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Periodic Solutions for a Four-Dimensional Coupled Polynomial System with N-Degree Homogeneous Nonlinearities

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Abstract: This paper studies the periodic solutions of a four-dimensional coupled polynomial system with N-degree homogeneous nonlinearities of which the unperturbed linear system has a center singular point in generalization resonance $1 : n$ at the origin. Considering arbitrary positive integers n and N with $n \leq N$ and $N \geq 2$, the new explicit expression of displacement function for the four-dimensional system is detected by introducing the technique on power trigonometric integrals. Then some precise and detailed results in comparison with the existing works, including the existence condition, the exact number, and the parameter control conditions of periodic solutions, are obtained, which can provide a new theoretical description and mechanism explanation for the phenomena of emergence and disappearance of periodic solutions. Results obtained in this paper improve certain existing results under some parameter conditions and can be extensively used in engineering applications. To verify the applicability and availability of the new theoretical results, as an application, the periodic solutions of a circular mesh antenna model are obtained by theoretical method and numerical simulations.

Keywords: four-dimensional coupled system; periodic solutions; existence condition; parameter control conditions; circular mesh antenna model

1. Introduction

Many problems in the fields of engineering and science can be described by nonlinear polynomial systems. Due to interaction between different variables, these systems often exhibit complicated dynamic characteristics and bifurcation behaviors [1–3]. One of the important ingredients for describing the dynamic behaviors of nonlinear systems is the bifurcation of periodic solutions, which is closely related to the second part of Hilbert's 16th problem [4]. Many scholars have done a lot of work in recent years, and some meaningful results for one-dimensional and planar systems have been obtained [5–9]. With the development of science and technology, the study of one-dimensional and planar systems cannot satisfy the need of practical applications. Hence, it is urgent to study the bifurcation of periodic solutions for high-dimensional polynomial systems. However, due to the complexity of geometric structure and numerical calculation of high-dimensional nonlinear polynomial systems, research on the bifurcation of periodic solutions is much more sophisticated than the one-dimensional and planar systems.

Up to now, some contributions have been made in the bifurcation theory of periodic solutions of high-dimensional nonlinear polynomial systems. Various classical and effective methods, such as the Poincaré map [10], the Melnikov method [11], the harmonic balance method [12], and the averaging method [13], were proposed to detect the periodic solutions. Further study on the bifurcation of periodic solutions for some types of polynomial systems was widely considered in [14] and references therein. Recently, to overcome the complex calculations appearing in the process of analyzing the periodic solutions of high-dimensional polynomial systems, some symbolic algorithms and programs were developed based on computer software, which can provide a convenient way to solve the problems in real applications [15,16].

The study of the existence and number of periodic solutions for high-dimensional polynomial systems is an important and hot topic in bifurcation theory that can help scientists better comprehend and analyze the complex periodic vibration phenomena exhibited in systems from different fields. Some results on the existence of periodic solutions for some certain types of systems, such as slow-fast systems and perturbed Hamilton systems, were obtained in [17–22]. In recent years, the study of the number of periodic solutions for high-dimensional systems has attracted much attention from researchers. The results can be mainly divided into two aspects: the lower bounds [23] and the upper bounds [24]. As we have seen, most of the studies deal with piecewise linear systems or three-dimensional nonlinear systems [25,26]. In fact, with the increasing of the dimensions of nonlinear systems, the amounts of calculation increase exponentially, and it is difficult to determine the number of periodic solutions. For a four or higher dimensional N-degree perturbation system, the upper bound of the number of periodic solutions bifurcating from the center singular point in certain resonance $1 : N$ or zero–Hopf singularity has been obtained [27–29]. However, the more general results on the existence and number of periodic solutions bifurcating from a four-dimensional center in generalization resonance $1 : n$, $n \in \mathbf{Z}_+$ have not been obtained.

Most of the mechanical models appearing in science and engineering applications, such as those exhibited in [1,3,30], are often multi-degree-of-freedom nonlinear dynamical systems. These systems can often be reduced to nonlinear dynamic systems with even dimensions and cannot be directly analyzed based on certain existing results. Inspired by the aforementioned works, to facilitate the practical applications, we are concerned with a four-dimensional coupled polynomial system with N-degree homogeneous nonlinearities of which the unperturbed linear system has a center singular point in generalization resonance $1 : n$, where $N, n \in \mathbf{Z}_+$. The upper bounds of the number of periodic solutions for $n = N$ and $n = 0$ have been obtained in previous literatures [27–29]. Our main aim is to bring the relevant studies to a more general case, $n \le N$. In this paper, some more precise and detailed results in comparison with the existing works, including the existence condition, exact number, and the parameter control conditions of periodic solutions, are obtained. The obtained results can be widely applied to engineering models, which can provide a detailed description for the periodic solutions when the coefficients are allowed to vary in a wide range of parameters. Periodic solutions of an engineering model show the complex periodic vibrations of the application devices, and our theoretical results may provide a parameter method for controlling periodic vibrations.

This paper is organized as follows: In Section 2, some preliminary lemmas that play an important role in our study of the periodic solutions of System (1) are presented. In Section 3, by detecting the new exact explicit expression of displacement function based on the Poincaré map and taking into account the complex coefficients, some results on the existence, number, and parameter control conditions of periodic solutions of the four-dimensional nonlinear system are obtained. In Section 4, as an application, the periodic solutions of a two-degree-of-freedom circular mesh antenna model subjected to thermal excitation are studied by theoretical results and numerical methods. In Section 5, conclusions of this paper are presented.

2. Preliminaries

Consider a four-dimensional coupled polynomial system with N-degree homogeneous nonlinearities of which the unperturbed linear system has a center singular point in generalization resonance $1 : n$ as follows,

$$\dot{x} = (A + \varepsilon\widetilde{A})x + F(x), \tag{1}$$

where $\varepsilon > 0$ is a small parameter,

$$\begin{aligned}
&x = (x_1, x_2, x_3, x_4)^{\mathrm{T}} \in \mathbf{R}^4,\ A = diag(A_1, A_2),\ \widetilde{A} = diag(\widetilde{A}_1, \widetilde{A}_2),\\
&A, \widetilde{A} \in \mathbf{R}^{4\times 4},\ A_k, \widetilde{A}_k \in \mathbf{R}^{2\times 2},\ A_k = \begin{pmatrix} 0 & -n^{k-1} \\ n^{k-1} & 0 \end{pmatrix},\ \widetilde{A}_k = (a_{2k-1}, a_{2k})^{\mathrm{T}},\\
&a_k = (a^k_{1000}, a^k_{0100}),\ a_j = (a^j_{0010}, a^j_{0001}),\ k = 1, 2,\ j = 3, 4,\\
&F = (F_1, F_2, F_3, F_4)^{\mathrm{T}},\ F_i(x) = \sum_{i_1+i_2+i_3+i_4=N}\ \sum_{i_1+i_2=N-2+[(i+1)/2]} a^i_{i_1 i_2 i_3 i_4} x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4},\\
&a^i_{i_1 i_2 i_3 i_4} \in \mathbf{R},\ i_1, i_2, i_3, i_4 \in \mathbf{N},\ i = 1,\ 2,\ 3,\ 4,\ n \le N,\ n, N \in \mathbf{Z}_+,
\end{aligned} \tag{2}$$

and $[\cdot]$ denotes the integer part. We are concerned with the existence condition, exact number, and parameter control conditions of periodic solutions of System (1). We all know that when $N = 1$, System (1) has no periodic solution. Hence, the degree of the nonlinear terms of System (1), $N \ge 2$, will be considered in this paper.

In this section, we will present some important lemmas on the transformations of System (1) and the exact formulas of power trigonometric integrals as preliminaries.

Rescaling System (1), the following lemma can be obtained:

Lemma 1. *By introducing the scale transformation*

$$x \to \varepsilon^{\frac{1}{N-1}} x, \tag{3}$$

System (1) can be rewritten as

$$\dot{x} = Ax + \varepsilon G(x), \tag{4}$$

where $G(x) = \widetilde{A}x + F(x) \in \mathbf{R}^4$.

Proof. Rescaling the variables of System (1) by Transformation (3), we obtain

$$\varepsilon^{\frac{1}{N-1}} \dot{x} = \varepsilon^{\frac{1}{N-1}} (A + \varepsilon\widetilde{A})x + \varepsilon^{\frac{N}{N-1}} F(x),$$

which can be reduced to

$$\dot{x} = (A + \varepsilon\widetilde{A})x + \varepsilon F(x).$$

This proof is completed. □

Writing System (4) in polar coordinates, the following lemma can be obtained:

Lemma 2. *Considering the transformation*

$$(x_1, x_2, x_3, x_4) = (r\cos\theta, r\sin\theta, \rho\cos(n(\theta + s)), \rho\sin(n(\theta + s))), \tag{5}$$

System (4) becomes

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \varepsilon H(\theta, y) + O(\varepsilon^2), \tag{6}$$

where $\theta, s \in \mathbf{S}^1$, $r, \rho \in \mathbf{R}$,

$$\begin{array}{l}
y = (r,\rho,s)^{\mathrm{T}},\ H = (H_1,H_2,H_3)^{\mathrm{T}},\\
H_1(\theta,y) = G_1(\theta,y)\cos\theta + G_2(\theta,y)\sin\theta,\\
H_2(\theta,y) = G_3(\theta,y)\cos(n(\theta+s)) + G_4(\theta,y)\sin(n(\theta+s)),\\
H_3(\theta,y) = \frac{1}{\rho n}(G_4(\theta,y)\cos(n(\theta+s)) - G_3(\theta,y)\sin(n(\theta+s)))\\
\quad -\frac{1}{r}(G_2(\theta,y)\cos\theta - G_1(\theta,y)\sin\theta),\\
G_k(\theta,y) = a^k_{1000} r\cos\theta + a^k_{0100} r\sin\theta + F_k(\theta,y),\ k = 1,2,\\
G_j(\theta,y) = a^j_{0010}\rho\cos(n(\theta+s)) + a^j_{0001}\rho\sin(n(\theta+s)) + F_j(\theta,y),\ j = 3,4,\\
F_i(\theta,y) = \sum\limits_{i_1+i_2+i_3+i_4=N}\ \sum\limits_{i_1+i_2=N-2+[(i+1)/2]} a^i_{i_1i_2i_3i_4} r^{i_1+i_2}\rho^{i_3+i_4}\cos^{i_1}\theta\sin^{i_2}\theta\\
\quad \times\cos^{i_3}(n(\theta+s))\sin^{i_4}(n(\theta+s)),\ i = 1,2,3,4.
\end{array}$$

Proof. Doing the change of variables from (x_1, x_2, x_3, x_4) to the new variables (θ, r, ρ, s) given by Transformation (5), System (4) becomes

$$\begin{array}{r}
\dot r\cos\theta - \dot\theta r\sin\theta = -r\sin\theta + \varepsilon G_1(\theta,y),\\
\dot r\sin\theta + \dot\theta r\cos\theta = r\cos\theta + \varepsilon G_2(\theta,y),\\
\dot\rho\cos(n(\theta+s)) - n(\dot\theta+\dot s)\rho\sin(n(\theta+s)) = -\rho n\sin(n(\theta+s)) + \varepsilon G_3(\theta,y),\\
\dot\rho\sin(n(\theta+s)) + n(\dot\theta+\dot s)\rho\cos(n(\theta+s)) = \rho n\cos(n(\theta+s)) + \varepsilon G_4(\theta,y),
\end{array} \tag{7}$$

where the expressions of $G_i(\theta, y)$, $i = 1,2,3,4$ are exhibited in Lemma 2. Hence,

$$\begin{array}{l}
\dot r = \varepsilon G_1(\theta,y)\cos\theta + \varepsilon G_2(\theta,y)\sin\theta,\\
\dot\rho = \varepsilon G_3(\theta,y)\cos(n(\theta+s)) + \varepsilon G_4(\theta,y)\sin(n(\theta+s)),\\
\dot s = \frac{\varepsilon}{\rho n}(G_4(\theta,y)\cos(n(\theta+s)) - G_3(\theta,y)\sin(n(\theta+s)))\\
\quad -\frac{\varepsilon}{r}(G_2(\theta,y)\cos\theta - G_1(\theta,y)\sin\theta),\\
\dot\theta = 1 - \frac{\varepsilon G_1(\theta,y)\sin\theta - \varepsilon G_2(\theta,y)\cos\theta}{r}.
\end{array} \tag{8}$$

