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On the Betti and Tachibana Numbers of Compact Einstein Manifolds

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Abstract: Throughout the history of the study of Einstein manifolds, researchers have sought relationships between the curvature and topology of such manifolds. In this paper, first, we prove that a compact Einstein manifold (M, g) with an Einstein constant $\alpha > 0$ is a homological sphere when the minimum of its sectional curvatures $> \alpha/(n+2)$; in particular, (M, g) is a spherical space form when the minimum of its sectional curvatures $> \alpha/n$. Second, we prove two propositions (similar to the above ones) for Tachibana numbers of a compact Einstein manifold with $\alpha < 0$.

Keywords: Einstein manifold; sectional curvature; Betti number; Tachibana number; spherical space form

MSC: 53C20; 53C43; 53C44

1. Introduction

The study of Einstein manifolds has a long history in Riemannian geometry. Throughout the history of the study of Einstein manifolds, researchers have sought relationships between curvature and topology of such manifolds. A. Besse [1] summarized the results. We present here some interesting facts related to the classification of all compact Einstein manifolds satisfying a suitable curvature inequality, which is one of the subjects of our research.

Recall that an n-dimensional $(n \geq 2)$ connected manifold M with a Riemannian metric g is said to be an *Einstein manifold* with *Einstein constant* α if its Ricci tensor satisfies $\mathrm{Ric} = \alpha\, g$; moreover, we have $\alpha = s/n$ for its scalar curvature s. Therefore, any Einstein manifold of dimensions two and three is a space form (i.e., has constant sectional curvature). The study of Einstein manifolds is more complicated in dimension four and higher (see [1] (p. 44)).

An important problem in differential geometry is to determine whether a smooth manifold admits an Einstein metric. When $\alpha > 0$, the example are symmetric spaces, which include the sphere $\mathbb{S}^n(1)$ with $\alpha = n-1$ and the sectional curvature $\sec = 1$, the product of two spheres $\mathbb{S}^n(1) \times \mathbb{S}^n(1)$ with $\alpha = n-1$ and $0 \leq \sec \leq 1$, and the complex projective space $\mathbb{C}P^m = \mathbb{S}^{2m+1}/\,\mathbb{S}^1$ with the Fubini–Study metric, $\alpha = 2m+2$ and $1 \leq \sec \leq 4$ (see [2] (pp. 86, 118, 149–150)). Recall that if (M, g) is a compact Einstein manifold with curvature bounds of the type $3n/(7n-4) < \sec \leq 1$, then (M, g) is isometric to a spherical space form. This might be not the best estimate: for $n = 4$ the sharp bound is $1/4$ (see [1] (p. 6)). In both these cases, the manifolds are real *homology spheres* (see [3] (p. XVI)). Therefore, any such manifold has the homology groups of an n-sphere; in particular, its Betti numbers are $b_1(M) = \ldots = b_{n-1}(M) = 0$.

One of the basic problems in Riemannian geometry was to classify Einstein four-manifolds with positive or nonnegative sectional curvature in the categories of either topology, diffeomorphism, or isometry (see, for example, [4–7]). It was conjectured that an Einstein four-manifold with $\alpha > 0$ and non-negative sectional curvature must be either $\mathbb{S}^4$, $\mathbb{C}\mathrm{P}^2$, $\mathbb{S}^2(1) \times \mathbb{S}^2(1)$ or a quotient. For example, if the maximum of the sectional curvatures of a compact Einstein four-manifold is bounded above by $(2/3)\,\alpha$, or if $\alpha = 1$ and the minimum of the sectional curvatures $\geq (1/6)(2-\sqrt{2})$, then the manifold is isometric to $\mathbb{S}^4$, $\mathbb{R}\mathrm{P}^4$ or $\mathbb{C}\mathrm{P}^2$ (see [6]). Classification of four-dimensional complete Einstein manifolds with $\alpha > 0$ and pinched sectional curvature was obtained in [7].

Here, we consider this problem from another side. Given a Riemannian manifold (M, g), the notion of symmetric *curvature operator* $\bar{R}$, acting on the space $\Lambda^2 M$ of 2-forms, is an important invariant of a Riemannian metric (see [2] (p. 83); [8,9]). The Tachibana Theorem (see [10]) asserts that a compact Einstein manifold (M, g) with $\bar{R} > 0$ is a spherical space form. Later on, it was proved that compact manifolds with $\bar{R} > 0$ are spherical space forms (see [11]).

