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Countably Expansiveness for Continuous Dynamical Systems

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Abstract: Expansiveness is very closely related to the stability theory of the dynamical systems. It is natural to consider various types of expansiveness such as countably-expansive, measure expansive, N-expansive, and so on. In this article, we introduce the new concept of countably expansiveness for continuous dynamical systems on a compact connected smooth manifold M by using the dense set D of M, which is different from the weak expansive flows. We establish some examples having the countably expansive property, and we prove that if a vector field X of M is C^1 stably countably expansive then it is quasi-Anosov.

Keywords: expansive; quasi-Anosov; quasi-transverality condition; Anosov

MSC: 37C20; 37C05; 37C29; 37D05

1. Introduction

Let X be a compact metric space with a metric d and $f : X \to X$ be a homeomorphism. Utz [1] introduced a dynamic property, which is called expansiveness. It means that, if two orbits stay within a small distance, then the orbits are the same. That is, a homeomorphism f is *expansive* if there is an expansive constant $e > 0$ such that for any $x \neq y \in X$ there is $i \in \mathbb{Z}$ satisfying $d(f^i(x), f^i(y)) > e$. From the definition of the expansiveness, it is possible to consider the set

$$\Phi^f_\delta(x) = \{y \in X : d(f^i(x), f^i(y)) \leq \delta \ \ \forall\, i \in \mathbb{Z}\}.$$

We can easily check that f is expansive if and only if $\Phi^f_\delta(x) = \{x\}$ for all $x \in X$.

Now, we have a natural question:

"Is $\Phi^f_\delta(x)$ finite or countable?"

Definition 1 ([2] Definition 2.8)**.** *Given $N \in \mathbb{N}$, a homeomorphism f of X is N-expansive on $A \subset X$ if there is an expansive constant $\delta > 0$ such that $\Phi^f_\delta(x)$ has at most N elements for all $x \in A$. If $A = X$. Then, we say that f is N-expansive.*

It is easy to see that if f is expansive then f is N-expansive. Now, we introduce another notion of expansiveness, which is a general notion of expansiveness.

Definition 2 ([2] Definition 1.6)**.** *We say that a homeomorphism f of X is countably expansive if there is an expansive constant $\delta > 0$ such that for all $x \in X$ the set $\Phi^f_\delta(x)$ is countable.*

Note that the relationship with among those notions is

$$\text{expansive} \Rightarrow N\text{-expansive} \Rightarrow \text{countably expansive}.$$

On the other hand, from the stochastic point of view, Morales and Sirvent [2] introduced a general notion of expansiveness by using a measure. For the Borel σ-algebra β on X, we denote $\mathcal{M}(X)$ the set of Borel probability measures on X endowed with the weak* topology. Let $\mathcal{M}^*(X)$ be the set of nonatomic measure $\mu \in \mathcal{M}(X)$.

Definition 3 ([2] Definition 1.3)**.** *We say that a homeomorphism f of X is μ-expansive if there exists an expansive constant $\delta > 0$ such that $\mu(\Phi^f_\delta(x)) = 0$ for all $x \in X$. We say that f is measure expansive if it is μ-expansive $\forall \mu \in \mathcal{M}^*(X)$.*

In among the notions, a remarkable notion is measure expansiveness (which was introduced by Morales [2]). It is exactly same as countably expansiveness (see [3]). That is, Artigue and Carrasco-Olivera [3] considered a relationship between the measure expansiveness and the countably expansiveness.

Remark 1 ([3] Theorem 2.1)**.** *Let $f : X \to X$ be a homeomorphism. Then,*

$$f \text{ is countably expansive} \iff f \text{ is measure expansive.}$$

Let M be a compact connected smooth manifold, and let $\mathrm{Diff}^1(M)$ be the space of diffeomorphisms of M endowed with the C^1 topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. In dynamical systems, the concept of expansiveness [1] is a useful notion for studying stability theory. In fact, Mañé [4] showed that if a diffeomorphism f of M is C^1 stably expansive then it is quasi-Anosov. Here, we say that f is *quasi-Anosov* if, for all $v \in TM \setminus \{\mathbf{0}\}$, the set $\{\|Df^n(v)\| : n \in \mathbb{Z}\}$ is unbounded.

Later, many mathematicians studied stability theory using the various types of expansiveness [5–8]. For instance, Moriaysu, Sakai, and Yamamoto [8] showed that if a diffeomorphism f of M is C^1 stably measure expansive then it is quasi-Anosov. The result is a generalization of the result of Mañé [4].

On the other hand, it is very important to extend from diffeomorphisms to vector fields (flows). In fact, many researchers studied the various aspects of flows, such as thermodynamics-Hamiltonian systems [9], nonlinear systems [10], and chaos systems [11,12].

From the result of [4], Moriyasu, Sakai, and Sun [7] extended expansive diffeomorphisms to vector fields about the C^1 stably point of view. That is, they showed that if a vector field X is C^1 stably expansive then it is quasi-Anosov. Lee and Oh [5] showed that if a vector field X is C^1 stably measure expansive then it is quasi-Anosov (We refer to the basic definitions related to the vector fields below.).

In this article, we introduce another type of countably expansive vector fields which is different than weak expansiveness in [6]. In addition, we establish some examples of the countably expansiveness for homeomorphisms and flows, such as shift map and suspension flow by applying the rotation map on the circle. Moreover, we prove that if a vector field X of a compact connected manifold M is C^1 stably countably expansive, then it is quasi-Anosov which is a general result of Moriyasu, Sakai, and Sun [7]. Furthermore, we have that if a vector field X of a compact connected manifold M is C^1 stably expansive, weak expansive, and countably expansive then it is quasi-Anosov.