Now System (8) can be rewritten as

$$\begin{array}{l}
\dot y = \varepsilon H(\theta,y),\\
\dot\theta = 1 - \frac{\varepsilon G_1(\theta,y)\sin\theta - \varepsilon G_2(\theta,y)\cos\theta}{r},
\end{array}$$

where $y = (r, \rho, s)^{\mathrm{T}}$, $H = (H_1, H_2, H_3)^{\mathrm{T}}$, and the expression $H(\theta, y)$ is exhibited in Lemma 2. Considering θ as a new independent variable, we obtain a non-autonomous system with the form

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}\theta} &= \varepsilon H(\theta,y)\left(1 + \sum_{j=1}^{\infty}\left(\frac{\varepsilon G_1(\theta,y)\sin\theta - \varepsilon G_2(\theta,y)\cos\theta}{r}\right)^j\right)\\
&= \varepsilon H(\theta,y) + O(\varepsilon^2).
\end{aligned}$$

This proof is completed. □

For convenience, two important notations that play an important role in our investigation are introduced;

$$\begin{array}{l}
J_{K_1}(s) = \int_0^{2\pi}\cos^{\lambda_1}\theta\sin^{\lambda_2}\theta\cos(n(\theta+s))\mathrm{d}\theta,\\
J_{K_2}(s) = \int_0^{2\pi}\cos^{\lambda_1}\theta\sin^{\lambda_2}\theta\sin(n(\theta+s))\mathrm{d}\theta,
\end{array}$$

where $\lambda_1, \lambda_2 \in \mathbf{N}$. Our next objective will be to discuss the exact expressions of $J_{K_1}(s)$ and $J_{K_2}(s)$.

Lemma 3. *Considering $\lambda_1 + \lambda_2 = N$, the following statements hold:*

(1) *If $N+n$ is odd, then $J_{K_1}(s) = J_{K_2}(s) = 0$.*
(2) *If $N+n$ is even, then*

 (i) *When λ_2 is odd,*

$$J_{K_1}(s) = -c(\lambda_1,\lambda_2)\sin(ns), J_{K_2}(s) = c(\lambda_1,\lambda_2)\cos(ns).$$

 (ii) *When λ_2 is even,*

$$J_{K_1}(s) = c(\lambda_1,\lambda_2)\cos(ns), J_{K_2}(s) = c(\lambda_1,\lambda_2)\sin(ns),$$

where

$$c(\lambda_1,\lambda_2) = \frac{(-1)^{[\frac{\lambda_2}{2}]}\pi}{2^{N-1}}\sum_{j_{11}=0}^{\lambda_1}\sum_{j_{21}=(N-n)/2-j_{11}}\sum_{0\le j_{21}\le\lambda_2}(-1)^{j_{21}}C_{\lambda_1}^{j_{11}}C_{\lambda_2}^{j_{21}}$$

and we set $C_0^0 = 1$.

Proof. When $\lambda_1+\lambda_2 = N$, note that

$$\begin{aligned}\cos^{\lambda_1}\theta\sin^{\lambda_2}\theta &= \left(\frac{e^{\mathrm{i}\theta}+e^{-\mathrm{i}\theta}}{2}\right)^{\lambda_1}\left(\frac{e^{\mathrm{i}\theta}-e^{-\mathrm{i}\theta}}{2\mathrm{i}}\right)^{\lambda_2}\\ &= \frac{1}{2^N\mathrm{i}^{\lambda_2}}\left(e^{\mathrm{i}\theta}+e^{-\mathrm{i}\theta}\right)^{\lambda_1}\left(e^{\mathrm{i}\theta}-e^{-\mathrm{i}\theta}\right)^{\lambda_2}\\ &= \frac{1}{2^N\mathrm{i}^{\lambda_2}}\sum_{j_1=0}^{\lambda_1}\sum_{j_2=0}^{\lambda_2}(-1)^{j_2}C_{\lambda_1}^{j_1}C_{\lambda_2}^{j_2}e^{\mathrm{i}(N-2(j_1+j_2))\theta}.\end{aligned}$$

Based on the analysis above, when $N-2(j_1+j_2) = n$ or $N-2(j_1+j_2) = -n$, the expressions $J_{K_1}(s)$ and $J_{K_2}(s)$ may not be equal to zero, where $j_1, j_2 \in Z_+$, $0 \le j_1 \le \lambda_1$, $0 \le j_2 \le \lambda_2$. Next, we discuss $J_{K_1}(s)$ and $J_{K_2}(s)$ by considering the parity of $N+n$.

(1) If $N+n$ is odd, there exists no $j_1, j_2 \in Z_+$ such that $N-2(j_1+j_2) = n$ and $N-2(j_1+j_2) = -n$, where $0 \le j_1 \le \lambda_1$, $0 \le j_2 \le \lambda_2$. Hence, $J_{K_1}(s) = J_{K_2}(s) = 0$ holds in this case.
(2) If $N+n$ is even, the expressions $J_{K_1}(s)$ and $J_{K_2}(s)$ may not be zero. In fact, supposing there exist $j_1 = j_{11}$ and $j_2 = j_{21}$ such that $N-2(j_1+j_2) = n$, there exist $j_1 = \lambda_1 - j_{11}$ and $j_2 = \lambda_2 - j_{21}$ such that $N-2(j_1+j_2) = -n$. Next, we discuss the exact expressions of $J_{K_1}(s)$ and $J_{K_2}(s)$.

 (i) The expression $J_{K_1}(s)$. Considering the introduced notation $J_{K_1}(s)$, together with the above analysis, we obtain

$$\begin{aligned}J_{K_1}(s) &= \frac{1}{2^{N+1}\mathrm{i}^{\lambda_2}}\int_0^{2\pi}\sum_{j_{11}=0}^{\lambda_1}\sum_{j_{21}=(N-n)/2-j_{11}}\sum_{0\le j_{21}\le\lambda_2}C_{\lambda_1}^{j_{11}}C_{\lambda_2}^{j_{21}}\left((-1)^{j_{21}}e^{\mathrm{i}n\theta}+(-1)^{\lambda_2-j_{21}}e^{-\mathrm{i}n\theta}\right)\left(e^{\mathrm{i}n(\theta+s)}+e^{-\mathrm{i}n(\theta+s)}\right)\mathrm{d}\theta\\ &= \frac{1}{2^{N+1}\mathrm{i}^{\lambda_2}}\int_0^{2\pi}\sum_{j_{11}=0}^{\lambda_1}\sum_{j_{21}=(N-n)/2-j_{11}}\sum_{0\le j_{21}\le\lambda_2}(-1)^{j_{21}}C_{\lambda_1}^{j_{11}}C_{\lambda_2}^{j_{21}}\left(e^{-\mathrm{i}ns}+(-1)^{\lambda_2}e^{\mathrm{i}ns}\right)\mathrm{d}\theta.\end{aligned}$$

The exact expression of $J_{K_1}(s)$ will be discussed based on the parity of λ_2.

 (a) When λ_2 is odd,

$$\begin{aligned}J_{K_1}(s) &= \frac{-(-1)^{\frac{\lambda_2-1}{2}}}{2^N}\int_0^{2\pi}\sum_{j_{11}=0}^{\lambda_1}\sum_{j_{21}=(N-n)/2-j_{11}}\sum_{0\le j_{21}\le\lambda_2}(-1)^{j_{21}}C_{\lambda_1}^{j_{11}}C_{\lambda_2}^{j_{21}}\sin(ns)\mathrm{d}\theta\\ &= \frac{-(-1)^{\frac{\lambda_2-1}{2}}\pi\sin(ns)}{2^{N-1}}\sum_{j_{11}=0}^{\lambda_1}\sum_{j_{21}=(N-n)/2-j_{11}}\sum_{0\le j_{21}\le\lambda_2}(-1)^{j_{21}}C_{\lambda_1}^{j_{11}}C_{\lambda_2}^{j_{21}}.\end{aligned}$$

(b) When λ_2 is even,

$$\begin{aligned} J_{K_1}(s) &= \frac{(-1)^{\frac{\lambda_2}{2}}}{2^N}\int_0^{2\pi}\sum_{j_{11}=0}^{\lambda_1}\sum_{j_{21}=(N-n)/2-j_{11}}\sum_{0\le j_{21}\le\lambda_2}(-1)^{j_{21}}C_{\lambda_1}^{j_{11}}C_{\lambda_2}^{j_{21}}\cos(ns)\mathrm{d}\theta \\ &= \frac{(-1)^{\frac{\lambda_2}{2}}\pi\cos(ns)}{2^{N-1}}\sum_{j_{11}=0}^{\lambda_1}\sum_{j_{21}=(N-n)/2-j_{11}}\sum_{0\le j_{21}\le\lambda_2}(-1)^{j_{21}}C_{\lambda_1}^{j_{11}}C_{\lambda_2}^{j_{21}}. \end{aligned}$$

(ii) The expression of $J_{K_2}(s)$. Thus:

$$\begin{aligned} J_{K_2}(s) &= \frac{1}{2^{N+1}\mathrm{i}^{\lambda_2+1}}\int_0^{2\pi}\sum_{j_{11}=0}^{\lambda_1}\sum_{j_{21}=(N-n)/2-j_{11}}\sum_{0\le j_{21}\le\lambda_2}C_{\lambda_1}^{j_{11}}C_{\lambda_2}^{j_{21}}\left((-1)^{j_{21}}e^{in\theta}+(-1)^{\lambda_2-j_{21}}e^{-\mathrm{i}n\theta}\right)\left(e^{\mathrm{i}n(\theta+s)}-e^{-\mathrm{i}n(\theta+s)}\right)\mathrm{d}\theta \\ &= \frac{1}{2^{N+1}\mathrm{i}^{\lambda_2+1}}\int_0^{2\pi}\sum_{j_{11}=0}^{\lambda_1}\sum_{j_{21}=(N-n)/2-j_{11}}\sum_{0\le j_{21}\le\lambda_2}(-1)^{j_{21}}C_{\lambda_1}^{j_{11}}C_{\lambda_2}^{j_{21}}\left(-e^{-\mathrm{i}ns}+(-1)^{\lambda_2}e^{\mathrm{i}ns}\right)\mathrm{d}\theta. \end{aligned}$$

Similar with case (i), we consider $J_{K_2}(s)$ by discussing the parity of λ_2.

(a) When λ_2 is odd,

$$J_{K_2}(s) = \frac{(-1)^{\frac{\lambda_2-1}{2}}\pi\cos(ns)}{2^{N-1}}\sum_{j_{11}=0}^{\lambda_1}\sum_{j_{21}=(N-n)/2-j_{11}}\sum_{0\le j_{21}\le\lambda_2}(-1)^{j_{21}}C_{\lambda_1}^{j_{11}}C_{\lambda_2}^{j_{21}}.$$

(b) When λ_2 is even,

$$J_{K_2}(s) = \frac{(-1)^{\frac{\lambda_2}{2}}\pi\sin(ns)}{2^{N-1}}\sum_{j_{11}=0}^{\lambda_1}\sum_{j_{21}=(N-n)/2-j_{11}}\sum_{0\le j_{21}\le\lambda_2}(-1)^{j_{21}}C_{\lambda_1}^{j_{11}}C_{\lambda_2}^{j_{21}}.$$

This proof is completed. □

3. Periodic Solutions of a Four-Dimensional Nonlinear System

In this section, the existence, exact number, and parameter control conditions of the periodic solutions of System (1) are investigated by detecting the new explicit expression of the displacement function based on the Poincaré map.