Denote by $\mathring{R}$ the symmetric *curvature operator of the second kind*, acting on the space $S_0^2 M$ of traceless symmetric two-tensors (see [1] (p. 52); [9,12]). Kashiwada (see [9]) proved that a compact Einstein manifold with $\mathring{R} > 0$ is a spherical space form. This statement is an analogue of the theorem of Tachibana in [10]. In contrast, if a complete Riemannian manifold (M, g) satisfies $\sec \geq \delta > 0$, then M is compact with $\operatorname{diam}(M, g) \leq \pi/\sqrt{\delta}$ (see [2] (p. 251)).

Remark 1 (By [2] (Theorem 10.3.7))**.** *There are manifolds with metrics of positive or nonnegative sectional curvature but not admitting any metric with* $\bar{R} \geq 0$ *(see also [2] (p. 352)). In particular, for three-dimensional manifolds the inequality* $\sec > 0$ *is equivalent to the inequality* $\bar{R} > 0$ *(see [9]).*

Using Kashiwada's theorem from [9] we can prove the following.

Theorem 1. *Let (M, g) be a compact Einstein manifold with Einstein constant $\alpha > 0$, and let δ be the minimum of its positive sectional curvature. If $\delta > \alpha/n$, then (M, g) is a spherical space form.*

We can present a generalization of above result in the following form.

Theorem 2. *Let (M, g) be a compact Einstein manifold with Einstein constant $\alpha > 0$ and let δ be the minimum of its positive sectional curvature. If $\delta > \alpha/(n+2)$, then (M, g) is a homological sphere.*

Obviously, $\mathbb{S}^n(1) \times \mathbb{S}^n(1)$ is not an example for Theorem 1 because the minimum of its sectional curvature is zero and $\alpha = n - 1$. On the other hand, the complex projective space $\mathbb{C}\mathrm{P}^m$ is an Einstein manifold with $\alpha = 2m + 2$ and sectional curvature bounded below by $\delta = 1$. Then the inequality $\alpha < (n+2)\,\delta$ can be rewritten in the form $\delta > 1$ because $n = 2m$. Therefore, $\mathbb{C}\mathrm{P}^m$ is not an example for Theorem 1. Moreover, all even dimensional Riemannian manifolds with positive sectional curvature have vanishing odd-dimensional homology groups. Thus, Theorem 1 complements this statement (see [2] (p. 328)).

Let (M, g) be an n-dimensional compact connected Riemannian manifold. Denote by $\Delta^{(p)}$ the *Hodge Laplacian* acting on differential p-forms on M for $p = 1, \ldots, n-1$. The spectrum of $\Delta^{(p)}$ consists of an unbounded sequence of nonnegative eigenvalues which starts from zero if and only if the p-th Betti number $b_p(M)$ of (M, g) does not vanish (see [13]). The sequence of positive eigenvalues of $\Delta^{(p)}$ is denoted by

$$0 < \lambda_1^{(p)} < \ldots < \lambda_m^{(p)} < \ldots \to \infty.$$

In addition, if $F_p(\omega) \geq \sigma > 0$ (see Equation (4) of F_p) at every point of M, then $\lambda_1^{(p)} \geq \sigma$ (see [13] (p. 342)). Using this and Theorem 1, we get the following.

Corollary 1. *Let (M, g) be a compact Einstein manifold with positive Einstein constant α and sectional curvature bounded below by a constant $\delta > 0$ such that $\delta > \alpha/(n+2)$. Then the first eigenvalue $\lambda_1^{(p)}$ of the Hodge Laplacian $\Delta^{(p)}$ satisfies the inequality $\lambda_1^{(p)} \geq (1/3)\,((n+2)\,\delta - \alpha)\,(n-p)$.*

Remark 2. *In particular, if (M, g) is a Riemannian manifold with curvature operator of the second kind bounded below by a positive constant $\rho > 0$, then using the main theorem from [14], we conclude that $\lambda_1^{(p)} \geq \rho\,(n-p)$.*

Conformal Killing p-forms $(p = 1, \ldots, n-1)$ were defined on Riemannian manifolds more than fifty years ago by S. Tachibana and T. Kashiwada (see [15,16]) as a natural generalization of conformal Killing vector fields.

