1,y),(e_j,E_j)_{2,p},(-\lambda-k,1),(1-\lambda-k+\mu,1)\\ \\ (f_j,F_j)_{1,q},(1-\lambda-k,1),(-\lambda-k-\mu,1)\end{array}\right]. \tag{16}
$$

provided that the conditions of incomplete H-function $\Gamma_{p,q}^{m,n}(z)$ in (6) are satisfied.

Proof. For the sake of convenience, let us denote the left hand side of the assertion (16) by $\mathfrak{I}$. Using (6) and (14) in the right side of (16)

$$
\mathfrak{I} = \int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\cdot\sum_{k=0}^\infty \frac{(\alpha_1)_k\cdots(\alpha_r)_k}{(\beta_1)_k\cdots(\beta_s)_k}\frac{1}{\Gamma(\rho k+\sigma)}\left(\frac{z_1}{(x+a+\sqrt{x^2+2ax})}\right)^k\cdot
$$

$$
\frac{1}{2\pi i}\int_{\mathfrak{L}}\left(\frac{z_2}{x+a+\sqrt{x^2+2ax}}\right)^{-\xi}f(\xi,y)d\xi\;dx
$$

where $f(\xi,y)$ is defined in (6).

Next, upon changing the order of summation, integration and contour integral involved therein (which is permissible under the stated conditions), we obtain

$$
\mathfrak{I} = \sum_{k=0}^\infty \frac{(\alpha_1)_k\cdots(\alpha_r)_k}{(\beta_1)_k\cdots(\beta_s)_k}\frac{z_1^k}{\Gamma(\rho k+\sigma)}\cdot\frac{1}{2\pi i}\int_{\mathfrak{L}}f(\xi,y)\,z_2^{-\xi}\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-(\lambda+k-\xi)}\,dx\;d\xi
$$

Finally, applying (15) in the above integral and reinterpreting it in the form of incomplete *H*-function $\Gamma_{p,q}^{m,n}(z)$, we arrive at the assertion (16). □

Theorem 2. *For* $\lambda, \mu \in \mathbb{C}$ *with* $0 < \Re(\mu) < \Re(\lambda)$ *and* $x > 0$, *then the following improper integral holds:*

$$
\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} {}_r\mathfrak{M}_s^{\rho,\sigma}\left[\begin{array}{c}\alpha_1,\cdots\alpha_r; \\[4pt] \beta_1,\cdots\beta_s;\end{array}\ \dfrac{z_1}{x+a+\sqrt{x^2+2ax}}\right]\cdot
$$
$$
\gamma_{p,q}^{m,n}\left[\dfrac{z_2}{x+a+\sqrt{x^2+2ax}}\ \left|\ \begin{array}{c}(e_1,E_1,y),(e_j,E_j)_{2,p} \\[8pt] (f_j,F_j)_{1,q}\end{array}\right.\right]dx = 2^{1-\mu}a^{\mu-\lambda}\Gamma(2\mu)\sum_{k=0}^\infty \dfrac{(\alpha_1)_k\cdots(\alpha_r)_k}{(\beta_1)_k\cdots(\beta_s)_k}\dfrac{\left(\frac{z_1}{a}\right)^k}{\Gamma(\rho k+\sigma)}\cdot
$$
$$
\gamma_{p+2,q+2}^{m,n+2}\left[\dfrac{z_2}{a}\ \left|\ \begin{array}{c}(e_1,E_1,y),(e_j,E_j)_{2,p},(-\lambda-k,1),(1-\lambda-k+\mu,1) \\[8pt] (f_j,F_j)_{1,q},(1-\lambda-k,1),(-\lambda-k-\mu,1)\end{array}\right.\right]. \tag{17}
$$

provided that the condition of incomplete H-function $\gamma_{p,q}^{m,n}(z)$ *in* (7) *are satisfied.*

Proof. For the sake of convenience, let us denote the left hand side of the assertion (17) by $\mathfrak{J}$. Using (7) and (14) in the left side of (17)

$$
\mathfrak{J} = \int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\cdot\sum_{k=0}^\infty\dfrac{(\alpha_1)_k\cdots(\alpha_r)_k}{(\beta_1)_k\cdots(\beta_s)_k}\dfrac{1}{\Gamma(\rho k+\sigma)}\left(\dfrac{z_1}{(x+a+\sqrt{x^2+2ax})}\right)^k\cdot
$$
$$
\dfrac{1}{2\pi i}\int_{\mathfrak{L}}\left(\dfrac{z_2}{x+a+\sqrt{x^2+2ax}}\right)^{-\xi}F(\xi,y)d\xi\ dx
$$

where $F(\xi,y)$ is defined in (7).