3.1. Displacement Function

System (1) is transformed into System (6) in Section 2, which implies that the periodic solutions of System (1) can be obtained by considering a Poincaré map for System (6). Denote by $y(\theta, z, \varepsilon)$ the solution of System (6) with the initial condition $y(0, z, \varepsilon) = z$, where $\theta \in \mathbf{S}^1$, $z = (r_0, \rho_0, s_0)^{\mathrm{T}} \in \mathbf{R}^2 \times \mathbf{S}^1$. Define a global cross section to vector field (6) by

$$\Sigma = \{(y, \theta) | \theta = 0\} \in \mathbf{R}^2 \times \mathbf{S}^2.$$

Note that $H(\theta, y)$, shown in System (6), is continuous and 2π periodic with respect to variable θ. Consider the Poincaré map of System (6),

$$P(z, \varepsilon) : \Sigma \to \Sigma : y(0, z, \varepsilon) \to y(2\pi, z, \varepsilon),$$

and define the displacement function as

$$\Phi(z,\varepsilon) = \frac{1}{\varepsilon}(y(2\pi,z,\varepsilon) - y(0,z,\varepsilon)) = \frac{1}{\varepsilon}(y(2\pi,z,\varepsilon) - z), \tag{9}$$

where $\Phi = (\Phi_1, \Phi_2, \Phi_3)^{\mathrm{T}}$. In what follows, we will detect the expression of displacement function.

Lemma 4. *The displacement function of System (6) with initial condition $y(0,z,\varepsilon) = z$ can be written as follows:*

$$\Phi(z,\varepsilon) = h(z) + O(\varepsilon),$$

where $h = (h_1, h_2, h_3)$, $h(z) = \int_0^{2\pi} H(\theta,z)\mathrm{d}\theta$.

Proof. Expand the solution of System (6), $y(\theta,z,\varepsilon)$, in the Taylor series in ε, i.e.,

$$y(\theta,z,\varepsilon) = z + \varepsilon y_1(\theta,z) + O(\varepsilon^2).$$

Clearly $y_1(\theta,z)$ is a real function that satisfies $y_1(0,z) = 0$, where $y_1 = (r_1, \rho_1, s_1)^{\mathrm{T}}$. Substituting $y(\theta,z,\varepsilon)$ into System (6) yields

$$\begin{array}{rcl} \frac{\partial y(\theta,z,\varepsilon)}{\partial\theta} & = & \varepsilon H(\theta, y(\theta,z,\varepsilon)) + O(\varepsilon^2) \\ & = & \varepsilon H\Big(\theta, z + \varepsilon y_1(\theta,z) + O(\varepsilon^2)\Big) + O(\varepsilon^2). \end{array} \tag{10}$$

The power series expansions of $\frac{\partial y(\theta,z,\varepsilon)}{\partial\theta}$ and $\varepsilon H\Big(\theta, z + \varepsilon y_1(\theta,z) + O(\varepsilon^2)\Big)$ at both ends of Equation (10) for ε, at $\varepsilon = 0$, can be expressed as

$$\begin{array}{rcl} \frac{\partial r(\theta,z,\varepsilon)}{\partial\theta} & = & \frac{\varepsilon\partial r_1(\theta,z)}{\partial\theta} + O(\varepsilon^2), \\ \frac{\partial \rho(\theta,z,\varepsilon)}{\partial\theta} & = & \frac{\varepsilon\partial \rho_1(\theta,z)}{\partial\theta} + O(\varepsilon^2), \\ \frac{\partial s(\theta,z,\varepsilon)}{\partial\theta} & = & \frac{\varepsilon\partial s_1(\theta,z)}{\partial\theta} + O(\varepsilon^2) \end{array} \tag{11}$$

and

$$\begin{array}{rcl} \varepsilon H_i\Big(\theta, z + \varepsilon y_1(\theta,z) + O(\varepsilon^2)\Big) & = & \varepsilon H_i(\theta,z) + \varepsilon \sum\limits_{\gamma=1}^{\infty} \frac{\partial^{\gamma} H_i(\theta,z)}{\gamma!} \\ & = & \varepsilon H_i(\theta,z) + O(\varepsilon^2), \end{array} \tag{12}$$

where

$$\begin{array}{rcl} \partial^{\gamma} H_i(\theta,z) & = & \sum\limits_{\gamma_1+\gamma_2+\gamma_3=\gamma} \frac{\partial^{\gamma_3}\partial^{\gamma_2}\partial^{\gamma_1} H_i(\theta,z)}{\partial r_0^{\gamma_3}\partial\rho_0^{\gamma_2}\partial s_0^{\gamma_1}} \Big(\varepsilon r_1(\theta,z) + O(\varepsilon^2)\Big)^{\gamma_1} \Big(\varepsilon \rho_1(\theta,z) + O(\varepsilon^2)\Big)^{\gamma_2} \\ & & \times\Big(\varepsilon s_1(\theta,z) + O(\varepsilon^2)\Big)^{\gamma_3},\ \gamma! = \gamma_1!\gamma_2!\gamma_3!,\ i = 1,2,3. \end{array} \tag{13}$$

Equating the coefficients of ε at both ends of Equation (10) yields

$$\frac{\partial y_1(\theta,z)}{\partial\theta} = H(\theta,z). \tag{14}$$

Based on $y_1(0,z) = 0$, the expression $y_1(\theta,z)$ is of the form

$$y_1(\theta,z) = \int_0^{\theta} H(\theta,z)\mathrm{d}\theta.$$

Now the expression of the solution $y(\theta,z,\varepsilon)$ becomes

$$y(\theta,z,\varepsilon) = z + \varepsilon\int_0^{\theta} H(\theta,z)\mathrm{d}\theta + O(\varepsilon^2).$$

Hence, based on Equation (9), the displacement function of System (6) can be obtained:

$$\begin{aligned}\Phi(z,\varepsilon) &= \tfrac{1}{\varepsilon}\Big(z+\varepsilon\int_0^{2\pi} H(\theta,z)\mathrm{d}\theta + O(\varepsilon^2) - z\Big)\\ &= \int_0^{2\pi} H(\theta,z)\mathrm{d}\theta + O(\varepsilon).\end{aligned}$$

This proof is completed. □

We note that a zero solution of the displacement function corresponds to a periodic solution of System (6), which implies that studying the exact expression of $\Phi(z,\varepsilon)$ is crucial for our investigation.

3.2. The Expression of $h(z)$

In this subsection, we study the important term, $h(z)$, of the displacement function $\Phi(z)$. Considering System (6) and Lemma 3, the following lemma can be obtained.

Lemma 5. *The expression of $h(z)$ can be obtained as follows:*

(1) *If $N+n$ is odd, then*

$$h(z) = \psi(z).$$

(2) *If $N+n$ is even, then*

$$h(z) = \psi(z) + \varphi(z),$$

where

$$\begin{aligned}
&\psi = (\psi_1,\psi_2,\psi_3)^{\mathrm{T}},\ \varphi = (\varphi_1,\varphi_2,\varphi_3)^{\mathrm{T}},\\
&\psi_1(z) = b_1 r_0,\ \psi_2(z) = b_2\rho_0,\ \psi_3(z) = b_3,\\
&\varphi_1(z) = r_0^{N-1}\rho_0(\mu_{11}\sin(ns_0) + \mu_{21}\cos(ns_0)),\\
&\varphi_2(z) = r_0^{N}(\mu_{12}\sin(ns_0) + \mu_{22}\cos(ns_0)),\\
&\varphi_3(z) = r_0^{N}\rho_0^{-1}n^{-1}(-\mu_{22}\sin(ns_0) + \mu_{12}\cos(ns_0))\\
&\quad + r_0^{N-2}\rho_0(\mu_{13}\sin(ns_0) + \mu_{23}\cos(ns_0)),\\
&b_1 = a^1_{1000}\pi + a^2_{0100}\pi,\ b_2 = a^3_{0010}\pi + a^4_{0001}\pi,\\
&b_3 = n^{-1}(a^4_{0010} - a^3_{0001})\pi - (a^2_{1000} - a^1_{0100})\pi,\\
&\mu_{i1} = \sum_{i_1=0}^{N-1}\sum_{i_2=N-1-i_1}\Bigg(\sum_{i_2+i\ is\ even}\Big((-1)^i c(i_1+1,i_2)a^1_{i_1i_210} + c(i_1,i_2+1)a^2_{i_1i_201}\Big)\\
&\quad + \sum_{i_2+i\ is\ odd}\Big(c(i_1+1,i_2)a^1_{i_1i_201} + (-1)^i c(i_1,i_2+1)a^2_{i_1i_210}\Big)\Bigg),\\
&\mu_{i2} = \sum_{i_1=0}^{N}\sum_{i_2=N-i_1}\Bigg(\sum_{i_2+i\ is\ even}(-1)^i c(i_1,i_2)a^3_{i_1i_200} + \sum_{i_2+i\ is\ odd}c(i_1,i_2)a^4_{i_1i_200}\Bigg),\\
&\mu_{i3} = \sum_{i_1=0}^{N-1}\sum_{i_2=N-1-i_1}\Bigg(\sum_{i_2+i\ is\ even}\Big(c(i_1,i_2+1)a^1_{i_1i_201} - (-1)^i c(i_1+1,i_2)a^2_{i_1i_210}\Big)\\
&\quad + \sum_{i_2+i\ is\ odd}\Big(-c(i_1+1,i_2)a^2_{i_1i_201} + (-1)^i c(i_1,i_2+1)a^1_{i_1i_210}\Big)\Bigg),\ i = 1,2.
\end{aligned}$$

Proof. Writing

$$H_i(\theta,z)=H_i^1(\theta,z)+H_i^N(\theta,z),\ h_i^m(z)=\int_0^{2\pi}H_i^m(\theta,z)\mathrm{d}\theta,\ i=1,2,3,\ m=1,N,$$

we obtain $h_i(z)=h_i^1(z)+h_i^N(z)$, where

$$\begin{aligned}
&H_1^m(\theta,z)=F_{1m}(\theta,z)\cos\theta+F_{2m}(\theta,z)\sin\theta,\\
&H_2^m(\theta,z)=F_{3m}(\theta,z)\cos(n(\theta+s_0))+F_{4m}(\theta,z)\sin(n(\theta+s_0)),\\
&H_3^m(\theta,z)=\tfrac{1}{\rho_0 n}(F_{4m}(\theta,z)\cos(n(\theta+s_0))-F_{3m}(\theta,z)\sin(n(\theta+s_0)))\\
&\qquad\qquad -\tfrac{1}{r_0}(F_{2m}(\theta,z)\cos\theta-F_{1m}(\theta,z)\sin\theta),\\
&F_{j1}(\theta,z)=G_j(\theta,z)-F_j(\theta,z),\ F_{jN}(\theta,z)=F_j(\theta,z),\ j=1,2,3,4.
\end{aligned}$$

Next, we discuss the exact expressions of $h_i^1(z)$ and $h_i^N(z)$.

