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# Constructing Some Logical Algebras with Hoops

**M. Aaly Kologani [1], Seok-Zun Song [2,*], R. A. Borzooei [3] and Young Bae Jun [3,4]**

[1] Hatef Higher Education Institute, Zahedan 9816848165, Iran; mona4011@gmail.com
[2] Department of Mathematics, Jeju National University, Jeju 63243, Korea
[3] Department of Mathematics, Shahid Beheshti University, Tehran 1983969411, Iran; borzooei@sbu.ac.ir (R.A.B.); skywine@gmail.com (Y.B.J.)
[4] Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea
* Correspondence: szsong@jejunu.ac.kr

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**Abstract:** In any logical algebraic structures, by using of different kinds of filters, one can construct various kinds of other logical algebraic structures. With this inspirations, in this paper by considering a hoop algebra or a hoop, that is introduced by Bosbach, the notion of co-filter on hoops is introduced and related properties are investigated. Then by using of co-filter, a congruence relation on hoops is defined, and the associated quotient structure is studied. Thus Brouwerian semilattices, Heyting algebras, Wajsberg hoops, Hilbert algebras and BL-algebras are obtained.

**Keywords:** hoop; co-filter; Brouwerian semilattice; Heyting algebra; Wajsberg hoop; Hilbert algebra; BL-algebra

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## 1. Introduction

Non-classical logics (or called alternative logics) are formal systems that differ in a significant way from standard logical systems such as propositional and predicate logic. Many-valued logics are non-classical logics which are similar to classical logic. Bosbach [1,2] proposed the concept of hoop which is a nice algebraic structure to research the many-valued logical system whose propositional value is given in a lattice. For various information on hoops, refer to [3–8].

In this paper, we introduce the notion of co-filter in hoops and we get some properties of it. Then we construct a congruence relation by using co-filters on hoops. Finally, we investigate under which conditions the quotient structure of this congruence relation will be Brouwerian semilattice, Heyting algebra, Wajsberg hoop, Hilbert algebra and BL-algebra.

## 2. Preliminaries

In this section, we recollect some definitions and results which will be used in the following and we shall not cite them every time they are used.

**Definition 1** ([9])**.** *A hoop is an algebraic structure* $(\mathbb{H}, \odot, \to, 1)$ *of type* $(2,2,0)$ *such that, for all* $\alpha, \beta, \gamma \in \mathbb{H}$ *it satisfies in the following conditions:*
$(HP1)$ $(\mathbb{H}, \odot, 1)$ *is a commutative monoid.*
$(HP2)$ $\alpha \to \alpha = 1$.
$(HP3)$ $(\alpha \odot \beta) \to \gamma = \alpha \to (\beta \to \gamma)$.
$(HP4)$ $\alpha \odot (\alpha \to \beta) = \beta \odot (\beta \to \alpha)$.

*On hoop $\mathbb{H}$, a binary relation $\leq$ is defined on $\mathbb{H}$ such that $\alpha \leq \beta$ iff $\alpha \to \beta = 1$ and $(\mathbb{H}, \leq)$ is a poset. If the least element $0 \in \mathbb{H}$ exists such that, for all $\alpha \in \mathbb{H}$, $0 \leq \alpha$, then $\mathbb{H}$ is called a* bounded hoop. *We let $\alpha^0 = 1$ and $\alpha^n = \alpha^{n-1} \odot \alpha$, for any $n \in \mathbb{N}$. If $\mathbb{H}$ is bounded, then, for all $\alpha \in \mathbb{H}$, the operation negation "$\,'\,$" is defined on $\mathbb{H}$ by, $\alpha' = \alpha \to 0$. If $(\alpha')' = \alpha$, for all $\alpha \in \mathbb{H}$, then $\mathbb{H}$ is said to have (DNP) property.*