The vector space of conformal Killing p-forms on a compact Riemannian manifold (M, g) has finite dimension $t_p(M)$ named the *Tachibana number* (see e.g., [17–19]). Tachibana numbers $t_1(M), \ldots, t_{n-1}(M)$ are conformal scalar invariants of (M, g) satisfying the duality condition $t_p(M) = t_{n-p}(M)$. The condition is an analog of the *Poincaré duality* for Betti numbers. Moreover, Tachibana numbers $t_1(M), \ldots, t_{n-1}(M)$ are equal to zero on a compact Riemannian manifold with negative curvature operator or negative curvature operator of the second kind (see [18,19]).

We obtain the following theorem, which is an analog of Theorem 1.

Theorem 3. *Let (M, g) be an Einstein manifold with sectional curvature bounded above by a negative constant $-\delta$ such that $\delta > -\alpha/(n+2)$ for the Einstein constant α. Then Tachibana numbers $t_1(M), \ldots, t_{n-1}(M)$ are zero.*

2. Proof of Results

Let (M, g) be an n-dimensional $(n \geq 2)$ Riemannian manifold and let R_{ijkl} and R_{ij} be, respectively, the components of the Riemannian curvature tensor and the Ricci tensor in orthonormal basis $\{e_1, \ldots, e_n\}$ of T_xM at any point $x \in M$. We consider an arbitrary symmetric two-tensor φ on (M, g). At any point $x \in M$, we can diagonalize φ with respect to g, using orthonormal basis $\{e_1, \ldots, e_n\}$ of T_xM. In this case, the components of φ have the form $\varphi_{ij} = \lambda_i\,\delta_{ij}$. Let $\sec(e_i, e_j)$ be the sectional curvature of the plane of T_xM generated by e_i and e_j. We can express $\sec(e_i, e_j)$ in the following form (see [1] (p. 436); [20]):

$$\frac{1}{2}\sum\nolimits_{i \neq j} \sec(e_i, e_j)\,(\lambda_i - \lambda_j)^2 = R_{ijlk}\varphi^{ik}\varphi^{jl} + R_{ij}\varphi^{ik}\varphi_k^j \tag{1}$$

If (M, g) is an Einstein manifold and its sectional curvature satisfies the inequality $\sec \geq \delta$ for a positive constant δ, then from Equation (1) we obtain the inequality

$$R_{ijlk}\varphi^{ik}\varphi^{jl} + \frac{s}{n}\,\varphi^{ik}\varphi_{ik} \geq (\delta/2)\sum\nolimits_{i \neq j}(\lambda_i - \lambda_j)^2. \tag{2}$$

If $\mathrm{trace}_g\,\varphi = \sum_i \lambda_i = 0$, then the identity holds $\sum_i (\lambda_i)^2 = -2\sum_{i<j}\lambda_i\lambda_j$. In this case, the following identities are true:

$$\frac{1}{2}\sum_{i \neq j}(\lambda_i - \lambda_j)^2 = (n-1)\sum_i(\lambda_i)^2 - 2\sum_{i<j}\lambda_i\lambda_j = n\sum_i(\lambda_i)^2 = n\|\,\varphi\,\|^2.$$

Then the inequality in Equation (2) can be rewritten in the form

$$R_{ijlk}\varphi^{ik}\varphi^{jl} + \frac{s}{n}\,\varphi^{ik}\varphi_{ik} \geq \; n\,\delta\|\varphi\|^2. \tag{3}$$

From Equation (3) we obtain the inequality

$$R_{ijlk}\varphi^{ik}\varphi^{jl} \geq (n\,\delta - \alpha)\|\varphi\,\|^2.$$

Then $\overset{\circ}{R} > 0$ for the case when $\alpha < n\,\delta$, where $\alpha = s/n$ is the Einstein constant of (M, g). If (M, g) is compact then it is a spherical space form (see [9]). Theorem 1 is proven.