(1) The expressions of $h_i^1(z)$, $i=1,2,3$. Thus

$$\begin{aligned}
h_1^1(z)\ &=\int_0^{2\pi}H_1^1(\theta,z)\mathrm{d}\theta\\
&=\int_0^{2\pi}(F_{11}(\theta,z)\cos\theta+F_{21}(\theta,z)\sin\theta)\mathrm{d}\theta\\
&=\int_0^{2\pi}\Big(\big(a_{1000}^1r_0\cos\theta+a_{0100}^1r_0\sin\theta\big)\cos\theta+\big(a_{1000}^2r_0\cos\theta+a_{0100}^2r_0\sin\theta\big)\sin\theta\Big)\mathrm{d}\theta\\
&=\big(a_{1000}^1+a_{0100}^2\big)\pi r_0=b_1r_0,\\
h_2^1(z)\ &=\int_0^{2\pi}H_2^1(\theta,z)\mathrm{d}\theta\\
&=\int_0^{2\pi}\Big(\big(a_{0010}^3\rho_0\cos(n(\theta+s_0))+a_{0001}^3\rho_0\sin(n(\theta+s_0))\big)\cos(n(\theta+s_0))\\
&\quad+\big(a_{0010}^4\rho_0\cos(n(\theta+s_0))+a_{0001}^4\rho_0\sin(n(\theta+s_0))\big)\sin(n(\theta+s_0))\Big)\mathrm{d}\theta\\
&=\big(a_{0010}^3+a_{0001}^4\big)\pi\rho_0=b_2\rho_0,\\
h_3^1(z)\ &=\int_0^{2\pi}H_3^1(\theta,z)\mathrm{d}\theta\\
&=\frac{\pi(a_{0010}^4-a_{0001}^3)}{n}-\pi(a_{1000}^2-a_{0100}^1)=b_3.
\end{aligned}$$

(2) The expressions of $h_i^N(z), i=1,2,3$. When $N+n$ is odd, then $h_1^N(z)=h_2^N(z)=h_3^N(z)=0$. When $N+n$ is even, the following expressions can be obtained based on Lemma 3:

$$
\begin{aligned}
h_1^N(z) &= \int_0^{2\pi} H_1^N(\theta,z)\mathrm{d}\theta \\
&= \int_0^{2\pi} (F_{1N}(\theta,z)\cos\theta + F_{2N}(\theta,z)\sin\theta)\mathrm{d}\theta \\
&= \sum_{i_1+i_2=N-1} r_0^{N-1}\rho_0 \int_0^{2\pi} \cos^{i_1+1}\theta \sin^{i_2}\theta \Big(a^1_{i_1 i_2 10}\cos(n(\theta+s_0)) + a^1_{i_1 i_2 01}\sin(n(\theta+s_0))\Big)\mathrm{d}\theta \\
&+ \sum_{i_1+i_2=N-1} r_0^{N-1}\rho_0 \int_0^{2\pi} \cos^{i_1}\theta \sin^{i_2+1}\theta \Big(a^2_{i_1 i_2 10}\cos(n(\theta+s_0)) + a^2_{i_1 i_2 01}\sin(n(\theta+s_0))\Big)\mathrm{d}\theta \\
&= r_0^{N-1}\rho_0 \sum_{i_1=0}^{N-1}\sum_{i_2=N-1-i_1}\Bigg(\Bigg(\sum_{i_2 \text{ is odd}}\Big(-c(i_1+1,i_2)a^1_{i_1 i_2 10} + c(i_1,i_2+1)a^2_{i_1 i_2 01}\Big) \\
&+ \sum_{i_2 \text{ is even}}\Big(c(i_1+1,i_2)a^1_{i_1 i_2 01} - c(i_1,i_2+1)a^2_{i_1 i_2 10}\Big)\Bigg)\sin(ns_0) \\
&+ \Bigg(\sum_{i_2 \text{ is odd}}\Big(c(i_1+1,i_2)a^1_{i_1 i_2 01} + c(i_1,i_2+1)a^2_{i_1 i_2 10}\Big) \\
&+ \sum_{i_2 \text{ is even}}\Big(c(i_1+1,i_2)a^1_{i_1 i_2 10} + c(i_1,i_2+1)a^2_{i_1 i_2 01}\Big)\Bigg)\cos(ns_0)\Bigg) \\
&= r_0^{N-1}\rho_0(\mu_{11}\sin(ns_0) + \mu_{21}\cos(ns_0)), \\
h_2^N(z) &= \int_0^{2\pi} H_2^N(\theta,z)\mathrm{d}\theta \\
&= \int_0^{2\pi} (F_{3N}(\theta,z)\cos(n(\theta+s_0)) + F_{4N}(\theta,z)\sin(n(\theta+s_0)))\mathrm{d}\theta \\
&= \sum_{i_1+i_2=N} r_0^N \int_0^{2\pi} \cos^{i_1}\theta \sin^{i_2}\theta \Big(a^3_{i_1 i_2 00}\cos(n(\theta+s_0)) + a^4_{i_1 i_2 00}\sin(n(\theta+s_0))\Big)\mathrm{d}\theta \\
&= r_0^N \sum_{i_1=0}^{N}\sum_{i_2=N-i_1} c(i_1,i_2)\Bigg(\Bigg(-\sum_{i_2 \text{ is odd}} a^3_{i_1 i_2 00} + \sum_{i_2 \text{ is even}} a^4_{i_1 i_2 00}\Bigg)\sin(ns_0) \\
&+ \Bigg(\sum_{i_2 \text{ is odd}} a^4_{i_1 i_2 00} + \sum_{i_2 \text{ is even}} a^3_{i_1 i_2 00}\Bigg)\cos(ns_0)\Bigg) \\
&= r_0^N(\mu_{12}\sin(ns_0) + \mu_{22}\cos(ns_0)), \\
h_3^N(z) &= \int_0^{2\pi} H_3^N(\theta,z)\mathrm{d}\theta \\
&= \int_0^{2\pi} \rho_0^{-1}n^{-1}(F_{4N}(\theta,z)\cos(n(\theta+s_0)) - F_{3N}(\theta,z)\sin(n(\theta+s_0)))\mathrm{d}\theta \\
&- \int_0^{2\pi} \frac{1}{r_0}(F_{2N}(\theta,z)\cos\theta - F_{1N}(\theta,z)\sin\theta)\mathrm{d}\theta \\
&= r_0^N\rho_0^{-1}n^{-1} \sum_{i_1=0}^{N}\sum_{i_2=N-i_1} c(i_1,i_2)\Bigg(\Bigg(-\sum_{i_2 \text{ is odd}} a^4_{i_1 i_2 00} - \sum_{i_2 \text{ is even}} a^3_{i_1 i_2 00}\Bigg)\sin(ns_0) \\
&+ \Bigg(\sum_{i_2 \text{ is even}} a^4_{i_1 i_2 00} - \sum_{i_2 \text{ is odd}} a^3_{i_1 i_2 00}\Bigg)\cos(ns_0)\Bigg) \\
&+ r_0^{N-2}\rho_0 \sum_{i_1=0}^{N-1}\sum_{i_2=N-1-i_1}\Bigg(\Bigg(\sum_{i_2 \text{ is odd}}\Big(c(i_1,i_2+1)a^1_{i_1 i_2 01} + c(i_1+1,i_2)a^2_{i_1 i_2 10}\Big) \\
&+ \sum_{i_2 \text{ is even}}\Big(-c(i_1+1,i_2)a^2_{i_1 i_2 01} - c(i_1,i_2+1)a^1_{i_1 i_2 10}\Big)\Bigg)\sin(ns_0) \\
&+ \Bigg(\sum_{i_2 \text{ is odd}}\Big(c(i_1,i_2+1)a^1_{i_1 i_2 10} - c(i_1+1,i_2)a^2_{i_1 i_2 01}\Big) \\
&+ \sum_{i_2 \text{ is even}}\Big(c(i_1,i_2+1)a^1_{i_1 i_2 01} - c(i_1+1,i_2)a^2_{i_1 i_2 10}\Big)\Bigg)\cos(ns_0)\Bigg) \\
&= r_0^N\rho_0^{-1}n^{-1}(-\mu_{22}\sin(ns_0) + \mu_{12}\cos(ns_0)) + r_0^{N-2}\rho_0(\mu_{13}\sin(ns_0) + \mu_{23}\cos(ns_0)).
\end{aligned}
$$

From the expressions $h_i(z) = h_i^1(z) + h_i^N(z)$, $i=1,2,3$, Lemma 5 holds.
This proof is completed. □

It is remarkable that the explicit expression $h(z)$ of System (1) for arbitrary positive integers n and N with $n \le N$ and $N \ge 2$ is obtained by introducing the technique on the exact formulas of power trigonometric integrals, which is new and plays an important role in detecting the exact number and parameter control conditions of the periodic solutions of System (1). Note that when $n = N$, we obtain $\mu_{23} = \mu_{11}$, $\mu_{21} = -\mu_{13}$ based on Lemma 5, then the exact expression $\varphi_3(z)$ is of the form

$$\begin{aligned}\varphi_3(z) \quad &= r_0^N \rho_0^{-1} n^{-1}(-\mu_{22}\sin(ns_0) + \mu_{12}\cos(ns_0)) \\ &+ r_0^{N-2}\rho_0(-\mu_{21}\sin(ns_0) + \mu_{11}\cos(ns_0)),\end{aligned}$$

which shows that the form of expression $h(z)$ in this case is in agreement with certain existing result in the previous literature [27], but we obtain all the coefficients in this paper. The result obtained in this section can help to provide more detailed bifurcation information about System (1) by considering the complex coefficients.

3.3. Periodic Solutions

In this subsection, we study the existence condition, exact number, and the parameter control conditions of the periodic solutions of System (1) by supposing $b_1 b_2 \neq 0$.

Theorem 1. *Based on the expression of $h(z)$, the following statements hold for $\varepsilon > 0$ sufficiently small.*

(1) *If there exists a solution $z^* = (r_0^*, \rho_0^*, s_0^*) \in \mathbf{R}^2 \times \mathbf{S}^1$ of $h(z) = 0$ such that*

$$\frac{\partial h(z)}{\partial z}|z^* \neq 0,\ r_0^* \geq 0,\ \rho_0^* \geq 0,\ (r_0^*)^2 + (\rho_0^*)^2 \neq 0.$$

System (1) has a periodic solution

$$x^*(t) = \varepsilon^{\frac{1}{N-1}}\Big(r_0^*\cos t, r_0^*\sin t, \rho_0^*\cos\big(n(t+s_0^*)\big), \rho_0^*\sin\big(n(t+s_0^*)\big)\Big) + O(\varepsilon^{\frac{N}{N-1}}).$$

(2) *The number of periodic solutions of System (1) can be provided by the number of solutions of $h(z) = 0$ that satisfy statement (1) and $s_0^* \in (0, 2\pi]$.*

Proof. We prove Theorem 1 by the following two steps.

(1) If there exists $z^* = (r_0^*, \rho_0^*, s_0^*) \in \mathbf{R}^2 \times \mathbf{S}^1$ such that $h(z^*) = 0$, we have $\Phi(z^*, 0) = h(z^*) + O(0) = 0$. Since

$$\frac{\partial \Phi(z,\varepsilon)}{\partial z}\bigg|(z^*,0) = \frac{\partial h(z)}{\partial z}|z^* \neq 0,$$

there exists a unique vector function $(z^*(\varepsilon), \varepsilon)$ in the neighborhood of $(z^*, 0)$ such that $\Phi(z^*(\varepsilon), \varepsilon) = 0$ for $\varepsilon > 0$ sufficiently small by the implicit function theorem. Hence, based on the definition of the Poincaré map, System (6) has a periodic solution in the neighborhood of z^*.

Recalling the variables transformation shown in Lemma 2 and properties of polar coordinates, when $\varepsilon > 0$ sufficiently small, System (4) has a periodic solution when the solution z^* satisfies $r_0^* \geq 0$, $\rho_0^* \geq 0$, $(r_0^*)^2 + (\rho_0^*)^2 \neq 0$, which is shown as

$$x^*(t) = \Big(r_0^*\cos t, r_0^*\sin t, \rho_0^*\cos\big(n(t+s_0^*)\big), \rho_0^*\sin\big(n(t+s_0^*)\big)\Big) + O(\varepsilon).$$

Based on the scale Transformation (3), when $\varepsilon > 0$ sufficiently small, System (1) has a periodic solution with the form

$$x^*(t) = \varepsilon^{\frac{1}{N-1}}\Big(r_0^*\cos t, r_0^*\sin t, \rho_0^*\cos\big(n(t+s_0^*)\big), \rho_0^*\sin\big(n(t+s_0^*)\big)\Big) + O(\varepsilon^{\frac{N}{N-1}}).$$

(2) A solution of $h(z) = 0$, which satisfies Statement (1), provides a periodic solution of System (1). Hence, the number of periodic solutions of System (1) can be obtained by discussing the number of solutions of $h(z) = 0$ that satisfy Statement (1) and $s_0^* \in (0, 2\pi]$.

This proof is completed. □

Theorem 1 provides a sufficient condition for analyzing the existence and number of periodic solutions of System (1). Next, we discuss the exact number of periodic solutions and the parameter control conditions based on Theorem 1.

Theorem 2. *For System (1), there is no periodic solution when $N + n$ is odd and $b_1 b_2 \neq 0$ based on the displacement function of order ε^0.*

Proof. When $N + n$ is odd, based on Lemma 5, we have

$$\begin{gathered} h_1(z) = b_1 r_0, \\ h_2(z) = b_2 \rho_0, \\ h_3(z) = b_3. \end{gathered}$$

Since $b_1 b_2 \neq 0$, there is no solution that satisfies $(r_0)^2 + (\rho_0)^2 \neq 0$ for $h(z) = 0$. Hence, there is no periodic solution for System (1) in this case and Theorem 2 holds.