2. Countably Expansiveness for Suspension Flows

In this paper, we focus on countably expansiveness which is defined as the following remark.

Remark 2. *In general, according to the Baire Category Theorem, there is a dense subset in a compact metric space X. Especially, we consider a dense subset to define the countably expansiveness on this space X.*

Definition 4. *We say that a homeomphism $f : X \to X$ is countably expansive if there is $\delta > 0$ such that for all $x \in X$ the set*

$$\Gamma_\delta^f(x) = \{y \in D \subset X : d(f^i(x), f^i(y)) \leq \delta \ \ \forall i \in \mathbb{Z} \text{ and } \overline{D} = X\}$$

is countable, where $\overline{A}$ is the closure of A.

Example 1. *Assume that X is a separable space and $f : X \to X$ is a homeomorphism. Then, it is clear that f is "countable expansive" (according to Definition 4).*

Example 2. *Let $f : S^1 \to S^1$ be an irrational rotation map. For all $x \in S^1$, we consider the set*

$$\Gamma_\delta^f(x) = \{y \in \mathbb{Q} \cap S^1 \subset S^1 : d(f^i x, f^i y) \leq \delta \ \ \forall\ i \in \mathbb{Z}\}$$

where $\mathbb{Q}$ is the rational numbers. Then, it is clear that $\overline{\mathbb{Q} \cap S^1} = S^1$, and $\#\Gamma_\delta^f(x) = \#(\mathbb{Q} \cap S^1)$: countable, denoted by $\#A$ the cardinality of a set A. Thus, f is countably expansive. However, $\Gamma_\delta^f(x) \neq \{x\}$ and $\Gamma_\delta^f(x)$ is not a finite set. Therefore, the map is neither expansive nor N-expansive.

Symbolic systems can be used to "code" some smooth systems. Indeed, to study of symbolic dynamics is the research of a specific class of the shift transformation in a sequence space. In addition, it provides more motivation of the relationships between topological and smooth dynamics. The properties of symbolic dynamical systems give a rich source of examples and counterexamples for topological dynamics and ergodic theory.

The set of all infinite sequences of 0s and 1s is called the sequence space of 0 and 1 or the symbol space of 0 and 1 is denoted by Σ_2. More precisely, $\Sigma_2 = \{(s_0 s_1 s_2 \cdots) | \ s_i = 0 \text{ or } s_i = 1 \ (\forall i)\}$. We often refer to elements of Σ_2 as points in Σ_2. Shift map $\sigma : \Sigma_2 \to \Sigma_2$ is defined by

$$\sigma(s_0 s_1 s_2 s_3 \cdots) = s_1 s_2 s_3 \cdots$$

In short, the shift map "deletets" is the first coordinate of the sequence, for example $\sigma(01110101\cdots) = 1110101\cdots$ (for more details, see [13]).

Definition 5 ([13] Definition 11.2)**.** *Let $s = s_0 s_1 s_2 s_3 \cdots$ and $t = t_0 t_1 t_2 t_3 \cdots$ be points in Σ_2. We denote the distance between s and t as $d(s,t)$ and define it by*

$$d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}.$$

Since $|s_i - t_i|$ is either 0 or 1, we know that

$$0 \leq d(s,t) \leq \sum_{i=0}^{\infty} \frac{1}{2^i} = 2.$$

The shift map is continuous; it is clear that two points are close if and only if their initial coordinates are same. The more the coordinates are the same before they are different, the closer they are to each other. Then, we know that the set of periodic points of the shift map is dense in Σ_2 (the shift map has 2^n periodic points of period n). We put the set

$$D = \{s_0 s_1 s_2 \cdots \ \in \ \Sigma_2 \mid \exists\, N \text{ such that } s_i = 0 \ \forall\ i \geq N\},$$

then $\overline{D} = \Sigma_2$ where $\overline{D}$ is a dense set of D. In fact, we can check the above facts by the following example.

Example 3. *Let $s, t \in \Sigma_2$ and $\delta = 1$. Then,*

$$\Gamma_\delta^\sigma(s) = \{y \in D \subset \Sigma : d(\sigma^n(s), \sigma^n(t)) \le \delta,\ \forall n \in \mathbb{Z} \text{ and } D \text{ is dense in } \Sigma\}$$

is countable.

Proof. Let $s = (s_0s_1s_2\cdots) \in \Sigma_2$ and $t = (t_0t_1t_2\cdots) \in \Sigma_2$. Then, we have two cases as follows.

(i) $d(\sigma^n(s), \sigma^n(t)) = 0$ for $\forall n$ implies $s = t$.
(ii) If $s \ne t$, then $0 < d(\sigma^n(s), \sigma^n(t)) \le \delta$ for $\forall n$ implies there exists $j \in \mathbb{Z}$ such that $s_{n+j} \ne t_{n+j}$. That is, the jth coordinate of $\sigma^n(s) = s_ns_{n+1}s_{n+2}\cdots s_{n+j}\cdots$ and $\sigma^n(t) = t_nt_{n+1}t_{n+2}\cdots t_{n+j}\cdots$ have at least one different coordinate components.

For two cases, we consider the cardinality of j, where j is countably many. This means that $\Gamma_\delta^\sigma(s)$ is a countable set. □

On the other hand, we need to check some properties of countably expansive homeomorphism, which are used to prove the lemmas and theorems below.