Define the quadratic form

$$F_p(\omega) = R_{ij}\,\omega^{i\,i_2\ldots\,i_p}\omega^{j}_{\;i_2\ldots\,i_p} - \frac{p-1}{2}\,R_{ijkl}\,\omega^{ij\,i_3\ldots\,i_p}\,\omega^{kl}_{\;i_2\ldots\,i_p} \tag{4}$$

for the components $\omega_{i_1\ldots i_p} = \omega(e_{i_1}, \ldots, e_{i_p})$ of an arbitrary differential p-form ω. If the quadratic form $F_p(\omega)$ is positive definite on a compact Riemannian manifold (M, g), then the p-th Betti number of the manifold vanishes (see [21] (p. 61); [3] (p. 88)). At the same time, in [22] the following inequality

$$F_p(\omega) \geq p\,(n-p)\,\varepsilon\,\|\omega\|^2 > 0$$

was proved for any nonzero p-form ω on a Riemannian manifold with $\bar{R} \geq \varepsilon > 0$. On the other hand, in [14] the inequality

$$F_p(\omega) \geq p(n-p)\,\delta\|\omega\|^2 > 0$$

was proved for any nonzero p-form ω on a Riemannian manifold with $\overset{\circ}{R} \geq \delta > 0$. In these cases, $b_1(M), \ldots, b_{n-1}(M)$ are zero (see [21]). We can improve these results for the case of Einstein manifolds. First, we will prove the following.

Lemma 1. *Let (M, g) be an Einstein manifold with Einstein constant α and sectional curvature bounded below by a constant $\delta > 0$. If $\alpha < (n+2)\delta$ then*

$$F_p(\omega) \geq (1/3)((n+2)\,\delta - \alpha)(n-p)\,\|\omega\|^2 > 0$$

for any nonzero p-form ω and an arbitrary $1 \leq p \leq n-1$.

Proof. Let $p \leq [n/2]$, then we can define the symmetric traceless two-tensor $\varphi^{(i_1 i_2 \ldots i_p)}$ with components (see [14])

$$\varphi^{(i_1 i_2 \ldots i_p)}_{jk} = \sum_{a=1}^{p}\left(\omega_{i_1\ldots i_{a-1} j i_{a+1}\ldots i_p} g_{k i_a} + \omega_{i_1\ldots i_{a-1} k i_{a+1}\ldots i_p} g_{j i_a}\right) - \frac{2p}{n}\,g_{jk}\,\omega_{i_1\ldots i_p}$$

for each set of values of indices $(i_1\, i_2 \ldots i_p)$ such that $1 \leq i_1 < i_2 < \ldots < i_p \leq n$. After long but simple calculations we obtain the identities (see also [14]),

$$\begin{aligned} R_{ijkl}\,\varphi^{il\,(i_1\ldots i_p)}\,\varphi^{jk}_{\;(i_1\ldots i_p)} &= p\left(\frac{2(n+4p)}{n}R_{ij}\,\omega^{i\,i_2\ldots i_p}\omega^{j}_{\;i_2\ldots i_p}\right. \\ &\left. -3\,(p-1)\,R_{ijkl}\,\omega^{ij\,i_3\ldots i_p}\omega^{kl}_{\;\;i_3\ldots i_p} - \frac{4p}{n^2}s\,\|\omega\|^2\right); \end{aligned} \tag{5}$$

$$\|\,\bar{\varphi}\,\|^2 = \frac{2p(n+2)(n-p)}{n}\,\|\omega\|^2, \tag{6}$$

where

$$\|\,\bar{\varphi}\,\|^2 = g^{ik}g^{jl}g_{i_1 j_1}\cdots g_{i_p j_p}\varphi^{(i_1\ldots i_p)}_{ij}\;\varphi^{(j_1\ldots j_p)}_{kl},$$
$$\|\omega\|^2 = \omega^{i_1 i_2\ldots i_p}\omega_{\;i_1 i_2\ldots i_p} = g^{i_1 j_1}\ldots g^{i_p j_p}\omega_{\;i_1\ldots i_p}\,\omega_{\;j_1\ldots j_p}$$

for $g^{ij} = (g^{-1})_{ij}$. If (M, g) is an Einstein manifold, then Equations (4) and (5) can be rewritten in the form

$$F_p(\omega) = \frac{s}{n}\,\|\omega\|^2 - \frac{p-1}{2}\,R_{ijkl}\,\omega^{ij\,i_3\ldots i_p}\,\omega^{kl}_{\;i_3\ldots i_p},$$

$$R_{ijkl}\,\varphi^{il(i_1\dots i_p)}\,\varphi^{jk}_{(i_1\dots i_p)} = p\Big(\frac{2n+4p}{n^2}\,s\,\|\omega\|^2 - 3(p-1)R_{ijkl}\,\omega^{ij\,i_3\dots i_p}\,\omega^{kl}_{i_3\dots i_p}\Big). \tag{7}$$