This proof is completed. □

Next, we discuss the number and parameter control conditions of the periodic solutions of System (1) when $N + n$ is even. For convenience, we introduce some notations:

$$\begin{aligned} &P = (b_1, b_2, b_3, \mu_{11}, \mu_{21}, \mu_{12}, \mu_{22}, n),\ l_1 = b_1 b_2 \mu_{11} \mu_{12},\ l_2 = b_1 b_2 \mu_{21} \mu_{22}, \\ &l_3(\lambda) = b_1 b_2 (\mu_{12}\lambda + \mu_{22})(\mu_{11}\lambda + \mu_{21}),\ l_4(\lambda) = b_1 b_2 (\mu_{12} + \mu_{22}\lambda)(\mu_{11} + \mu_{21}\lambda), \\ &q_1 = \mu_{12}\mu_{21} - \mu_{11}\mu_{22},\ q_2 = \mu_{21}\mu_{13} - \mu_{23}\mu_{11}, \\ &q_3 = (\mu_{12}\mu_{23} - \mu_{22}\mu_{13}) b_1 \mu_{22} + (\mu_{12}^2 + \mu_{22}^2)\mu_{21} b_2 n^{-1},\ q_4 = b_1 \mu_{13} - b_2 n^{-1} \mu_{21}, \\ &q_5 = b_1 \mu_{23} + b_2 n^{-1} \mu_{11},\ q_6 = b_1 \mu_{23} - b_3 \mu_{21},\ q_7 = b_1 \mu_{13} - b_3 \mu_{11}, \\ &B_1 = -b_3 \mu_{12} \mu_{11} + b_1 \mu_{13} \mu_{12} - b_2 n^{-1} \mu_{22} \mu_{11}, \\ &B_2 = -b_3 \mu_{12} \mu_{21} - b_3 \mu_{22} \mu_{11} - b_2 n^{-1} \mu_{22} \mu_{21} + b_2 n^{-1} \mu_{12} \mu_{11} + b_1 \mu_{13} \mu_{22} + b_1 \mu_{23} \mu_{12}, \\ &B_3 = -b_3 \mu_{22} \mu_{21} + b_2 n^{-1} \mu_{12} \mu_{21} + b_1 \mu_{23} \mu_{22},\ \Delta = B_2^2 - 4 B_1 B_3,\ k_0 = -\frac{B_3 \mu_{11}}{B_1 \mu_{21}}, \\ &k_i = \frac{-B_2 + (-1)^i \sqrt{\Delta}}{2B_1}, k_m = \frac{-B_2 + (-1)^m \sqrt{\Delta}}{2B_3}, i = 1, 2, m = 3, 4, \end{aligned} \tag{15}$$

and some important sets:

$$\begin{aligned} &P_0 = \left\{P \middle| \mu_{11}\mu_{12}\mu_{21}\mu_{22} \neq 0\right\},\ P_{1ij} = \left\{P \middle| \mu_{ij} = 0, \mu_{i(3-j)}\mu_{(3-i)j}\mu_{(3-i)(3-j)} \neq 0\right\}, \\ &P_{21i} = \left\{P \middle| \mu_{1i} = \mu_{2(3-i)} = 0, \mu_{1(3-i)}\mu_{2i} \neq 0\right\},\ P_{3i} = \left\{P \middle| \mu_{1i} = \mu_{2i} = 0\right\}, \\ &P_{22i} = \left\{P \middle| \mu_{i1} = \mu_{i2} = 0, \mu_{(3-i)1}\mu_{(3-i)2} \neq 0\right\},\ W_{v1} = \left\{P \middle| q_v \neq 0\right\},\ W_{v2} = \left\{P \middle| q_v = 0\right\}, \\ &Q_{1j_1} = \left\{P \middle| B_{j_1} = 0, B_{j_2}B_{j_3} \neq 0, j_1, j_2, j_3 \in \{1, 2, 3\}, j_1 \neq j_2 \neq j_3\right\}, \\ &Q_{2j_1j_2} = \left\{P \middle| B_{j_1} = B_{j_2} = 0, B_{j_3} \neq 0, j_1, j_2, j_3 \in \{1, 2, 3\}, j_1 \neq j_2 \neq j_3, j_1 < j_2\right\}, \\ &Q_0 = \{P | B_1 B_2 B_3 \neq 0\},\ Q_3 = \{P | B_1 = B_2 = B_3 = 0\},\ L_{i1} = \{P | l_i > 0\}, \\ &L_{i2} = \{P | l_i < 0\},\ L_{31} = \left\{P \middle| l_3(k_0) > 0\right\},\ L_{32} = \left\{P \middle| l_3(k_0) < 0\right\},\ W_1 = W_{11} \cap W_{21}, \\ &U_1 = \left\{j \middle| \Delta > 0, l_3(k_j) > 0, j \in \{1, 2\}\right\},\ U_2 = \left\{m \middle| \Delta > 0, l_4(k_m) > 0, m \in \{3, 4\}\right\}, \\ &i, j = 1, 2, v = 1, 2, 3, 4, 5, 6, 7. \end{aligned} \tag{16}$$

Denote $\mathfrak{R}_1(U_i)$ as the elements number of set $U_i (i = 1,2)$ and write

$$
\begin{aligned}
&\Gamma_{01} = (P_0 \cup P_{11} \cup P_{12} \cup P_{21}) \cap W_1,\ \Gamma_{02} = (P_0 \cup P_{112} \cup P_{122}) \cap W_{22},\ Q_1 = Q_0 \cup Q_{12} \cup Q_{13},\\
&\Gamma_{11} = \Gamma_{01} \cap Q_1,\ \Gamma_{12} = \Gamma_{01} \cap (Q_{11} \cup Q_{213}),\ \Gamma_{1i}(l) = \left\{P \middle| P \in \Gamma_{1i}, \mathfrak{R}_1(U_i) = l, l \in \{0,1,2\}\right\},\\
&\Gamma_{21}(i) = P_0 \cap W_{12} \cap L_{1i},\ \Gamma_{22}(i) = \Gamma_{02} \cap Q_1 \cap L_{3i},\ \Gamma_{23}(i) = \Gamma_{02} \cap Q_{11} \cap L_{1i},\\
&\Gamma_3(i) = (P_{221} \cap W_{41} \cap L_{2i}) \cup (P_{222} \cap W_{51} \cap L_{1i}),\\
&\Gamma_4 = (P_{221} \cap W_{42} \cap W_{61}) \cup (P_{222} \cap W_{52} \cap W_{71}),\\
&\Gamma_{51} = \{P|\Delta < 0\},\ \Gamma_{52} = \{P|\Delta = 0\},\ \Gamma_5 = W_1 \cap \Gamma_{51},\\
&\Gamma_6 = W_1 \cap \Gamma_{52},\ P_{ij} = P_{ij1} \cup P_{ij2},\ i,j = 1,2,\ l = 0,1,2.
\end{aligned}
\tag{17}
$$

Denoting $\mathfrak{R}$ as the number of periodic solutions of System (1), the following result can be obtained.

Theorem 3. *Considering System (1) and supposing $P \notin Q_3$, when $N + n$ is even and $b_1 b_2 \neq 0$, the following statements hold for $\varepsilon > 0$ sufficiently small based on the displacement function of order ε^0.*

(1) $\mathfrak{R} = 2n$ *if* $P \in \Gamma_1(2)$.
(2) $\mathfrak{R} = n$ *if* $P \in \Gamma_1(1) \cup \Gamma_2(1) \cup \Gamma_3(1)$.
(3) $\mathfrak{R} = 0$ *if* $P \in \Gamma_1(0) \cup \Gamma_2(2) \cup \Gamma_3(2) \cup \Gamma_4 \cup \Gamma_5 \cup P_{31}$.
(4) *The number of periodic solutions of System (1) cannot be obtained if $P \in \Gamma_6$, where $\Gamma_1(l) = \Gamma_{11}(l) \cup \Gamma_{12}(l)$, $\Gamma_2(j) = \Gamma_{21}(j) \cup \Gamma_{22}(j) \cup \Gamma_{23}(j)$, $l = 0,1,2$, $j = 1,2$.*

Proof. When $N + n$ is even, the exact expression of $h(z)$ can be rewritten as:

$$
\begin{aligned}
&\widetilde{h_1}(\widetilde{z}) = b_1 r_0 + r_0^{N-1}\rho_0(\mu_{11}u + \mu_{21}v),\\
&\widetilde{h_2}(\widetilde{z}) = b_2 \rho_0 + r_0^{N}(\mu_{12}u + \mu_{22}v),\\
&\widetilde{h_3}(\widetilde{z}) = b_3 + r_0^{N}\rho_0^{-1}n^{-1}(-\mu_{22}u + \mu_{12}v) + r_0^{N-2}\rho_0(\mu_{13}u + \mu_{23}v),\\
&\widetilde{h_4}(\widetilde{z}) = u^2 + v^2 - 1,
\end{aligned}
\tag{18}
$$

where $\widetilde{z} = (r_0, \rho_0, u, v)$, $u = \sin(ns_0)$, $v = \cos(ns_0)$. Next, we discuss the number of periodic solutions of System (1) by considering the real solutions of $\widetilde{h}(\widetilde{z}) = 0$ based on the following two cases, where $\widetilde{h} = (\widetilde{h}_1, \widetilde{h}_2, \widetilde{h}_3, \widetilde{h}_4)^{\mathrm{T}}$.

(1) If $\mu_{11} = \mu_{21} = 0$, i.e., $P \in P_{31}$, the equation $\widetilde{h}(\widetilde{z}) = 0$ has no solution with respect to $\widetilde{z}$ since $b_1 b_2 \neq 0$, and we obtain $\mathfrak{R} = 0$ in this case. If $\mu_{12} = \mu_{22} = 0$, i.e., $P \in P_{32}$, now $P \in Q_3$, and we will not discuss the periodic solutions of System (1) in this case.
(2) If $P \notin P_{31} \cup P_{32}$, equation $\widetilde{h}(\widetilde{z}) = 0$ has real solutions with respect to $\widetilde{z}$ only in the case of $r_0 \rho_0 \neq 0$. Hence, based on Equation (18), the following equations can be obtained:

$$
\frac{B_1 u^2 + B_2 uv + B_3 v^2}{-(\mu_{12}u + \mu_{22}v)(\mu_{11}u + \mu_{21}v)} = 0 \tag{19}
$$

and

$$
r_0^{N-1} = \frac{-b_2 \rho_0^2}{(\mu_{12}u + \mu_{22}v) r_0^2},\ \frac{\rho_0^2}{r_0^2} = \frac{b_1(\mu_{12}u + \mu_{22}v)}{b_2(\mu_{11}u + \mu_{21}v)}, \tag{20}
$$

where the expressions of B_1, B_2, and B_3 are shown in (15). Based on Equation (19), the relationship between u and v can be obtained, which reveals the value of $\frac{\rho_0^2}{r_0^2}$. When $\frac{\rho_0^2}{r_0^2}$ is positive, combining with the equation $u^2 + v^2 = 1$, a set of solution (u, v), which provides a positive r_0^{N-1}, can be obtained. Hence, when a positive $\frac{\rho_0^2}{r_0^2}$ is obtained, the number of solutions of $\widetilde{h}(\widetilde{z}) = 0$ that satisfy $r_0, \rho_0 > 0$ is one, so the equation $h(z) = 0$ has n solutions with respect to z for $s_0 \in (0, 2\pi]$, which reveals $\mathfrak{R} = n$ if the Jacobian of $h(z)$ is nonzero at the solution. Next, we discuss the periodic solutions of system (1) by considering the solutions of $\widetilde{h}(\widetilde{z}) = 0$ based on the following two cases:

Case 1. $P \in P_0 \cup P_{11} \cup P_{12} \cup P_{21}$. The periodic solutions of System (1) will be discussed by the following three subcases:

(1) If $\mu_{12}\mu_{21} - \mu_{11}\mu_{22} = 0$, which only exists for $P \in P_0$, then

$$\begin{aligned} \frac{\rho_0^2}{r_0^2} &= \frac{b_1\mu_{12}}{b_2\mu_{11}}, \\ B_1\mu_{21}\mu_{22} - B_3\mu_{11}\mu_{12} &= -\frac{\mu_{12}\mu_{21}q_3}{\mu_{22}}, \\ \frac{B_1u^2 + B_2uv + B_3v^2}{(\mu_{12}u + \mu_{22}v)(\mu_{11}u + \mu_{21}v)} &= \frac{B_1\mu_{21}u + B_3\mu_{11}v}{\mu_{11}\mu_{21}(\mu_{12}u + \mu_{22}v)}. \end{aligned} \tag{21}$$

In this subcase, if $B_1 = 0$, we obtain $B_2 = \frac{q_3}{\mu_{22}}$, $B_3 = \frac{q_3}{\mu_{12}}$; if $B_3 = 0$, we obtain $B_1 = -\frac{\mu_{12}q_3}{\mu_{22}^2}$, $B_2 = -\frac{q_3}{\mu_{22}}$. Hence, the parameter condition $P \notin Q_3$ holds only when $q_3 \neq 0$, which indicates $B_1\mu_{21}\mu_{22} - B_3\mu_{11}\mu_{12} \neq 0$ when $P \notin Q_3$. Now Equation (19) can be reduced to

$$B_1\mu_{21}u + B_3\mu_{11}v = 0. \tag{22}$$

In fact, it is easy to verify that there exists no parameter condition such that $P \in Q_{212} \cup Q_{213} \cup Q_{223} \cup Q_3$ due to $q_3 \neq 0$. Hence, combining Equation (22) with $u^2 + v^2 = 1$, the number of solutions of $\widetilde{h}(\widetilde{z}) = 0$ that satisfy $r_0, \rho_0 > 0$ is one for $b_1b_2\mu_{11}\mu_{12} > 0$ and zero for $b_1b_2\mu_{11}\mu_{12} < 0$.

(2) If $\mu_{21}\mu_{13} - \mu_{23}\mu_{11} = 0$, which only exists for $P \in P_0 \cup P_{112} \cup P_{122}$, then

$$\begin{aligned} \frac{\rho_0^2}{r_0^2} &= \frac{b_1(\mu_{12}u + \mu_{22}v)}{b_2(\mu_{11}u + \mu_{21}v)}, \\ B_1\mu_{21}\mu_{22} - B_3\mu_{11}\mu_{12} &= -(\mu_{22}^2 + \mu_{12}^2)b_2n^{-1}\mu_{11}\mu_{21}, \\ \frac{B_1u^2 + B_2uv + B_3v^2}{(\mu_{12}u + \mu_{22}v)(\mu_{11}u + \mu_{21}v)} &= \frac{B_1\mu_{21}u + B_3\mu_{11}v}{\mu_{11}\mu_{21}(\mu_{12}u + \mu_{22}v)}. \end{aligned} \tag{23}$$

We obtain $B_1\mu_{21}\mu_{22} - B_3\mu_{11}\mu_{12} \neq 0$. So Equation (19) can be reduced to

$$B_1\mu_{21}u + B_3\mu_{11}v = 0. \tag{24}$$

In fact, it is easy to verify that there exists no parameter condition such that $P \in Q_{212} \cup Q_{213} \cup Q_{223} \cup Q_3$. Next, we discuss the periodic solutions of System (1) by discussing the following cases:

(I) If $P \in Q_0 \cup Q_{12} \cup Q_{13}$, from Equations (23)–(24), we obtain

$$u = k_0v,\ \frac{\rho_0^2}{r_0^2} = l_3(k_0),$$

where $k_0 = -\frac{B_3\mu_{11}}{B_1\mu_{21}}$, $l_3(k_0) = \frac{b_1(\mu_{12}k_0+\mu_{22})}{b_2(\mu_{11}k_0+\mu_{21})}$. Combining Equation (22) with the equation $u^2 + v^2 = 1$, the number of the solutions of $\widetilde{h}(\widetilde{z}) = 0$ that satisfy $r_0, \rho_0 > 0$ is one for $l_3(k_0) > 0$ and zero for $l_3(k_0) < 0$.

(II) If $P \in Q_{11}$, then

$$v = 0,\ \frac{\rho_0^2}{r_0^2} = \frac{b_1\mu_{12}}{b_2\mu_{11}}.$$

The number of solutions of $\widetilde{h}(\widetilde{z}) = 0$ that satisfy $r_0, \rho_0 > 0$ is one for $l_1 > 0$ and zero for $l_1 < 0$.

The Jacobian of $h(z) = 0$ at each solution is nonzero when $P \notin Q_3$. Hence, we obtain $\Re = n$ for $P \in \Gamma_{21}(1) \cup \Gamma_{22}(1) \cup \Gamma_{23}(1)$ and $\Re = 0$ for $P \in \Gamma_{21}(2) \cup \Gamma_{22}(2) \cup \Gamma_{23}(2)$.

(3) If $(\mu_{12}\mu_{21} - \mu_{11}\mu_{22})(\mu_{21}\mu_{13} - \mu_{23}\mu_{11}) \neq 0$ when $P \in P_0 \cup P_{11} \cup P_{12} \cup P_{21}$, Equation (19) is irreducible. To obtain the relationship between u and v, we need to discuss the following equation:

$$B_1u^2 + B_2uv + B_3v^2 = 0. \tag{25}$$

When $\Delta = B_2^2 - 4B_1B_3 = 0$, the Jacobian of $h(z) = 0$ is zero at each solution, which implies that the periodic solutions of System (1) cannot be obtained by Theorem 1 under this condition; when $\Delta < 0$, there exists no real relationship between u and v, which shows that System (1) has no periodic solution under this condition. Hence, only when $\Delta > 0$, two real expressions of u with respect to v can be obtained and System (1) may exist periodic solutions. Next, we discuss the periodic solutions of System (1) by the following three cases:

(I) If $P \in Q_{223} \cup Q_{212}$, it is always $\Delta = 0$.

(II) If $P \in Q_0 \cup Q_{12} \cup Q_{13}$, when $\Delta > 0$, from Equation (25), then

$$u = k_iv,\ k_i = \frac{-B_2 + (-1)^i\sqrt{\Delta}}{2B_1},\ i = 1, 2.$$

Hence, in this case, the number of solutions of $\widetilde{h}(\widetilde{z}) = 0$ that satisfy $r_0, \rho_0 > 0$ is: (a) two for $l_3(k_1) > 0, l_3(k_2) > 0$; (b) one for $l_3(k_1)l_3(k_2) < 0$; (c) zero for $l_3(k_1) < 0, l_3(k_2) < 0$.

(III) If $P \in Q_{11} \cup Q_{213}$, when $\Delta > 0$, from Equation (25), then

$v = k_mu, k_m = \frac{-B_2+(-1)^m\sqrt{\Delta}}{2B_3}, m = 3, 4.$

Hence, in this case, the number of solutions of $\widetilde{h}(\widetilde{z}) = 0$ which satisfy $r_0, \rho_0 > 0$ is: (a) two for $l_4(k_3) > 0, l_4(k_4) > 0$; (b) one for $l_4(k_3)l_4(k_4) < 0$; (c) zero for $l_4(k_3) < 0, l_4(k_4) < 0$.

It is easy to verify that there exists no parameter condition such that $P \in Q_3$. Together with the cases (I)–(III), the Jacobian of $h(z) = 0$ is nonzero at each solution when $\Delta \neq 0$. Hence, we obtain $\Re = 2\,n$ for $P \in \Gamma_{11}(2) \cup \Gamma_{12}(2)$; $\Re = n$ for $P \in \Gamma_{11}(1) \cup \Gamma_{12}(1)$; $\Re = 0$ for $P \in \Gamma_{11}(0) \cup \Gamma_{12}(0) \cup \Gamma_{53}$; and the number of periodic solutions of System (1) cannot be obtained for $P \in \Gamma_{61}$, where

$$\Gamma_{53} = (P_0 \cup P_{11} \cup P_{12} \cup P_{21}) \cap (W_{11} \cap W_{21}) \cap \Gamma_{51},$$
$$\Gamma_{61} = (P_0 \cup P_{11} \cup P_{12} \cup P_{21}) \cap (W_{11} \cap W_{21}) \cap \Gamma_{52}.$$

Case 2. $P \in P_{22}$. Next, we discuss the periodic solutions of System (1) by the following two subcases.

(I) When $P \in P_{221}$, we obtain $B_1 = 0$. If $B_2 \neq 0$, i.e., $P \in W_{41}$, we obtain

$$B_2u + B_3v = 0,$$
$$\frac{\rho_0^2}{r_0^2} = \frac{b_1\mu_{22}}{b_2\mu_{21}}.$$

Now the number of the solutions of $\widetilde{h}(\widetilde{z}) = 0$ that satisfy $r_0, \rho_0 > 0$ with respect to $\widetilde{z}$ is one for $b_1 b_2 \mu_{21} \mu_{22} > 0$ and zero for $b_1 b_2 \mu_{21} \mu_{22} < 0$. If $B_2 = 0$, $B_3 \neq 0$, i.e., $P \in W_{42} \cap W_{61}$, the equation $\widetilde{h}(\widetilde{z}) = 0$ has no solution.

(II) When $P \in P_{222}$, we obtain $B_3 = 0$. If $B_2 \neq 0$, i.e., $P \in W_{51}$, we obtain

$$\begin{gathered} B_1 u + B_2 v = 0, \\ \frac{\rho_0^2}{r_0^2} = \frac{b_1 \mu_{12}}{b_2 \mu_{11}}. \end{gathered}$$

Now the number of the solutions of $\widetilde{h}(\widetilde{z}) = 0$ which satisfy $r_0, \rho_0 > 0$ with respect to $\widetilde{z}$ is one for $b_1 b_2 \mu_{11} \mu_{12} > 0$ and zero for $b_1 b_2 \mu_{11} \mu_{12} < 0$. If $B_2 = 0$, $B_1 \neq 0$, i.e., $P \in W_{52} \cap W_{71}$, the equation $\widetilde{h}(\widetilde{z}) = 0$ has no solution.

Hence, based on cases (I)–(II), we obtain $\mathfrak{R} = n$ for $P \in \Gamma_3(1)$ and $\mathfrak{R} = 0$ for $P \in \Gamma_3(2) \cup \Gamma_4$.

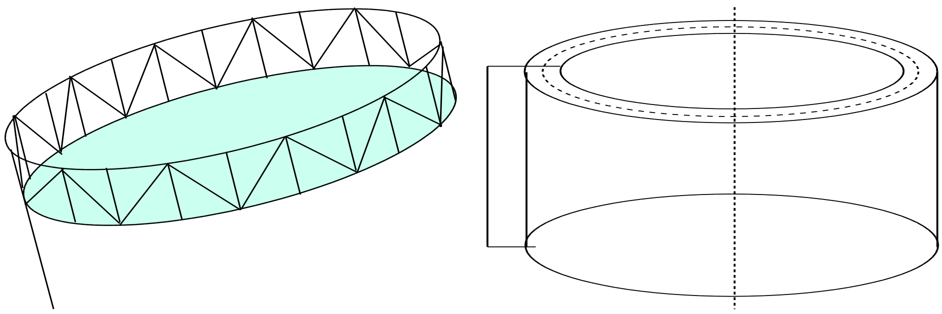
In summary, together with the above cases, we can obtain the fact that $(P_{22} \cup P_{31}) \cap W_{11} = \varnothing$, which implies $\Gamma_{53} = \Gamma_5$ and $\Gamma_{61} = \Gamma_6$. Hence, the exact number of the periodic solutions of System (1) and the parameter control conditions can be obtained as shown in Theorem 3.

This proof is completed. □

In fact, the upper bounds of the number of periodic solutions for a four-dimensional system with all the N-degree homogeneous nonlinearities in the cases of $n = N$ and $n = 0$ have been obtained [27,29]. However, if some coefficients of the N-degree terms are zero and the system can be reduced to Systems (1) or (3), some more precise and detailed results, including the existence condition, exact number, and parameter control conditions of the periodic solutions are obtained for $n \leq N$, $N \geq 2$. Considering a special case, $n = N$, for System (1), the exact number of periodic solutions obtained in Theorem 3 verifies and improves the upper bound of the number of periodic solutions obtained in the previous literature. Results shown in Theorems 1–3 present a new theoretical description and mechanism explanation for the phenomena of the emergence and disappearance of periodic solutions, which can be widely and directly applied to engineering applications of form (1) and provides engineers the parameter method for vibration control.