Lemma 1. *If a homeomorphism f of X is countably expansive, then $f|_A : A \to A$ is countably expansive for a closed subset $A \subset X$.*

Proof. Since f is countably expansive, there exists $\delta > 0$ such that $\Gamma_\delta^f(x)$ is countable set for all $x \in X$. Let A be a closed subset of X. Then, $D \cap A$ is dense in X. If $y' \in D \cap A \subset X$ satisfying $d(f^i(x), f^i(y')) \le \delta$ for all $i \in \mathbb{Z}$ and for all $x \in X$, then $y' \in \Gamma_\delta^f(x)$. This means that $\Gamma_\delta^{f|_A}(x) \subset \Gamma_\delta^f(x)$. As $\Gamma_\delta^{f|_A}(x)$ is countable, $f|_A$ is countably expansive. □

Example 4. *Let f be the identity of the interval $[0, 1]$ and $A = \{0\}$. It is clear that $f : A \to A$ is expansive but f is not expansive in the whole interval. Thus, the converse of Lemma 1 cannot hold.*

Lemma 2. *Let $f : X \to X$ be a homeomorphism. Then,*

(a) f is countably expansive if and only f^k is countably expansive, for some $k \in \mathbb{Z} \setminus \{0\}$.
(b) If f is the identity map $\mathtt{I_d}$, then f is not countably expansive.

Proof of (a). ($\Rightarrow$) For fixed k, since X is compact we can choose $0 < \epsilon < \delta$ such that if $d(x, y) \le \epsilon$ then $d(f^i(x), f^i(y)) \le \delta$ for all $-k \le i \le k$. We have $\Gamma_\epsilon^{f^k}(x) \subset \Gamma_\delta^f(x)\ \forall x \in X$. Since $\Gamma_\delta^f(x)$ is countable, $\Gamma_\epsilon^{f^k}(x)$ is countable as well. Therefore, f^k is countably expansive.

($\Leftarrow$) Let $D \subset X$ be dense in X and $\delta > 0$ be the countably expansive constant for f^k for some $k \in \mathbb{Z} \setminus \{0\}$. For all $x \in X$, $\Gamma_\delta^{f^k}(x) = \{y \in D : d((f^k)^i(x), (f^k)^i(y)) \le \delta\ \ \forall i \in \mathbb{Z}\}$. Let $\Gamma_{i\frac{\delta}{2}}^f(x) = \{y \in D : d(f^i(x), f^i(y)) \le \delta/2\ \ 0 \le i \le k\}$. Then,

$$\bigcup_{i \in \mathbb{Z}} \Gamma_{i\frac{\delta}{2}}^f(x) = \Gamma_{\frac{\delta}{2}}^f(x) \subset \Gamma_\delta^{f^k}(x).$$

This means that f is countably expansive. □

Proof of (b). Suppose that f is the identity map $\mathtt{I_d}$. Then, there exists $x \in X$ such that $\Gamma_\delta^{\mathtt{I_d}}(x) = B[x, \delta]$ for any $\delta > 0$. As we know that the set is uncountable, f is not countably expansive. □

Remark 3. *In Lemma 2, (b) says that if the identity is countable expansive then the space is countable (there are countable compact metric spaces).*

Lee, Morales, and Thach [6] characterized the countably expansive flows in measure-theoretical terms, which is extended the result of [3] in the discrete case, called weak expansive flows. They showed that a flow is countably expansive if and only if the flow is weak measure expansive.

Definition 6 ([6] Definition 1.1)**.** *A flow ϕ on a compact metric space X is countably expansive if there is an expansive constant $\delta > 0$ such that for any $x \in X$ and $c \in C$ there is an at most countable subset $B \subseteq X$ satisfying $\Gamma^{\phi}_{\delta,c}(x) \subseteq \phi_{\mathbb{R}}(x)$, where*

$$\Gamma^{\phi}_{\delta,c}(x) = \bigcap_{t\in\mathbb{R}} \phi_{-c(t)}(B[\phi_t(x), \delta]).$$

Here, C denotes the set of continuous maps $c : \mathbb{R} \to \mathbb{R}$ with $c(0) = 0$ and $\phi_{\mathbb{R}}(x) = \{\phi_t(x) : t \in \mathbb{R}\}$.

Now, we introduce the new notion of *countably expansive flows* by using a dense subset D of X and consider the examples showing the countably expansive property, very well. A continuous *flow* of X such that

1. $\phi : X \times \mathbb{R} \longrightarrow X$ satisfying $\phi(x, 0) = x$,
2. $\phi(\phi(x, s), t) = \phi(x, s + t)$ for $x \in X$ and $s,\ t \in \mathbb{R}$.

Denote by

$$\phi(x, s) = \phi_s(x) \text{ and } \phi_{(a,b)}(x) = \{\phi_t(x) : t \in (a, b)\}.$$

Definition 7. *We say that a flow ϕ of X is countably expansive if there exist an expansive constant $\delta > 0$ and dense subset D of X such that $\Gamma^{\phi}_{\delta}(x)$ is a countable set, where*

$$\Gamma^{\phi}_{\delta}(x) = \{y \in D \subset X : d(\phi_t(x), \phi_{h(t)}(y)) \le \delta \ \exists\, h \in \mathcal{H} \text{ and } \forall\, t \in \mathbb{R}\}$$

and $\mathcal{H}$ denotes the set of increasing continuous maps $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$.