**Proposition 1** ([1,2])**.** *Let $(\mathbb{H}, \odot, \to, 1)$ be a hoop. Then, for all $\alpha, \beta, \gamma \in \mathbb{H}$, it satisfies in the following conditions:*
$(i)$ $(\mathbb{H}, \leq)$ *is a meet-semilattice with* $\alpha \wedge \beta = \alpha \odot (\alpha \to \beta)$.
$(ii)$ $\alpha \odot \beta \leq \gamma$ *iff* $\alpha \leq \beta \to \gamma$.
$(iii)$ $\alpha \odot \beta \leq \alpha, \beta$ *and* $\alpha^n \leq \alpha$, *for any* $n \in \mathbb{N}$.
$(iv)$ $\alpha \leq \beta \to \alpha$.
$(v)$ $1 \to \alpha = \alpha$ *and* $\alpha \to 1 = 1$.
$(vi)$ $\alpha \odot (\alpha \to \beta) \leq \beta$.
$(vii)$ $\alpha \to \beta \leq (\beta \to \gamma) \to (\alpha \to \gamma)$.
$(viii)$ $\alpha \leq \beta$ *implies* $\alpha \odot \gamma \leq \beta \odot \gamma$, $\gamma \to \alpha \leq \gamma \to \beta$ *and* $\beta \to \gamma \leq \alpha \to \gamma$.

**Proposition 2** ([1,2])**.** *Let $\mathbb{H}$ be a bounded hoop. Then, for any $\alpha, \beta \in \mathbb{H}$, the following conditions hold:*
$(i)$ $\alpha \leq \alpha''$ *and* $\alpha \odot \alpha' = 0$
$(ii)$ $\alpha' \leq \alpha \to \beta$.
$(iii)$ $\alpha''' = \alpha'$.
$(iv)$ *If $\mathbb{H}$ has (DNP), then* $\alpha \to \beta = \beta' \to \alpha'$.
$(v)$ *If $\mathbb{H}$ has (DNP), then* $(\alpha \to \beta) \to \beta = (\beta \to \alpha) \to \alpha$.

**Proposition 3** ([10])**.** *Let $\mathbb{H}$ be a hoop and for any $\alpha, \beta \in \mathbb{H}$, define the operation $\vee$ on $\mathbb{H}$ as follows,*

$$\alpha \vee \beta = ((\alpha \to \beta) \to \beta) \wedge ((\beta \to \alpha) \to \alpha).$$

*Then, for all $\alpha, \beta, \gamma \in \mathbb{H}$, the following conditions are equivalent:*
$(i)$ $\vee$ *is associative,*
$(ii)$ $\alpha \leq \beta$ *implies* $\alpha \vee \gamma \leq \beta \vee \gamma$,
$(iii)$ $\alpha \vee (\beta \wedge \gamma) \leq (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$,
$(iv)$ $\vee$ *is the join operation on* $\mathbb{H}$.

*A hoop $\mathbb{H}$ is said to a $\vee$-hoop, if it satisfies one of the above equivalent conditions.*

**Proposition 4** ([10])**.** *Let $\mathbb{H}$ be a $\vee$-hoop and $\alpha, \beta, \gamma \in \mathbb{H}$. Then $\vee$-hoop $(\mathbb{H}, \vee, \wedge)$ is a distributive lattice and $(\alpha \vee \beta) \to \gamma = (\alpha \to \gamma) \wedge (\beta \to \gamma)$.*

**Definition 2** ([10])**.** *A non-empty subset $F$ of a hoop $\mathbb{H}$ is called a filter of $\mathbb{H}$ if, for any $\alpha, \beta \in \mathbb{H}$, the following condition hold:*
$(F1)$ $\alpha, \beta \in F$ *implies* $\alpha \odot \beta \in F$.
$(F2)$ $\alpha \leq \beta$ *and* $\alpha \in F$ *imply* $\beta \in F$.

*The set of all filters of $\mathbb{H}$ is denoted by $\mathcal{F}(\mathbb{H})$. Clearly, for any filter $F$ of $\mathbb{H}$, $1 \in F$. $F$ is called a proper filter if $F \neq \mathbb{H}$. So, if $\mathbb{H}$ is a bounded hoop, then a filter is proper iff it does not contain $0$. It is easy to see that $F \in \mathcal{F}(\mathbb{H})$ iff, for any $\alpha, \beta \in \mathbb{H}$, $1 \in F$ and if $\alpha, \alpha \to \beta \in F$, then $\beta \in F$.*

## 3. Co-Filters in Hoops

From here on, if there is no mention, $\mathbb{H}$ denotes a bounded hoop.