4. Application

To demonstrate the applicability and effectiveness of our theoretical results, the periodic breath vibrations of a two-degree-of-freedom mechanical model of the circular mesh antenna subjected to the thermal excitation are investigated. An equivalent circular cylindrical shell model is regarded as a simplified model of the circular mesh antenna, as shown in Figure 1 [30].



(**a**) (**b**)

Figure 1. (**a**) The circular mesh antenna; (**b**) The equivalent circular cylindrical shell.

Based on Reddy's first-order shear deformation theorem, Hamilton's principle, and the Galerkin procedure, a two-degree-of-freedom dynamic system of the circular cylindrical shell is obtained as follows [30]:

$$\begin{aligned}
&\ddot{w}_1 + \gamma_{10}\dot{w}_1 + \omega_1^2 w_1 + \gamma_{11} f_T \cos(\Omega t) w_2 + \gamma_{12} w_2 + \gamma_{13} w_1^3 + \gamma_{14} w_1^2 w_2 + \gamma_{15} w_1 w_2^2 \\
&+\gamma_{16} w_2^3 = \gamma_{17} + \gamma_{18} f_T \cos(\Omega t), \\
&\ddot{w}_2 + \gamma_{20}\dot{w}_2 + \omega_2^2 w_2 + \gamma_{21} f_T \cos(\Omega t) w_1 + \gamma_{22} w_1 + \gamma_{23} w_2^3 + \gamma_{24} w_1 w_2^2 + \gamma_{25} w_1^2 w_2 \\
&+\gamma_{26} w_1^3 = \gamma_{27} + \gamma_{28} f_T \cos(\Omega t),
\end{aligned} \tag{26}$$

where γ_{10} and γ_{20} are frictional coefficients, f_T is thermal excitation, and ω_1 and ω_2 are two linear frequencies.

Considering the case of primary parameter resonance and 1:1 internal resonance, we have the relations

$$\omega_1^2 = \Omega^2 + \varepsilon\sigma_1,\ \omega_2^2 = \Omega^2 + \varepsilon\sigma_2,$$

where σ_i, $i = 1, 2$ are detuning parameters. Supposing $\Omega = 1$ for convenience, introducing the scale transformations

$$\begin{gathered}
\gamma_{i0} \to \varepsilon\gamma_{i0},\ \gamma_{14} \to \varepsilon\gamma_{14},\ \gamma_{26} \to \varepsilon\gamma_{26},\ \gamma_{ij} \to \varepsilon^2\gamma_{ij}, \\
\gamma_{16} \to \varepsilon^2\gamma_{16},\ \gamma_{24} \to \varepsilon^2\gamma_{24},\ i = 1, 2,\ j = 1, 2, 3, 5, 7, 8
\end{gathered}$$

and denoting the asymptotic solutions of System (26) as

$$w_i(t, \varepsilon) = w_{i0}(T_0, T_1) + \varepsilon w_{i1}(T_0, T_1) + \cdots,\ i = 1, 2,$$

we obtain

$$w_{i0}(T_0, T_1) = \Lambda_i(T_1) e^{\mathrm{i}T_0} + \overline{\Lambda}_i(T_1) e^{-\mathrm{i}T_0},\ i = 1, 2,$$

where $T_0 = t$, $T_1 = \varepsilon t$. Writing

$$\Lambda_i(T_1) = x_{2i-1}(T_1) + \mathrm{i}x_{2i}(T_1),\ i = 1, 2,$$

the averaged equations are obtained based on normal form theory and the method of multiple scales:

$$\begin{aligned}
\dot{x}_1 &= -\tfrac{1}{2}\gamma_{10}x_1 - \tfrac{1}{2}\sigma_1 x_2 - \tfrac{1}{2}\gamma_{14}x_4(x_1^2 + 3x_2^2) - \gamma_{14}x_1x_2x_3, \\
\dot{x}_2 &= -\tfrac{1}{2}\gamma_{10}x_2 + \tfrac{1}{2}\sigma_1 x_1 + \tfrac{1}{2}\gamma_{14}x_3(3x_1^2 + x_2^2) + \gamma_{14}x_1x_2x_4, \\
\dot{x}_3 &= -\tfrac{1}{2}\gamma_{20}x_3 - \tfrac{1}{2}\sigma_2 x_4 - \tfrac{3}{2}\gamma_{26}x_2(x_1^2 + x_2^2), \\
\dot{x}_4 &= -\tfrac{1}{2}\gamma_{20}x_4 + \tfrac{1}{2}\sigma_2 x_3 + \tfrac{3}{2}\gamma_{26}x_1(x_1^2 + x_2^2).
\end{aligned} \tag{27}$$

By introducing the transformations

$$\tau = \frac{\sigma_1 t}{2},\ \gamma_{10} \to \varepsilon\gamma_{10},\ \gamma_{20} \to \varepsilon\gamma_{20},$$

if $\sigma_1 \neq 0$, System (27) can be rewritten as

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}\tau} &= -x_2 - \frac{\varepsilon\gamma_{10}}{\sigma_1}x_1 - \frac{\gamma_{14}}{\sigma_1}x_4(x_1^2 + 3x_2^2) - \frac{2\gamma_{14}}{\sigma_1}x_1x_2x_3, \\
\frac{\mathrm{d}x_2}{\mathrm{d}\tau} &= x_1 - \frac{\varepsilon\gamma_{10}}{\sigma_1}x_2 + \frac{\gamma_{14}}{\sigma_1}x_3(3x_1^2 + x_2^2) + \frac{2\gamma_{14}}{\sigma_1}x_1x_2x_4, \\
\frac{\mathrm{d}x_3}{\mathrm{d}\tau} &= -\frac{\sigma_2}{\sigma_1}x_4 - \frac{\varepsilon\gamma_{20}}{\sigma_1}x_3 - \frac{3\gamma_{26}}{\sigma_1}x_2(x_1^2 + x_2^2), \\
\frac{\mathrm{d}x_4}{\mathrm{d}\tau} &= \frac{\sigma_2}{\sigma_1}x_3 - \frac{\varepsilon\gamma_{20}}{\sigma_1}x_4 + \frac{3\gamma_{26}}{\sigma_1}x_1(x_1^2 + x_2^2).
\end{aligned} \tag{28}$$

Now System (28) is in the form of System (1) and the degree of its homogeneous terms is three, which shows that $N = 3$. Supposing $\gamma_{10}\gamma_{20}\gamma_{14}\gamma_{26}\sigma_1 \neq 0$, we discuss the periodic solutions of System (28) by considering the cases of $n = 1$, $n = 2$, and $n = 3$, where: $n = \frac{\sigma_2}{\sigma_1}$.

(1) $n = 1$. Based on Lemma 5, we obtain:

$$b_1 = -\frac{2\gamma_{10}}{\sigma_1}\pi,\ b_2 = -\frac{2\gamma_{20}}{\sigma_1}\pi,\ \mu_{11} = -\frac{2\gamma_{14}}{\sigma_1}\pi,\ \mu_{12} = \frac{6\gamma_{26}}{\sigma_1}\pi,$$
$$\mu_{21} = \mu_{22} = b_3 = \mu_{13} = 0,\ \mu_{23} = -\frac{6\gamma_{14}}{\sigma_1}\pi.$$

The following two statements hold for System (28): (1) If $\gamma_{20} + 3\gamma_{10} = 0$, then $P \in Q_3$ and the periodic solutions cannot be obtained by Theorem 3; (2) If $\gamma_{20} + 3\gamma_{10} \neq 0$, then $b_1\mu_{23} + b_2 n^{-1}\mu_{11} \neq 0$ and the number of periodic solutions is 1 and 0 if $l_1 = b_1 b_2 \mu_{11}\mu_{12} > 0$ and $l_1 = b_1 b_2 \mu_{11}\mu_{12} < 0$, respectively.

(2) $n = 2$. Now $N + n$ is odd, and System (28) has no periodic solution based on Theorem 2.

(3) $n = 3$. Based on Lemma 5, we obtain $\mu_{11} = \mu_{21} = 0$. Hence, System (28) has no periodic solution in this case based on Theorem 3.

To verify the effectiveness of our results, we detect the phase portraits of the periodic solutions of System (28) under a group of parameter conditions via numerical simulations. Considering the parameter condition

$$\mathrm{PC} = (\sigma_2, \gamma_{10}, \gamma_{20}, \gamma_{14}) = (1, 0.5, 0.5, 1),$$

when $n = 1$, System (28) has one periodic solution for $\gamma_{26} < 0$ and no periodic solution for $\gamma_{26} > 0$; when $n = 2$ and $n = 3$, System (28) has no periodic solution. Next, we show the phase portraits of the one periodic solution of System (28) in the case of $\sigma_1 = 1$ and $\gamma_{26} < 0$. Figure 2a,b respectively demonstrate the projections of the periodic solution on the plans (x_1, x_2) and (x_3, x_4), and Figure 2c,d show the phase portraits in space (x_1, x_2, x_3) and (x_1, x_3, x_4) for $\gamma_{26} = -1$.

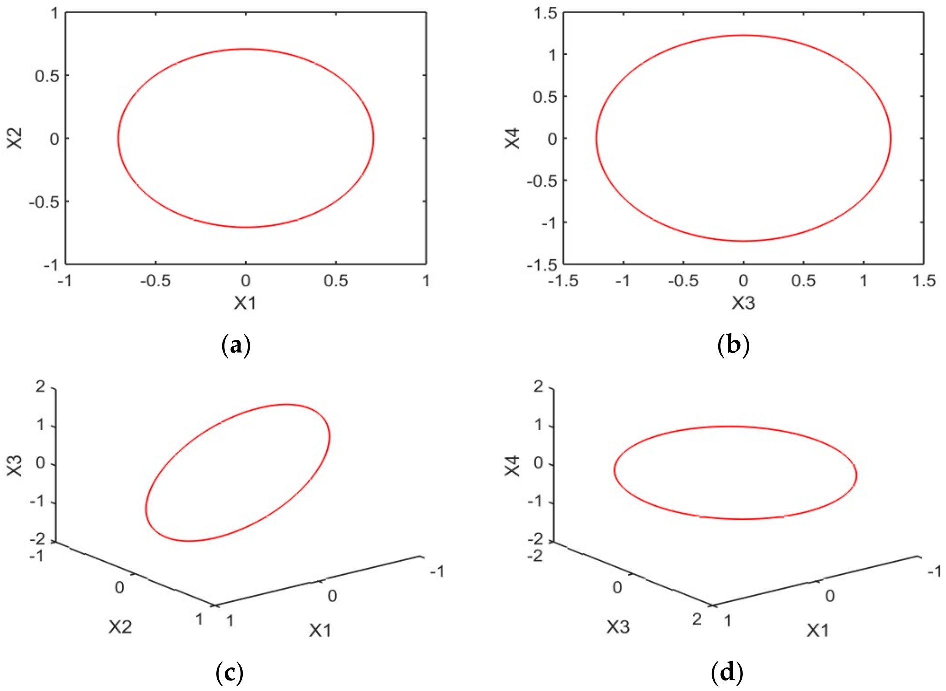


Figure 2. Periodic solution of System (28) with $\gamma_{26} = -1$: (**a**) on the plane (x_1, x_2); (**b**) on the plane (x_3, x_4); (**c**) in space (x_1, x_2, x_3); (**d**) in space (x_1, x_3, x_4).

Figure 3 shows the phase portraits of the periodic solution of System (28) for $\sigma_1 = 1$ and $\gamma_{26} = -0.5 < 0$.