Now, let $\tau : \Sigma_2 \to \mathbb{R}$ be a continuous function and consider the space

$$Y^{\tau,f} = \{(x, t) \in \Sigma_2 \times \mathbb{R} : 0 \le t \le \tau(x)\}$$

with $(x, \tau(x)) \sim (\sigma(x), 0)$ for each $x \in \Sigma_2$. The *suspension flow* over σ with *height function* τ is the flow $(\phi^{\tau,f}_t)_{t\in\mathbb{R}}$ on $Y^{\tau,f}$ defined by

$$\phi^{\tau,f}_t(x, s) = (x, s + t) \text{ whenever } s + t \in [0, \tau(x)].$$

More precisely, for all $x \in \Sigma_2$, $\sigma^0(x) = 0$ and $\sigma^{n+1}(x) = \sigma^n(x) + \sigma(\tau(x))$ for all $n \in \mathbb{Z}$. For all $(x, s) \in Y^{\tau,f}$ and $t \in \mathbb{R}$, there is a unique $n \in \mathbb{Z}$ such that $\sigma^n(x) \le s + t < \sigma^{n+1}(x)$; we set

$$\phi^{\tau,f}_t(x, s) = (\tau(x), s + t - \sigma^n(x)).$$

The Bowen–Walters distance (Definition 2 of [14]) makes $Y^{\tau,f}$ a compact metric space where a neighborhood of a point $(x, s) \in Y$ contains all the points of $\phi^{\tau,f}_t(w, s) \in Y^{\tau,f}$ where $|t|$ is small and w is close to x. With respect to the topology generated, $\phi^{\tau,f}_t$ is a homeomorphism on $Y^{\tau,f}$ for all $t \in \mathbb{R}$.

Carrasco-Olivera and Morales [15] extended the concept of expansive measure from homeomorphism [2] to flows. They (respectively [6]) showed that a homeomorphism of a compact metric space is measure expansive (respectively, countable expansive) if its suspension flow is. The following theorem says that the case of countably expansiveness, which is defined in this paper, is also satisfied.

Theorem 1. *Let $f : X \to X$ be a homeomorphism. Then, f is countably expansive if and only if there is a continuous map $\tau : X \to (0,\infty)$ such that the suspension flow $\phi_t^{\tau,f}$ on $Y^{\tau,f}$ over f with height function τ is countably expansive.*

Proof. It is enough to show that *"only if"* part. Let $\tau : X \to (0,\infty)$ be given by $\tau(x) = 1$ for all $x \in X$. Then, the quotient space $Y^{\tau,f}$ associated with X and τ, the suspension flow $\phi_t^{1,f}$ on $Y^{\tau,f}$ given by

$$\phi_t^{1,f}(x,s) = (x, t+s) \text{ for } (x,s) \in Y^{\tau,f} \text{ with } 0 \le t+s \le 1.$$

Now, we claim that $\phi_t^{\tau,f}$ is countably expansive. It is sufficient to show that there are a constant $\delta > 0$ and a dense subset D' of $Y^{\tau,f}$ such that

$$\Gamma_\delta^{\phi_t^{1,f}}(x,s) = \{(y,t) \in D' \subset Y^{\tau,f} : d(\phi_t^{1,f}(x,s), \phi_{h(t)}^{1,f}(y,t)) \le \delta\}$$

for all $t \in \mathbb{R}$ and some $h \in \mathcal{H}$. Since $D' = D \times [0,1]$ and D is a dense subset of $Y^{\tau,f}$, $\Gamma_\delta^{\phi_t^{1,f}}(x,s) \subseteq \Gamma_\delta^f(x) \times [0,1]$. Thus, $\Gamma_\delta^{\phi_t^{1,\sigma}}(x,s)$ is a countable set. Therefore, the suspension flow $\phi_t^{\tau,f}$ on $Y^{\tau,f}$ over f with height function τ is countably expansive. □

By the following examples, it is easy to see that a suspension flow over an irrational rotation map on the unit circle S^1 is countably expansive by applying the Theorem 1.

Example 5. *Consider a flow ϕ on the unit circle S^1 given by*

$$\phi(e^{is}, t) = e^{(s+t)i} \ \ for\ e^{is} \in S^1, \ \ t \in \mathbb{R}.$$

Then ϕ is countably expansive.

In addition, we can see that the following example satisfies Theorem 1.

Example 6. *If $f : S^1 \longrightarrow S^1$ is an irrational rotation map then there is a continuous map $\tau : S^1 \longrightarrow (0,\infty)$ such that the suspension flow $\phi_t^{\tau,f}$ on $Y^{\tau,f}$ over f with height function τ is countably expansive.*

On the other hand, we can find a dense subset of Σ_2 as following remark.