We introduce the notion of co-filters on hoops, and it is proved that co-filters are not filters and some properties of them are studied. Moreover, a congruence relation is defined by them and is investigated the quotient structure of this congruence relation.

**Definition 3.** *A subset $I$ of $\mathbb{H}$ is said to be a co-filter of $\mathbb{H}$ if, for any $\alpha, \beta \in \mathbb{H}$,*
$(CF_1)$ $0 \in I$.
$(CF_2)$ $(\alpha \to \beta)' \in I$ *and* $\beta \in I$ *imply* $\alpha \in I$.

**Example 1.** *Let $\mathbb{H} = \{0, a, b, c, d, 1\}$. Define the operations $\odot$ and $\to$ on $\mathbb{H}$ as below,*

| $\to$ | 0 | a | b | c | d | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| a | d | 1 | d | 1 | d | 1 |
| b | c | c | 1 | 1 | 1 | 1 |
| c | b | c | d | 1 | d | 1 |
| d | a | a | b | c | 1 | 1 |
| 1 | 0 | a | b | c | d | 1 |

| $\odot$ | 0 | a | b | c | d | 1 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | 0 | a | 0 | a |
| b | 0 | 0 | 0 | 0 | b | b |
| c | 0 | a | 0 | a | b | c |
| d | 0 | 0 | b | b | d | d |
| 1 | 0 | a | b | c | d | 1 |

*Then $(\mathbb{H}, \odot, \to, 0, 1)$ is a hoop and $I = \{0, b, d\}$ is a co-filter of $\mathbb{H}$, which is not a filter of $\mathbb{H}$ because $1 \notin I$.*

**Note.** For $S \subseteq \mathbb{H}$, define $S' = \{\alpha \in \mathbb{H} \mid \alpha' \in S\}$.

**Proposition 5.** *If $\mathbb{H}$ has (DNP) and $\emptyset \neq I \subseteq \mathbb{H}$, then $I$ is a filter of $\mathbb{H}$ iff $I'$ is a co-filter of $\mathbb{H}$.*

**Proof.** $(\Rightarrow)$ Suppose $I \in \mathcal{F}(\mathbb{H})$. Then $1 \in I$, and so $0 = 1' \in I'$. Let $\alpha, \beta \in \mathbb{H}$ such that $(\alpha \to \beta)' \in I'$ and $\beta \in I'$. Since $\mathbb{H}$ has (DNP), by Proposition 2(iv), $\beta' \to \alpha' = \alpha \to \beta \in I$, $\beta' \in I$ and since $I \in \mathcal{F}(\mathbb{H})$, by Definition 2, $\alpha' \in I$, and so $\alpha'' = \alpha \in I'$. Hence, $I'$ is a co-filter of $\mathbb{H}$.
$(\Leftarrow)$ Let $I'$ be a co-filter of $\mathbb{H}$. Then $0 \in I'$, and so $1 \in I$. Now, suppose $\alpha, \beta \in A$ such that $\alpha, \alpha \to \beta \in I$. Thus $(\alpha \to \beta)' \in I'$ and $\alpha' \in I'$. Since $\mathbb{H}$ has (DNP), by Proposition 2(iv), $(\beta' \to \alpha')' \in I'$, $\alpha' \in I'$ and since $I'$ is a co-filter of $\mathbb{H}$, by definition, $\beta' \in I'$. Hemce, by (DNP), $\beta \in I$. Therefore, $I \in \mathcal{F}(\mathbb{H})$. □

If $\mathbb{H}$ does not have (DNP), then Proposition 5 is not true, in general. We show this in the following example.