Next, upon changing the order of summation, integration and contour integral involved therein (which is permissible under the stated conditions), we obtain

$$
\mathfrak{J} = \sum_{k=0}^\infty\dfrac{(\alpha_1)_k\cdots(\alpha_r)_k}{(\beta_1)_k\cdots(\beta_s)_k}\dfrac{z_1^k}{\Gamma(\rho k+\sigma)}\cdot\dfrac{1}{2\pi i}\int_{\mathfrak{L}}F(\xi,y)\ z_2^{-\xi}\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-(\lambda+k-\xi)}\ dx\ d\xi
$$

Finally, applying (15) in the above integral and reinterpreting it in the form of incomplete *H*-function $\gamma_{p,q}^{m,n}(z)$, we arrive at the assertion (17). $\square$

3. Special Cases and Remarks

In this section, we recorded some interesting special cases of main results (Theorems 1 and Theorem 2):

Corollary 1. *If* $\lambda,\mu \in \mathbb{C}$ *with* $0 < \Re(\mu) < \Re(\lambda)$, $x > 0$ *and incomplete H-function reduces into Fox H-function with the help of* (8), *then the following integral formula holds:*

$$
\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}{}_r\mathfrak{M}_s^{\rho,\sigma}\left[\begin{array}{c}\alpha_1,\cdots\alpha_r; \\[4pt] \beta_1,\cdots\beta_s;\end{array}\ \dfrac{z_1}{x+a+\sqrt{x^2+2ax}}\right]\mathcal{H}_{p,q}^{m,n}\left[\dfrac{z_2}{x+a+\sqrt{x^2+2ax}}\ \left|\ \begin{array}{c}(e_j,E_j)_{1,p} \\[8pt] (f_j,F_j)_{1,q}\end{array}\right.\right]dx
$$
$$
= 2^{1-\mu}a^{\mu-\lambda}\Gamma(2\mu)\dfrac{\prod_{j=1}^s\Gamma(\beta_j)}{\prod_{j=1}^r\Gamma(\alpha_j)}\mathcal{H}_{2,2:r+1,s+2:p,q}^{0,2:1,r+1:m,n}\left[\begin{array}{c}-\frac{z_1}{a} \\[8pt] \frac{z_2}{a}\end{array}\ \left|\ \begin{array}{c}A^* \\[8pt] B^*\end{array}\right.\right]. \tag{18}
$$

where

$$A^* = (-\lambda : 1, 1), (1 - \lambda + \mu : 1, 1) : (1 - \alpha_j, 1)_{1,p}, (0, 1) : (e_j, E_j)_{1,p}$$
$$B^* = (1 - \lambda : 1, 1), (-\lambda - \mu : 1, 1) : (0, 1), (1 - \beta_j, 1)_{1,q}, (1 - \sigma, \rho) : (f_j, F_j)_{1,q}$$

provided that members of both side exist.