Remark 4. *As we know that*

$$D = \{s = s_0 s_1 s_2 \cdots \in \Sigma_2 \ : \ \text{there is } N \text{ satisfying } s_i = 0 \text{ for all } i \ge N\}$$

is a dense subset of Σ_2, we can check that $D' = D \times [0,1]$ is a dense subset of $\Sigma_2 \times [0,1]$. Fix $0 < \delta < \frac{1}{4}$, let $y = (x,t),\ y_0 = (x_0, t_0)$ in D'. Then, there exists $h \in \mathcal{H}$ such that $d^{1,\sigma}(\phi_t^{1,\sigma}(y), \phi_{h(t)}^{1,f}(y_0)) \le \delta$ for all $t \in \mathbb{R}$. Since

$$\begin{aligned} \phi_1^{1,\sigma}(y) &= \phi_1^{1,\sigma}(x,t) = (x, t+1-\sigma^n(x)) \\ \phi_1^{1,\sigma}(y_0) &= \phi_{h(1)}^{1,\sigma}(x_0,t_0) = (x_0, t_0 + h(1) - \sigma^n(x_0)) \ \text{ and} \end{aligned}$$

$d^{1,f}(\phi_1^{1,\sigma}(y), \phi_{h(1)}^{1,\sigma}(y_0)) \le \delta$. *Therefore*

$$\begin{aligned} |t+1-\sigma^n(x) - t_0 - h(1) - \sigma^n(y)| &\le |t+1-t_0-h(1)| + |\sigma^n(x) - \sigma^n(x_0)| \\ &\le \delta. \end{aligned}$$

In fact there exists $N > 0$ such that $d(\sigma^n(x), \sigma^n(x_0)) \leq \frac{1}{2^n}$ for all $n \geq N$ then $B_\epsilon(y) \cap D' \neq \emptyset$. This means that D' is a dense subset of $\Sigma_2 \times [0,1]$.

Moreover, we can show that Theorem 1 holds by using the above remark for the case of the suspension flow over a shift map as following example.

Example 7. *If $\sigma : \Sigma_2 \to \Sigma_2$ is countably expansive if and only if there is a continuous map $\tau : \Sigma_2 \to (0, \infty)$ such that the suspension flow $\phi_t^{\tau,\sigma}$ on $Y^{\tau,\sigma}$ over σ with height function τ is countably expansive.*

Proof. Let $\tau : \Sigma_2 \to (0, \infty)$ be given by $\tau(x) = 1$ for all $x \in \Sigma_2$. Then, the quotient space $Y^{\tau,\sigma}$ corresponding to Σ and τ, the suspension flow $\phi_t^{1,\sigma}$ on $Y^{\tau,\sigma}$ given by

$$\phi_t^{1,\sigma}(x,s) = (x, t+s) \quad \text{for } (x,s) \in Y^{\tau,\sigma} \text{ with } 0 \leq t+s \leq 1.$$

Now, we claim that $\phi_t^{\tau,\sigma}$ is countably expansive. It is enough to show that there are a constant $\delta > 0$ and a dense subset D' of $Y^{\tau,\sigma}$ such that

$$\Gamma_\delta^{\phi_t^{1,\sigma}}(x,s) = \{(y,t) \in D' \subset Y^{\tau,\sigma} : d(\phi_t^{1,\sigma}(x,s), \phi_{h(t)}^{1,\sigma}(y,t)) \leq \delta\}$$

for all $t \in \mathbb{R}$ and some $h \in \mathcal{H}$. Since $D' = D \times [0,1]$ and D is a dense subset of Σ_2,

$$\Gamma_\delta^{\phi_t^{1,\sigma}}(x,s) \subseteq \Gamma_\delta^{\sigma}(x) \times [0,1].$$

Thus $\Gamma_\delta^{\phi_t^{1,\sigma}}(x,s)$ is a countable set. Therefore, the suspension flow $\phi_t^{\tau,\sigma}$ on $Y^{\tau,\sigma}$ over σ with height function τ is countably expansive. □

3. C^1 Stably Countably Expansive Vector Fields

Recall that M is a compact connected smooth manifold, d is the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Denote by $\mathfrak{X}^1(M)$ the set of all C^1 vector fields of M endowed with the C^1 topology. Then, every $X \in \mathfrak{X}^1(M)$ generates a C^1 flow $X_t : M \times \mathbb{R} \to M$ satisfying $X_s \circ X_t = X_{s+t}$ for all $s,t \in \mathbb{R}$, $X_0 = 1_d$ and $dX_t(x)/dt|_{t=0} = X(x)$ for any $x \in M$. Here, X_t is called the *integrated flow* of X. Throughout this paper, for $X, Y, \ldots \in \mathfrak{X}^1(M)$, denote the integrated flows by $X_t, Y_t, \ldots$, respectively.

Note: *To study of dynamical systems, the properties of orbits (or points) are important: singular, periodic, non-wandering, etc. If a flow has a periodic orbit or singularity, then it causes a chaos phenomenon (for example, Geometric Lorenz attractor). This means that we cannot control the system. Especially, the countably expansive flow which we present in this paper does not have a singularity. Thus, we could investigate the stability of countably expansive flows.*

For $x \in M$, denote by $O_X(x)$ the orbit $\{X_t(x) : t \in \mathbb{R}\}$ of the flow X_t (or X) through x. A point $x \in M$ is *singular* of X if $X(x) = \mathbf{0}_x$, $\mathrm{Sing}(X)$ denotes the set of singular points of X. It is said that a point p is *periodic* if $X_{\pi(p)}(p) = p$ for some $\pi(p) > 0$, but $X_t(p) \neq p$ for all $0 < t < \pi(p)$, $PO(X_t)$ denotes the set of periodic points of X. A point p is *regular* if $x \notin PO(X_t)$ and $x \notin \mathrm{Sing}(X)$. The set of non-wandering points of X, denoted by $\Omega(X_t)$ then we can see that

$$\mathrm{Sing}(X) \cup PO(X_t) \subset \Omega(X_t).$$

A flow X_t of M is *expansive* if for given $\epsilon > 0$ there is a expansive constant $\delta > 0$ such that if $x, y \in X$ satisfying $d(X_t(x), X_{h(t)}(y)) \leq \delta$ for some $h \in \mathcal{H}$ and all $t \in \mathbb{R}$ then $y \in X_{[-\epsilon,\epsilon]}(x)$, where $\mathcal{H}$ denotes the set of increasing continuous maps $h : \mathbb{R} \to \mathbb{R}$ fixing 0.