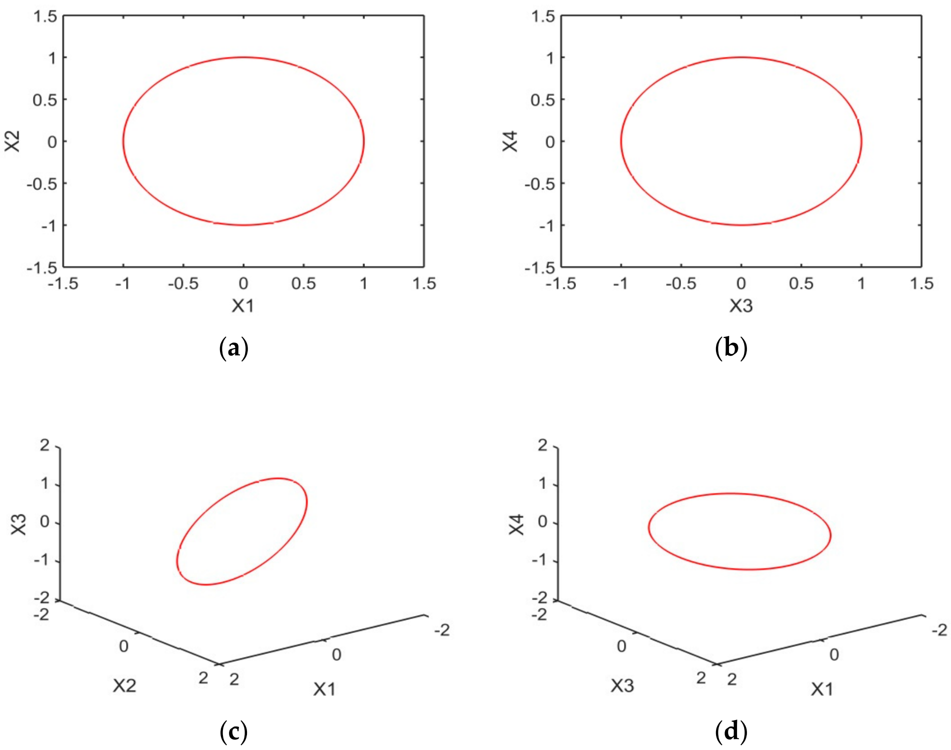


Figure 3. Periodic solution of System (28) with $\gamma_{26} = -0.5$: (**a**) on the plane (x_1, x_2); (**b**) on the plane (x_3, x_4); (**c**) in space (x_1, x_2, x_3); (**d**) in space (x_1, x_3, x_4).

The exact number and parameter control conditions of the periodic solutions of System (28) are obtained based on theoretical and numerical methods, which can provide a detailed description for the periodic solutions when the coefficients are allowed to vary in a wide range of parameters. Based on numerical simulations, under the parameter condition (PC), the phase portraits of the periodic solutions with different values of γ_{26} can be observed from Figures 2 and 3, which can verify the theoretical and numerical results. The obtained results show new understanding on the periodic motions of the circular mesh antenna and provide a method for vibration control.

5. Conclusions

In this paper, the periodic solutions for a four-dimensional coupled polynomial system with N-degree homogeneous nonlinearities are investigated. When the unperturbed linear part of the four-dimensional system has a center singular point in generalization resonance $1 : n$ at the origin for $n \in \mathbf{Z}_+$, $n \leq N$, $N \geq 2$, the new explicit expression of the displacement function of order ε^0 is obtained by introducing the exact formulas on power trigonometric integrals, which is in agreement with certain existing results, but we obtain the more detailed expression by including the exact coefficients. By considering the zero solutions of $h(z)$ with complex coefficient relations, the results on the existence condition, exact number, and parameter control conditions of the periodic solutions of System (1) are obtained, which shows that the parity of $N + n$ has a great influence on the periodic solutions. Results obtained in this paper provide more precise and detailed bifurcation information for System (1) than the existing results in the case of $n = 0, N$, which can be widely applied to various engineering applications and help engineers better analyze the complex periodic vibrations exhibited in reality. Theorems 1–3 verify and improve the upper bound of the number of periodic solutions obtained in the previous literature for System (1), which presents the effects of important parameters on the phenomena of the emergence and disappearance of periodic solutions and shows a new theoretical explanation on vibration mechanisms.

As an application, the periodic solutions of a two-degree-of-freedom circular mesh antenna model subjected to the thermal excitation are investigated by theoretical and numerical methods. The results on the exact number, parameter control conditions, and the relative positions of the periodic solutions are obtained, which verify the applicability and validity of the theoretical results and show the complicated periodic motions of circular mesh antenna. Periodic vibrations with high amplitudes may lead to serious damage to the device, and the theoretical results obtained in this paper may provide a parameter method for vibration control of circular mesh antenna.

It is remarkable that the method for detecting the existence of periodic solutions of System (1) can also be extended to the four-dimensional system with m-degree nonhomogeneous terms:

$$\dot{x} = (A + \varepsilon\widetilde{A})x + \varepsilon F(x), \tag{29}$$

where $F_i(x) = \sum_{N=n}^{m} \sum_{i_1+i_2+i_3+i_4=N} \sum_{i_1+i_2=N-2+[(i+1)/2]} a^i_{i_1 i_2 i_3 i_4} x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}$, $m \geq n$, and the other expressions x, A, and $\widetilde{A}$ have the same meaning as those exhibited in System (1). Lemma 5 provides a convenient way to compute the expression $h(z)$, where

$$h(z) = \psi(z) + \sum_{N+n \ even} \sum_{N=n}^{m} \varphi(N,z),\ \varphi(N,z) = \varphi(z).$$

Hence, we can obtain a result similar to Theorem 1. However, due to the complex coefficients of $h(z)$, research on the exact number and parameter control conditions of periodic solutions of System (29) is still a difficult topic that will be investigated in a further study. In addition, some higher dimensional (odd or even) systems have arisen more and more frequently in the fields of science and engineering in recent years, so we will try to study their periodic solutions as well as the stability in later work.

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References

1. Avramov, K.V.; Mikhlin, Y.V.; Kurilov, E. Asymptotic analysis of nonlinear dynamics of simply supported cylindrical shells. *Nonlinear Dyn.* **2007**, *47*, 331–352. [CrossRef]
2. Zhang, H.Y.; Huang, T.S.; Dai, L.M. Nonlinear dynamic analysis and characteristics diagnosis of seasonally perturbed predator–prey systems. *Commun. Nonlinear Sci. Numer. Simul.* **2015**, *22*, 407–419. [CrossRef]
3. Mergen, H.G. Dynamical analysis of multilayered cantilevers. *Commun. Nonlinear Sci. Numer. Simul.* **2019**, *71*, 244–253.
4. Li, J.B. Hilbert's 16th problem and bifurcations of planar polynomial vector fields. *Int. J. Bifurc. Chaos* **2003**, *13*, 47–106. [CrossRef]
5. An, Y.L.; Han, M.A. On the number of limit cycles near a homoclinic loop with a nilpotent singular point. *J. Differ. Equ.* **2015**, *258*, 3194–3247. [CrossRef]
6. Yang, J.M.; Ding, W. Limit cycles of a class of Lienard systems with restoring forces of seventh degree. *Appl. Math. Comput.* **2018**, *316*, 422–437. [CrossRef]
7. Da Cruz, L.P.C.; Novaes, D.D.; Torregrosa, J. New lower bound for the Hilbert number in piecewise quadratic differential systems. *J. Differ. Equ.* **2019**, *266*, 4170–4203. [CrossRef]
8. Bey, M.; Badi, S.; Fernane, K.; Makhlouf, A. The number of limit cycles bifurcating from the periodic orbits of an isochronous center. *Math. Methods Appl. Sci.* **2019**, *42*, 821–829. [CrossRef]
9. Kadry, S.; Alferov, G.; Ivanov, G.; Korolev, V.; Selitskaya, V. A new method to study the periodic solutions of the ordinary differential equations using functional analysis. *Mathematics* **2019**, *7*, 677. [CrossRef]
10. Poincaré, H. *Les Méthodes Nouvelles de la Mécanique Celeste*; Gauthier: Paris, France, 1899.

11. Veerman, P.; Holmes, P. The existence of arbitrarily many distinct periodic orbits in a two degree of freedom Hamiltonian system. *Physica D* **1985**, *14*, 177–192. [CrossRef]
12. Nayfeh, A.H.; Balachandran, A.B. *Applied Nonlinear Dynamics*; Wiley: New York, NY, USA, 1995.
13. Sanders, J.A.; Verhulst, F.; Murdock, J. *Averaging Methods in Nonlinear Dynamical Systems*; Springer: New York, NY, USA, 2007.
14. Mahdi, A.; Romanovski, V.G.; Shafer, D.S. Stability and periodic oscillations in the Moon-Rand systems. *Nonlinear Anal. Real World Appl.* **2013**, *14*, 294–313. [CrossRef]
15. Wang, L.X.; Zhang, J.M.; Shao, H.J. Existence and global stability of a periodic solution for a cellular neural network. *Nonlinear Sci. Numer. Simul.* **2014**, *19*, 2983–2992. [CrossRef]
16. Zhang, J.M.; Zhang, L.J.; Bai, Y.Z. Stability and bifurcation analysis on a predator–prey system with the weak Allee effect. *Mathematics* **2019**, *7*, 432. [CrossRef]
17. Wiggins, S.; Holmes, P. Homoclinic orbits in slowly varying oscillators. *SIAM J. Appl. Math.* **1987**, *18*, 592–611. [CrossRef]
18. Ye, Z.Y.; Han, M.A. Bifurcations of invariant tori and subharmonic solutions of singularly perturbed system. *Chin. Ann. Math. Ser. B* **2007**, *28*, 135–148. [CrossRef]
19. Chiba, H. Periodic orbits and chaos in fast-slow systems with Bogdanov-Takens type fold points. *J. Differ. Equ.* **2011**, *250*, 112–160. [CrossRef]
20. Sourdis, C. On periodic orbits in a slow-fast system with normally elliptic slow manifold. *Math. Mehtods Appl. Sci.* **2014**, *37*, 270–276. [CrossRef]
21. Tian, H.H.; Han, M.A. Bifurcation of periodic orbits by perturbing high-dimensional piecewise smooth integrable systems. *J. Differ. Equ.* **2017**, *263*, 7448–7474. [CrossRef]
22. Sun, M.; Zhang, W.; Chen, J.E.; Yao, M.H. Subharmonic Melnikov theory for degenerate resonance systems and its application. *Nonlinear Dyn.* **2017**, *89*, 1173–1186. [CrossRef]
23. Wang, Q.L.; Huang, W.T.; Liu, Y.R. Multiple limit cycles bifurcation from the degenerate singularity for a class of three-dimensional systems. *Acta Math. Appl. Sin. Engl. Ser.* **2016**, *32*, 73–80. [CrossRef]
24. Li, J.; Quan, T.T.; Zhang, W. Bifurcation and number of subharmonic solutions of a 4D non-autonomous slow–fast system and its application. *Nonlinear Dyn.* **2018**, *92*, 721–739. [CrossRef]
25. Cheng, Y.Y. Bifurcation of limit cycles of a class of piecewise linear differential systems in R^4 with three zones. *Discrete Dyn. Nat. Soc.* **2013**, *2013*. [CrossRef]
26. Navarro, J.F.; Poveda, R. Computation of periodic orbits in three-dimensional Lotka-Volterra systems. *Math. Methods Appl. Sci.* **2017**, *40*, 7185–7200. [CrossRef]
27. Barreira, L.; Llibre, J.; Valls, C. Bifurcation of limit cycles from a 4-dimensional center in R^m in resonance 1:N. *J. Math. Anal. Appl.* **2012**, *389*, 754–768. [CrossRef]
28. Chen, Y.M.; Liang, H.H. Zero-zero-Hopf bifurcation and ultimate bound estimation of a generalized Lorenz–Stenflo hyperchaotic system. *Math. Methods. Appl. Sci.* **2017**, *40*, 3424–3432. [CrossRef]
29. Barreira, L.; Llibre, J.; Valls, C. Limit cycles bifurcating from a zero–Hopf singularity in arbitrary dimension. *Nonlinear Dyn.* **2018**, *92*, 1159–1166. [CrossRef]
30. Zhang, W.; Chen, J.; Wang, Y.F.; Yang, X.D. Continuous model and nonlinear dynamic responses of circular mesh antenna clamped at one side. *Eng. Struct.* **2017**, *151*, 115–135. [CrossRef]