Proof. Taking $y = 0$ in the result (16), we get the desired result. □

Corollary 2. *For $\lambda, \mu \in \mathbb{C}$ with $0 < \Re(\mu) < \Re(\lambda)$, $x > 0$ and incomplete H-functions reduces into Incomplete Wright functions, then the following integral formula holds:*

$$\int_0^\infty x^{\mu-1} (x + a + \sqrt{x^2 + 2ax})^{-\lambda} {}_r\mathfrak{M}_s^{\rho,\sigma} \left[\begin{array}{c} \alpha_1, \cdots \alpha_r; \\ \\ \beta_1, \cdots \beta_s; \end{array} \frac{z_1}{x + a + \sqrt{x^2 + 2ax}} \right]$$

$$_p\Psi_q^{(\Gamma)} \left[\begin{array}{c} (e_1, E_1, y), (e_j, E_j)_{2,p}; \\ \\ (f_j, F_j)_{1,q}; \end{array} \frac{z_2}{x + a + \sqrt{x^2 + 2ax}} \right] dx$$

$$= 2^{1-\mu} a^{\mu-\lambda} \Gamma(2\mu) \sum_{k=0}^\infty \frac{(\alpha_1)_k \cdots (\alpha_r)_k}{(\beta_1)_k \cdots (\beta_s)_k} \frac{\left(\frac{z_1}{a}\right)^k}{\Gamma(\rho k + \sigma)}$$

$$_{p+2}\Psi_{q+2}^{(\Gamma)} \left[\left. \frac{z_2}{a} \right| \begin{array}{c} (e_1, E_1, y), (e_j, E_j)_{2,p}, (1 + \lambda + k, 1), (\lambda + k - \mu, 1) \\ \\ (f_j, F_j)_{1,q}, (\lambda + k, 1), (1 + \lambda + k + \mu, 1) \end{array} \right]. \tag{19}$$

and

$$\int_0^\infty x^{\mu-1} (x + a + \sqrt{x^2 + 2ax})^{-\lambda} {}_r\mathfrak{M}_s^{\rho,\sigma} \left[\begin{array}{c} \alpha_1, \cdots \alpha_r; \\ \\ \beta_1, \cdots \beta_s; \end{array} \frac{z_1}{x + a + \sqrt{x^2 + 2ax}} \right]$$

$$_p\Psi_q^{(\gamma)} \left[\begin{array}{c} (e_1, E_1, y), (e_j, E_j)_{2,p}; \\ \\ (f_j, F_j)_{1,q}; \end{array} \frac{z_2}{x + a + \sqrt{x^2 + 2ax}} \right] dx$$

$$= 2^{1-\mu} a^{\mu-\lambda} \Gamma(2\mu) \sum_{k=0}^\infty \frac{(\alpha_1)_k \cdots (\alpha_r)_k}{(\beta_1)_k \cdots (\beta_s)_k} \frac{\left(\frac{z_1}{a}\right)^k}{\Gamma(\rho k + \sigma)}$$

$$_{p+2}\Psi_{q+2}^{(\gamma)} \left[\left. \frac{z_2}{a} \right| \begin{array}{c} (e_1, E_1, y), (e_j, E_j)_{2,p}, (1 + \lambda + k, 1), (\lambda + k - \mu, 1) \\ \\ (f_j, F_j)_{1,q}, (\lambda + k, 1), (1 + \lambda + k + \mu, 1) \end{array} \right]. \tag{20}$$

provided that each member of assertion (19) and (20) are exist.

Proof. Assuming (9) and (10), we get the required result here from those in Theorems 1 and 2. □

Corollary 3. *If* $\lambda, \mu \in \mathbb{C}$ *with* $0 < \mathfrak{R}(\mu) < \mathfrak{R}(\lambda)$, $x > 0$ *and incomplete H-function reduces into Fox–Wright function with the help of* (11), *then the following integral formula holds:*

$$
\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} {}_r\mathfrak{M}_s^{\rho,\sigma}
\left[
\begin{array}{c}
\alpha_1, \cdots \alpha_r; \\
\beta_1, \cdots \beta_s;
\end{array}
\ \frac{z_1}{x+a+\sqrt{x^2+2ax}}
\right]
$$

$$
{}_p\Psi_q
\left[
\begin{array}{c}
(e_1, E_1)_{1,p}; \\
(f_j, F_j)_{1,q};
\end{array}
\ \frac{z_2}{x+a+\sqrt{x^2+2ax}}
\right] dx
$$