Definition 8. *We say that $X \in \mathfrak{X}^1(M)$ is countably expansive if there exists $\delta > 0$ such that*

$$\Gamma_\delta(x) \text{ is a countable set for all } x \in M,$$

where $\Gamma_\delta(x) = \{y \in D \subset M : d(X_{h(t)}(y), X_t(x)) \le \delta, \text{ for all } t \in \mathbb{R}, \text{ some } h \in \mathcal{H} \text{ and } D \text{ is dense in } M\}$.

First, we can check some properties of countably expansive flows as following lemmas.

Lemma 3. *If $X \in \mathfrak{X}^1(M)$ is countably expansive, then* $\mathrm{Sing}(X)$ *is totally disconnected.*

Proof. Assume that $\mathrm{Sing}(X)$ is not totally disconnected. Take $x, y \in M$. For any $\delta > 0$, let $\mathcal{I}$ be a closed small arc with two end points x and y such that the length of $\mathcal{I}$ is less than δ. Let $e = \delta/2$ be an expansive constant. We can take a local chart (U, ϕ) satisfying $\mathcal{I} \subset U(\subset \mathbb{R}^n)$. Then, $\mathcal{I} \cap \mathbb{Q}^c = D$. Thus, we consider the dense set $D \subset \mathcal{I}$ and we can easily see that $\overline{D} = \mathcal{I}$ and D is uncountable. For any $x \in \mathcal{I}$,

$$\Gamma_e(x) = \{y \in D : d(X_t(x), X_{h(t)}(y)) \le e \ \ \forall t \in \mathbb{R},\ h \in \mathcal{H}\} : \text{uncountable}.$$

This means that X is not countably expansive. This contradicts to complete the proof. □

We can see that the singular points of countably expansive flows are isolated by the below lemma.

Lemma 4. *Let $X \in \mathfrak{X}^1(M)$. If the flow X_t is countably expansive, then every singular points of X is isolated.*

Proof. Suppose that there exist $x \ne y \in \mathrm{Sing}(X)$ which are not isolated. Let $\mathcal{I}$ be a closed small arc with two endpoints x and y. By Lemma 3, this is a contradiction. Thus, every singular points is isolated. □

A closed X_t-invariant subset Λ is *hyperbolic* if there exist constants $C > 0$, $\lambda > 0$ and a splitting $T_xM = E^s_x \oplus \langle X(x)\rangle \oplus E^u_x$ $(x \in \Lambda)$ satisfying the tangent flow $DX_t : TM \to TM$ has invariant continuous splitting and

$$\|DX_t|_{E^s_x}\| \le Ce^{\lambda t} \text{ and } \|DX_{-t}|_{E^u_x}\| \le Ce^{-\lambda t}$$

for $x \in \Lambda$ and $t > 0$. We say that $X \in \mathfrak{X}^1(M)$ is *Anosov* when M is hyperbolic for X_t.

We say that a vector field X is *Axiom A* if $PO(X_t)$ is dense in $\Omega(X_t) \setminus \mathrm{Sing}(X)$ and $\Omega(X_t)$ is hyperbolic. For Axiom A vector field X, we know that $\Omega(X_t)$ is equal to the union of each *basic set* Λ_i $(1 \le i \le l)$ of X. Note that the basic set is closed, invariant, and transitive. A collection of basic sets $\Lambda_{i_1}, \cdots, \Lambda_{i_k}$ of X is called a *cycle* if, for each $j = 1, 2, \cdots, k$, there exists $a_j \in \Omega(X_t)$ such that $\alpha(a_j) \subset \Lambda_{i_j}$ and $\omega(a_j) \subset \Lambda_{i_{j+1}}$ $(k+1 \equiv 1)$. We say that a vector field X has no *cycle* if there exist no cycles among the basic sets of X.

For any hyperbolic periodic point x of X, the sets

$$\begin{aligned} W^s(x) =& \{y \in M : d(X_t(x), X_t(y)) \to 0 \text{ as } \ t \to \infty\} \ \text{ and} \\ W^u(x) =& \{y \in M : d(X_t(x), X_t(y)) \to 0 \text{ as } \ t \to -\infty\} \end{aligned}$$

are the *stable manifold* and *unstable manifold* of x, respectively. For Axiom A vector field $X \in \mathfrak{X}^1(M)$, we say that X has the *quasi-transversality condition* if $T_xW^s(x) \cap T_xW^u(x) = \{\mathbf{0_x}\}$ for any $x \in M$.

The *exponential map* defined by $\exp_x : T_xM(1) \to M$ for all $x \in M$ where $T_xM(r) = \{v \in T_xM : \|v\| \le r\}$. Let $M_X = \{x \in M : X(x) \ne \mathbf{0}_x\}$. For any $x \in M_X$, we set

$$N_x = (\mathrm{Span}X(x))^\perp \subset T_xM \ \text{ and } \ \Pi_{x,r} = \exp_x(N_x(r)),$$

where $N_x(r) = N_x \cap T_xM(r)$ for $0 < r \le 1$.