**Example 2.** *Let $\mathbb{H} = \{0, a, b, 1\}$ be a chain such that $0 \leq a \leq b \leq 1$ and two binary operations $\odot$ and $\to$ which are given below,*

| $\to$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| a | a | 1 | 1 | 1 |
| b | 0 | a | 1 | 1 |
| 1 | 0 | a | b | 1 |

| $\odot$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | a | a |
| b | 0 | a | b | b |
| 1 | 0 | a | b | 1 |

*By routine calculations, $(\mathbb{H}, \odot, \to, 0, 1)$ is a hoop that does not have (DNP). It is clear that $\mathbb{H}$ is a co-filter of $\mathbb{H}$ but $\mathbb{H}' = \{0, a, 1\}$ is not a filter of $\mathbb{H}$.*

**Note.** If $F$ is a proper filter of $\mathbb{H}$, then by Definition 2, $0 \notin F$. Thus, $F$ is not a co-filter of $\mathbb{H}$. On the other hand, for any proper co-filter $I$ of $\mathbb{H}$, if $1 \notin I$, then $I \notin \mathcal{F}(\mathbb{H})$.

**Proposition 6.** *Let $I$ be a co-filter of $\mathbb{H}$. Then the following statements hold:*
$(i)$ *If $\alpha \leq \beta$ and $\beta \in I$, then $\alpha \in I$, for any $\alpha, \beta \in \mathbb{H}$.*
$(ii)$ *If $\alpha \in I$, then $\alpha \odot \beta \in I$, for any $\beta \in \mathbb{H}$.*
$(iii)$ *If $\mathbb{H}$ is a $\vee$-hoop with (DNP), then $\alpha \vee \beta \in I$, for any $\alpha, \beta \in I$.*

**Proof.** $(i)$ Let $\alpha, \beta \in \mathbb{H}$ such that $\alpha \leq \beta$ and $\beta \in I$. Then $\alpha \to \beta = 1$, and so $(\alpha \to \beta)' = 0 \in I$. Since $I$ is a co-filter of $\mathbb{H}$, $(\alpha \to \beta)' \in I$ and $\beta \in I$, we have $\alpha \in I$.
$(ii)$ Let $\alpha, \beta \in \mathbb{H}$ and $\alpha \in I$. By Proposition 1(iii), $\alpha \odot \beta \leq \alpha$. Since $\alpha \in I$, by (i), $\alpha \odot \beta \in I$.
$(iii)$ Suppose $\alpha, \beta \in I$. By Proposition 4,

$$(\alpha \vee \beta) \to \beta = (\alpha \to \beta) \wedge (\beta \to \beta) = \alpha \to \beta,$$

then $((\alpha \vee \beta) \to \beta)' = (\alpha \to \beta)'$. By Proposition 2(ii), $\alpha' \leq \alpha \to \beta$, and so, by Proposition 1(viii) and (DNP), $(\alpha \to \beta)' \leq \alpha'' = \alpha$. Hence, $((\alpha \vee \beta) \to \beta)' \leq \alpha$. From $I$ is a co-filter of $\mathbb{H}$ and $\alpha \in I$, by (i) $((\alpha \vee \beta) \to \beta)' \in I$. Moreover, by assumption, $\beta \in I$ and $I$ is a co-filter of $\mathbb{H}$. Therefore, $\alpha \vee \beta \in I$. □

**Corollary 1.** *If $I$ is a co-filter of $\mathbb{H}$ and $1 \in I$, then $I = \mathbb{H}$.*

**Proof.** By Proposition 6(i), the proof is straightforward. □

We provide conditions for a nonempty subset to be a co-filter.