$$
= 2^{1-\mu} a^{\mu-\lambda} \Gamma(2\mu) \frac{\prod\limits_{j=1}^{s} \Gamma(\beta_j)}{\prod\limits_{j=1}^{r} \Gamma(\alpha_j)} \mathcal{H}_{2,2:r+1,s+2:p,q+1}^{0,2:1,r+1:1,p}
\left[
\begin{array}{c|c}
-\frac{z_1}{a} & A^* \\
\frac{z_2}{a} & B^*
\end{array}
\right] \tag{21}
$$

where

$$
A^* = (-\lambda : 1, 1), (1-\lambda+\mu : 1, 1) : (1-\alpha_j, 1)_{1,p}, (0,1) : (1-e_j, E_j)_{1,p}
$$

$$
B^* = (1-\lambda : 1, 1), (1-\lambda-\mu : 1, 1) : (0,1), (1-\beta_j, 1)_{1,q}, (1-\sigma, \rho) : (0,1), (1-f_j, F_j)_{1,q}
$$

provided that improper integral exist.

Proof. Further, setting y = 0 in (19), we get the desired result. □

Corollary 4. *If* $\lambda, \mu \in \mathbb{C}$ *with* $0 < \mathfrak{R}(\mu) < \mathfrak{R}(\lambda)$, $x > 0$ *and incomplete H-function reduces into incomplete hypergeometric function with the help of* (12) *and* (13), *then the following integral formula holds:*

$$
\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} {}_r\mathfrak{M}_s^{\rho,\sigma}
\left[
\begin{array}{c}
\alpha_1, \cdots \alpha_r; \\
\beta_1, \cdots \beta_s;
\end{array}
\ \frac{z_1}{x+a+\sqrt{x^2+2ax}}
\right]
$$

$$
{}_p\Gamma_q
\left[
\begin{array}{c}
(e_1, y), e_2, \cdots, e_p; \\
f_1, \cdots, f_q;
\end{array}
\ \frac{z_2}{x+a+\sqrt{x^2+2ax}}
\right] dx
$$

$$
= 2^{1-\mu} a^{\mu-\lambda} \Gamma(2\mu) \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_r)_k}{(\beta_1)_k \cdots (\beta_s)_k} \frac{\left(\frac{z_1}{a}\right)^k}{\Gamma(\rho k + \sigma)}
$$

$$
{}_{p+2}\Gamma_{q+2}
\left[
\frac{z_2}{a}
\ \middle|
\begin{array}{c}
(e_1, y), e_2, \cdots, e_p, -\lambda-k, 1-\lambda-k+\mu \\
f_1, \cdots f_q, 1-\lambda-k, -\lambda-k-\mu
\end{array}
\right]. \tag{22}
$$

and

$$
\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} {}_r\mathfrak{M}_s^{\rho,\sigma} \left[\begin{array}{c} \alpha_1,\cdots\alpha_r; \\ \\ \beta_1,\cdots\beta_s; \end{array} \; \frac{z_1}{x+a+\sqrt{x^2+2ax}} \right]
$$

$$
{}_p\Upsilon_q \left[\begin{array}{c} (e_1,y),e_2,\cdots,e_p; \\ \\ f_1,\cdots,f_q; \end{array} \; \frac{z_2}{x+a+\sqrt{x^2+2ax}} \right] dx
$$

$$
= 2^{1-\mu}a^{\mu-\lambda}\Gamma(2\mu) \sum_{k=0}^\infty \frac{(\alpha_1)_k\cdots(\alpha_r)_k}{(\beta_1)_k\cdots(\beta_s)_k} \frac{\left(\frac{z_1}{a}\right)^k}{\Gamma(\rho k+\sigma)}
$$

$$
{}_{p+2}\Upsilon_{q+2} \left[\frac{z_2}{a} \; \middle| \; \begin{array}{c} (e_1,y),e_2,\cdots,e_p,-\lambda-k,1-\lambda-k+\mu \\ \\ f_1,\cdots f_q,1-\lambda-k,-\lambda-k-\mu \end{array} \right]. \tag{23}
$$

provided that both integrals exist.