Let $\mathcal{N} = \bigcup_{x \in M_X} N_x$ be the normal bundle on M_X. Then, we present a *linear Poincaré flow* for X on $\mathcal{N}$ by

$$\Psi_t : \mathcal{N} \to \mathcal{N}, \ \Psi_t|_{N_x} = \pi_{N_x} \circ D_x X_t|_{N_x},$$

where $\pi_{N_x} : T_x M \to N_x$ is the natural projection along the direction of $X(x)$, and $D_x X_t$ is the derivative map of X_t. We say that $X \in \mathfrak{X}^1(M)$ is *quasi-Anosov* if $\sup_{t \in \mathbb{R}} \|\Psi_t(v)\| < \infty$ for $v \in \mathcal{N}$ then $v = \mathbf{0}$.

Definition 9. *We say that the integrated flow X_t of $X \in \mathfrak{X}^1(M)$ is C^1 stably countably expansive if there is a C^1 neighborhood $\mathcal{U}(X)$ of X such that the integrated flow Y_t of $Y \in \mathcal{U}(X)$ is countably expansive.*

The main theorem of this paper stated as follows.

Theorem 2. *If a vector field X is C^1 stably countably expansive, then it satisfies Axiom A without cycle condition.*

Now, let us prove the above theorem. To show this, we first need following lemma.

Lemma 5. *Suppose that $X \in \mathfrak{X}^1(M)$, $\mathrm{Sing}(X) = \emptyset$ and $p \in \mathcal{O}_X(p) \in PO(X_t)$ $(X_T(p) = p)$. For the Poincaré map $f : \Pi_{p,r_0} \to \Pi_p$ $(r_0 > 0)$, let $\mathcal{U}(X) \subset \mathfrak{X}^1(M)$ be a C^1 neighborhood of X and given $0 < r \le r_0$. Then, there are $\delta_0 > 0$ and $0 < \epsilon_0 < \frac{r}{2}$ such that for a map $\mathcal{O} : N_p \to N_p$ with $\|\mathcal{O} - D_p f\| < \delta_0$, there is $Y \in \mathcal{U}(X)$ satisfying*

$$\begin{aligned} &\text{(i) } Y(x) = X(x), \ \text{ if } x \notin F_p(X_t, r, \frac{T}{2}), \\ &\text{(ii) } p \in \mathcal{O}_X(p) \in PO(Y_t), \\ &\text{(iii) } g(x) = \begin{cases} exp_p \circ D_p f \circ exp_p^{-1}(x), & \text{if } x \in B[p, \frac{\epsilon_0}{4}] \cap \Pi_{p,r} \\ f(x), & \text{if } x \notin B[p, \epsilon_0] \cap \Pi_{p,r}, \end{cases} \end{aligned}$$

where $F_p(X_t, r, \frac{T}{2}) = \{X_t(y) : y \in \Pi_{p,r}$ and $0 \le t \le T/2\}$ and $g : \Pi_{p,r} \to \Pi_p$ is the Poincaré map of Y_t.

Proof. By Lemma 1.3 of [16]. □

Denote $\mathfrak{X}^*(M)$ as the set of $X \in \mathfrak{X}^1(M)$ with the property that there is a C^1 neighborhood $\mathcal{U}(X)$ of X such that every $\gamma \in \mathrm{PO}(Y_t)$ is hyperbolic for $Y \in \mathcal{U}(X)$. It was proved by [17] that $X \in \mathfrak{X}^*(M)$ if and only if X satisfies Axiom A without cycle condition.

Proof of Theorem 2. Let X be C^1 stably countably expansive. Then, the proof is completed by showing $X \in \mathfrak{X}^*(M)$. Suppose there exists $X \notin \mathfrak{X}^*(M)$. Then, there are $Y \in \mathcal{U}(X)$ and Y has a non-hyperbolic periodic point $p \in \mathcal{O} \in PO(Y_t)$.

Let $T = \pi(p)$ and $f : \Pi_{p,r_0} \to \Pi_p$ (for $r_0 > 0$) be the Poincaré map of Y_t at p. As p is a non-hyperbolic fixed point of f, there is an eigenvalue λ of $D_p f$ with $|\lambda| = 1$. Let $\delta_0 > 0$ and $0 < \epsilon_0 < r_0$ be given by Lemma 5 for $\mathcal{U}(X)$ and r_0. Then, for the linear isomorphism $\mathcal{O} = D_p f : N_p \to N_p$, there is $Z \in \mathcal{U}(Y)$ satisfying

$$\begin{aligned} &\text{(i) } Z(x) = Y(x), \ \text{ if } x \notin F_p(Y_t, r_0, \frac{T}{2}), \\ &\text{(ii) } g(x) = \begin{cases} \exp_p \circ D_p f \circ \exp_p^{-1}(x), & \text{if } x \in B[p, \frac{\epsilon_0}{4}] \cap \Pi_{p,r_0} \\ f(x) & \text{if } x \notin B[p, \epsilon_0] \cap \Pi_{p,r_0}. \end{cases} \end{aligned}$$