On the other hand, for a fixed set of values of indices $(i_1, i_2, \dots, i_p)$ such that $1 \le i_1 < i_2 < \dots < i_p \le n$, the equality in Equation (3) can be rewritten in the form

$$R_{ijkl}\,\varphi^{il\,(i_1\dots i_p)}\,\varphi^{jk(i_1\dots i_p)} + \frac{s}{n}\,\varphi^{ik\,(i_1\dots i_p)}\,\varphi_{ik}^{(i_1\dots i_p)} \ge n\,\delta\,\varphi^{kl\,(i_1\dots i_p)}\,\varphi_{kl}^{(i_1\dots i_p)}. \tag{8}$$

Then from Equation (8) we obtain the inequality

$$R_{ijkl}\,\varphi^{il\,(i_1\dots i_p)}\,\varphi^{jk}{}_{(i_1\dots i_p)} \ge \Big(n\delta - \frac{s}{n}\Big)\,\|\,\bar{\varphi}\,\|^2. \tag{9}$$

Using Equation (9) we deduce from Equation (7) the following inequality:

$$6p\,F_p(\omega) \ge \Big(n\,\delta - \frac{s}{n+2}\Big)\,\|\,\bar{\varphi}\,\|^2. \tag{10}$$

Thus, using Equation (6) we can rewrite Equation (10) in the following form:

$$F_p(\omega) \ge (1/3)((n+2)\,\delta - \alpha)\,(n-p)\|\omega\|^2. \tag{11}$$

It is obvious that if the sectional curvature of an Einstein manifold (M, g) satisfies the inequality $\sec \ge \delta$ for a positive constant δ, then the scalar curvature of (M, g) satisfies the inequality $s \ge n(n-1)\,\delta > 0$. In this case, if $(n-1)\,\delta \le \alpha < (n+2)\,\delta$, then from Equation (11) we deduce that the quadratic form $F_p(\omega)$ is positive definite for any $p \le [n/2]$. It is known [23] that $F_p(\omega) = F_{n-p}(*\,\omega)$ and $\|\omega\|^2 = \|\,*\,\omega\|^2$ for any p-form ω with $1 \le p \le n-1$ and the Hodge star operator $* : \Lambda^p M \to \Lambda^{n-p} M$ acting on the space of p-forms $\Lambda^p M$. Therefore, the inequality in Equation (11) holds for any $p = 1, \dots, n-1$. □

Recall that if on an n-dimensional compact Riemannian manifold (M, g) the quadratic form $F_p(\omega)$ is positive definite for any smooth p-form ω with $p = 1, \dots, n-1$, then the Betti numbers $b_1(M), \dots, b_{n-1}(M)$ vanish (see [3] (p. 88); [13] (pp. 336–337)). In this case, Theorem 2 directly follows from Lemma 1.

If the curvature of an Einstein manifold (M, g) satisfies $\sec \le -\delta < 0$ for a positive constant δ, then the Einstein constant of (M, g) satisfies the the obvious inequality $\alpha \le -(n-1)\,\delta < 0$. On the other hand, from Equation (1) we deduce the inequality $R_{ijlk}\varphi^{ik}\varphi^{jl} \le -(n\,\delta + \alpha)\,\|\varphi\,\|^2$. Therefore, if $\delta > -\alpha/n$, then $\mathring{R} < 0$. In this case, the Tachibana numbers $t_1(M), \dots, t_{n-1}(M)$ are equal to zero (see [19]). We proved the following.

Proposition 1. *Let (M^n, g) be an Einstein manifold with sectional curvature bounded above by a negative constant $-\delta$ such that $\delta > -\alpha/n$ for the Einstein constant α. Then the Tachibana numbers $t_1(M), \dots, t_{n-1}(M)$ are zero.*

We can complete this result. If an Einstein manifold (M^n, g) satisfies the curvature inequality $\sec \le -\delta < 0$ for a positive constant δ, then from Equations (3) and (7) we deduce the inequality $F_p(\omega) \le -\frac{1}{3}((n+2)\,\delta + \alpha)(n-p)\|\omega\|^2$ for any $p = 1, \dots, n-1$. Therefore, the Tachibana numbers $t_1(M), \dots, t_{n-1}(M)$ of a compact Einstein manifold with sectional curvature bounded above by a negative constant $-\delta$ such that $\delta \ge -\alpha/(n+2)$ are zero.

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