Proof. Again, taking $E_j = F_j = 1$ in (16) and (17), we arrive the required result. □

Corollary 5. *For $\lambda, \mu \in \mathbb{C}$ with $0 < \Re(\mu) < \Re(\lambda)$, $x > 0$ and incomplete H-function reduces into Fox–Wright generalized hypergeometric function, then the following integral formula holds:*

$$
\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} {}_p\Psi_q \left[\begin{array}{c} (e_j,E_j)_{1,p}; \\ \\ (f_j,F_j)_{1,q}; \end{array} \; \frac{z}{x+a+\sqrt{x^2+2ax}} \right] dx
$$

$$
= 2^{1-\mu}a^{\mu-\lambda}\Gamma(2\mu) {}_{p+2}\Psi_{q+2} \left[\frac{z}{a} \; \middle| \; \begin{array}{c} (e_j,E_j)_{1,p},(1+\lambda,1),(\lambda-\mu,1) \\ \\ (f_j,F_j)_{1,q},(\lambda,1),(1+\lambda+\mu,1) \end{array} \right]. \tag{24}
$$

provided that each member of assertion (24) are exist.

Proof. Assuming (16), M-series reduces into unity and with the help of (11), we get the required result here from those in Theorem 1. □

Remark 1. *If M-series reduces into unity and incomplete H-function into Bessel function of the first kind $J_\nu(z)$ in (16), then result recorded by Choi et al. [27].*

Remark 2. *If M-series reduces into unity and incomplete H-function into generalized Bessel function of the first kind $W_\nu(z)$ in (16), then result recorded by Choi et al. [28].*

4. Conclusions

In this work, we have derived some interesting integrals involving the product of M-series and incomplete *H*-functions, which are expressed in terms of incomplete *H*-functions. We have also given some special cases by specializing the parameters of M-series and incomplete *H*-functions (for example, Fox's *H*-Function, Incomplete Fox–Wright functions, Fox–Wright functions and Incomplete generalized hypergeometric functions) and also listed few known results. The results derived in present investigation are general in nature and can be used to obtain many integrals having real world applications.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bansal, M.K.; Choi, J. A Note on Pathway Fractional Integral Formulas Associated with the Incomplete *H*-functions. *Int. J. Appl. Comput. Math.* **2019**, *5*, 133. [CrossRef]
2. Bansal, M.K.; Jolly, N.; Jain, R.; Kumar, D. An integral operator involving generalized Mittag-Leffler function and associated fractional calculus results. *J. Anal.* **2019**, *27*, 727–740. [CrossRef]
3. Chaurasia, V.B.L.; Kumar, D. The integration of certain product involving special functions. *Sci. Ser. A Math. Sci.* **2010**, *19*, 7–12.
4. Chaurasia, V.B.L.; Singh, J. Certain integral properties pertaining to special functions. *Sci. Ser. A Math. Sci.* **2010**, *19*, 1–6.
5. Kumar, D.; Ayant, F.; Kumar, D. A new class of integrals involving generalized hypergeometric function and multivariable Aleph-function. *Kragujevac J. Math.* **2020**, *44*, 539–550.
6. Brychkov, Y.A. *Handbook of Special Functions: Derivatives, Integrals, Series and Other Formulas*; CRC Press: Boca Raton, FL, USA, 2008.
7. Garg, M.; Mittal, S. On a new unified integral. *Proc. Indian Acad. Sci. Math. Sci.* **2004**, *114*, 99–101. [CrossRef]
8. Srivastava, H.M.; Exton, H. A generalization of the Weber-Schafheitlin integral. *J. Reine Angew. Math.* **1979**, *309*, 1–6.
9. Epstein, B. Some Applications of the Mellin Transform in Statistics. *Ann. Math. Stat.* **1948**, *19*, 370–379. [CrossRef]
10. Yao K. Spherically Invariant Random Processes: Theory and Applications. In *Communications, Information and Network Security*; The Springer International Series in Engineering and Computer Science (Communications and Information Theory); Bhargava V.K., Poor H.V., Tarokh V., Yoon S., Eds.; Springer: Berlin/Heidelberg, Germany, 2003; Volume 712, pp. 315–332.
11. Yao, K.; Simon, M.K.; Biglieri, E. A Unified Theory on Wireless Communications Fading Statistics based on SIRP. In Proceedings of the Fifth IEEE Workshopon Signal Processing Advances in Wireless Communications, Lisbon, Portugal, 11–14 July 2004; pp. 135–139.
12. Chaurasia, V.B.L.; Kumar, D. Application of special functions and SIRP in wireless communication fading statistics. *Glob. J. Sci. Front. Res.* **2010**, *10*, 14–19.
13. Srivastava, H.M.; Chaudhary, M.A.; Agarwal, R.P. The Incomplete Pochhammer Symbols and Their Applications to Hypergeometric and Related Functions. *Integral Transform. Spec. Funct.* **2012**, *23*, 659–683. [CrossRef]
14. Srivastava, H.M.; Saxena, R.K.; Parmar, R.K. Some Families of the Incomplete *H*-functions and the Incomplete $\overline{H}-$functions and Associated Integral Transforms and Operators of Fractional Calculus with Applications. *Russ. J. Math. Phys.* **2018**, *25*, 116–138. [CrossRef]
15. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; (North-Holland Mathematical Studies); Elsevier (North Holland) Science Publishers: Amsterdam, The Netherland; London, UK; New York, NY, USA, 2006; p. 204.
16. Mathai, A.M.; Saxena, R.K. *The H-Function with Applications in Statistics Other Disciplines*; Wiley Eastern: New Delhi, India; Wiley Halsted: New York, NY, USA, 1978.
17. Mathai, A.M.; Saxena, R.K.; Haubold, H.J. *The H-function: Theory and Applications*; Springer: New York, NY, USA, 2009.
18. Srivastava, H.M.; Gupta, K.C.; Goyal, S.P. *The H-Functions of One and Two Variables with Applications*; South Asian Publishers: New Delhi, India, 1982.
19. Sharma, M.; Jain, R. A Note on a Generalized M-Series as a SpecialL Function of Fractional Calculus. *Fract. Calc. Appl. Anal.* **2009**, *12*, 449–452.