Here, g is the Poincaré map of Z. Since the eigenvalue λ of $D_p g$ is 1, we can take a vector v $(v \neq 0)$ associated to λ such that $\|v\| \le \frac{\epsilon_0}{4}$ and $\exp_p(v) \in B[p, \frac{\epsilon_0}{4}]$. Then,

$$g(\exp_p(v)) = \exp_p \circ D_p f \circ \exp_p^{-1}(\exp_p(v)) = \exp_p(v).$$

Put $I_v = \{\eta v : 0 < \eta < \frac{\epsilon_0}{8}\}$ and $\exp_p(I_v) = \mathcal{I}_p$. Then, $\mathcal{I}_p$ is an invariant small arc such that

$$\mathcal{I}_p \subset B[p, \epsilon_0] \cap \Pi_{p,r_0} \text{ and } g(x) = x \ (x \in \mathcal{I}_p),$$

and so $Z_T(\mathcal{I}_p) = \mathcal{I}_p$, where Z_T is the time T-map of the flow Z_t. Since Z_T is the identity on $\mathcal{I}_p$, Z_T is not countably expansive. Thus, this contradict to that X is C^1 stably countably expansive. Therefore, we completed the proof. □

For given $p \in M_X,\ t \in \mathbb{R}$, take a constant $r > 0$, a C^1 map $\tau : \Pi_{p,r} \to \mathbb{R}$ such that $\tau(p) = t$ and $X_{\tau(y)}(y) \in \Pi_{X_t(p),1}$ for $\forall y \in \Pi_{p,r}$. The *Poincaré map* $f_{p,t} : \Pi_{p,r_0} \to \Pi_{X_t(p),1}$ is given by

$$f_{p,t}(y) = X_{\tau(y)}(y) \ \text{ for } \ y \in \Pi_{p,r_0}.$$

For given $\epsilon > 0$, we denote by $\mathcal{N}_\epsilon(\Pi_{p,r})$ the set of diffeomorphisms $\xi : \Pi_{p,r} \to \Pi_{p,r}$ satisfying $\text{supp}(\xi) \subset \Pi_{p,\frac{r}{2}}$ and $d_{C^1}(\xi, \ \mathtt{I_d}) < \epsilon$. Here, d_{C^1} is the C^1 metric, $\mathtt{I_d} : \Pi_{p,r} \to \Pi_{p,r}$ is the identity map and $\text{supp}(\xi) = \overline{\Pi_{p,r}}$ where it differs from $\mathtt{I_d}$.

Lemma 6. *Suppose that $X \in \mathfrak{X}^1(M)$ and $Sing(X) = \emptyset$. For the Poincaré map $f : \Pi_{x,r_0} \to \Pi_{x'}(x' = X_{t_0}(x))$ and $X_t(x) \neq x$ $(0 < t \leq t_0)$, let $\mathcal{U}(X) \subset \mathfrak{X}^1(M)$ be C^1 neighborhood of X and $0 < r \leq r_0$. Then, there is $\epsilon > 0$ with the property that for any $\xi \in \mathcal{N}_\epsilon(\Pi_{x,r})$, there exists $Y \in \mathcal{U}(X)$ such that*

$$\begin{cases} Y(y) = X(y), & \text{if } y \notin F_x(X_t, r, t_0) \\ f_Y(y) = f \circ \xi(y), & \text{if } y \in \Pi_{x,r}. \end{cases}$$

where $f_Y : \Pi_{x,r} \to \Pi_{x'}$ is the Poincaré map of Y_t.

Proof. By Remark 2 of [18]. □

Theorem 3. *If a vector field X is C^1 stably countably expansive, then X is quasi-Anosov.*

Proof. It is enough to show that if the flow X_t of $X \in \mathfrak{X}^1(M)$ is C^1 stably countably expansive then X satisfies the quasi-transversality condition by applying Theorem A of [7] and Theorem 2.

Assume that there exists X such that it does not satisfy the quasi-transversality condition. Then, there exists $x \in M$ such that

$$T_x W^s(x) \cap T_x W^u(x) \neq \{\mathbf{0}_x\},$$

and thus we have $x \notin \Omega(X_t)$. By Lemma 6 with a small C^1 perturbation of X at x, we can construct $Y \in \mathcal{U}(X)$ and an arc $\mathcal{L}_x$ centered at x. There exists a local chart (U, Y_t) such that $\mathcal{L}_x \subset U$ and $\mathcal{L}_x$ is diffeomorphic to $[0,1]$. Now, we consider the set $\Gamma_\delta^{f_Y}(x)$, where f_Y is the Poincaré map defined by Y_t. That is,

$$\Gamma_\delta^{f_Y}(x) = \{y \in [0,1] \cap \mathbb{Q}^C : d({f_Y}^i(x), {f_Y}^i(y)) \leq \delta \ \ \forall i \in \mathbb{Z}\}.$$

We can check that the set is uncountable, easily. Therefore, Y_t is not countably expansive. The contradiction completes the proof. □

4. Conclusions

The theory of dynamical systems is motivated by the search of knowledge about the orbits of a given dynamical systems. To describe the dynamics on the underlying space, it is usual to use the notion of expansiveness. In the various type of expansiveness for a homeomorphism of a compact metric space X, Artigue and Carrasco-Olivera proved that a homeomorphism $f : X \to X$ is countably expansive if and only if f is measure expansive (Theorem 2.1 [3]).

In this article, we extend the countably expansiveness to the continuous dynamical systems. However, there is a problem to define the countably expansiveness for flows caused by reparameterization for each point in $\Gamma_\delta(x)$ of Definition 8. Therefore, we define a suitable concept of countably expansiveness for flows as an improvement of the problem. By using this concept, we apply the results for expansiveness and measure expansiveness to the countably expansive flows. More precisely, we prove that, if a vector field X is C^1 stably countably expansive, then it satisfies Axiom A without cycle condition. Furthermore, it is quasi-Anosov.

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