20. Oberhettinger, F. *Tables of Mellin Transforms*; Springer: New York, NY, USA, 1974.
21. Choi, J.; Hasanov, A.; Srivastava, H.M.; Turaev, M. Integral representations for Srivastava's triple hypergeometric functions. *Taiwan. J. Math.* **2011**, *15*, 2751–2762. [CrossRef]
22. Khan, N.U.; Kashmin, T. Some integrals for the generalized Bessel-Maitland functions. *Electron. J. Math. Anal. Appl.* **2016**, *4*, 139–149.
23. Nisar, K.S.; Rahman, G.; Mubeen, S.; Arshad, M. Certain new integral formulas involving the generalized k-Bessel function. *Commun. Numer. Anal.* **2017**, *2017*, 84–90. [CrossRef]
24. Bansal, M.K.; Kumar, D.; Jain, R. Interrelationships Between Marichev–Saigo–Maeda Fractional Integral Operators, the Laplace Transform and the $\bar{H}$-Function. *Int. J. Appl. Comput. Math.* **2019**, *5*, 103. [CrossRef]
25. Bansal, M.K.; Kumar, D.; Jain, R. A Study of Marichev-Saigo-Maeda Fractional Integral Operators Associated with S-Generalized Gauss Hypergeometric Function. *KYUNGPOOK Math. J.* **2019**, *59*, 433–443.
26. Srivastava, H.M.; Bansal, M.K.; Harjule, P. A study of fractional integral operators involving a certain generalized multi-index Mittag-Leffler function. *Math. Meth. Appl. Sci.* **2018**, *41*, 6108–6121. [CrossRef]
27. Choi, J.; Agarwal, P. Certain unified integrals associated with Bessel functions. *Bound. Value Probl.* **2013**, *2013*, 95. [CrossRef]
28. Choi, J.; Agarwal, P.; Mathur, S.; Purohit, S.D. Certain new integral formulas involving the generalized Bessel functions. *Bull. Korean Math. Soc.* **2014**, *4*, 995–1003. [CrossRef]