**Proposition 7.** *Let $\alpha, \beta \in \mathbb{H}$ and $\emptyset \neq I \subseteq \mathbb{H}$ such that $I$ has the following properties,*
*$(i)$ if $\alpha, \beta \in I$, then $\alpha' \to \beta \in I$,*
*$(ii)$ if $\alpha \leq \beta$ and $\beta \in I$, then $\alpha \in I$.*
*Then $I$ is a co-filter of $\mathbb{H}$.*

**Proof.** Let $\alpha \in I$. Since, for all $\alpha \in \mathbb{H}$, $0 \leq \alpha$, by (ii), $0 \in I$. Suppose $\alpha, \beta \in \mathbb{H}$ such that $(\alpha \to \beta)' \in I$ and $\beta \in I$. Then by (i),

$$\beta' \to (\alpha \to \beta)' = \beta' \to ((\alpha \to \beta) \to 0) \in I.$$

By (HP3), $(\alpha \to \beta) \to \beta'' \in I$. Moreover, by Propositions 2(i) and 1(viii), $\beta \leq \beta''$ and so $(\alpha \to \beta) \to \beta \leq (\alpha \to \beta) \to \beta''$. Since $(\alpha \to \beta) \to \beta'' \in I$, by (ii), $(\alpha \to \beta) \to \beta \in I$. Also, by Proposition 1(vi), $\alpha \leq (\alpha \to \beta) \to \beta$, and by (ii), $\alpha \in I$. Hence $I$ is a co-filter of $\mathbb{H}$. □

By below example, we show that the converse of Proposition 7, is not true.

**Example 3.** *Let $\mathbb{H} = \{0, a, b, c, 1\}$ be a set with the following Cayley tabels:*

| $\to$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| a | b | 1 | 0 | 0 | 1 |
| b | c | 0 | 1 | 0 | 1 |
| c | c | 0 | 0 | 1 | 1 |
| 1 | 0 | a | b | c | 1 |

| $\odot$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | 0 | 0 | a |
| b | 0 | 0 | b | 0 | b |
| c | 0 | 0 | 0 | c | c |
| 1 | 0 | a | b | c | 1 |

*Then $(\mathbb{H}, \odot, \to, 0, 1)$ is a hoop and $I = \{0, a\}$ is a co-filter of $\mathbb{H}$ but $a' \to 0 = b \to 0 = c \notin I$.*

**Proposition 8.** *Let $\mathbb{H}$ has (DNP). Then $I$ is a co-filter of $\mathbb{H}$ iff for any $\alpha, \beta \in \mathbb{H}$, $I$ has the following properties,*
*$(i)$ if $\alpha, \beta \in I$, then $\alpha' \to \beta \in I$.*
*$(ii)$ if $\alpha \leq \beta$ and $\beta \in I$, then $\alpha \in I$.*

**Proof.** $(\Rightarrow)$ Let $I$ be a co-filter of $\mathbb{H}$. Then by Proposition 6(i), item (ii) is clear. Suppose $\alpha, \beta \in I$. By Proposition 1(vi) and (viii), $((\alpha' \to \beta) \to \beta)' \leq \alpha''$. Since $\mathbb{H}$ has (DNP), $((\alpha' \to \beta) \to \beta)' \leq \alpha$. By assumption, $\alpha \in I$, and so by Proposition 6(i), $((\alpha' \to \beta) \to \beta)' \in I$. Moreover, since $\beta \in I$ and $I$ is a co-filter of $\mathbb{H}$, $\alpha' \to \beta \in I$.
$(\Leftarrow)$ The proof is similar to the proof of Proposition 7. □

**Theorem 1.** *Let $I$ be a co-filter of $\mathbb{H}$. Then, for all $\alpha, \beta, \gamma \in \mathbb{H}$, the following statements hold:*
*$(i)$ $((\alpha \to \beta) \to \alpha)' \in I$ and $\alpha \in I$ imply $\beta \in I$.*
*$(ii)$ If $\alpha \to (\alpha \to \beta) \in I$, then $\alpha \to \beta \in I$.*
*$(iii)$ $((\beta \to (\beta \to \alpha)) \to \gamma)' \in I$ and $\gamma \in I$ imply $\beta \to \alpha \in I$.*
*$(iv)$ If $(\alpha \to \beta)' \in I$, then $((\alpha \to \beta)' \to \beta)' \in I$.*

**Proof.** $(i)$ Let $\alpha, \beta \in \mathbb{H}$ such that $((\alpha \to \beta) \to \alpha)' \in I$ and $\alpha \in I$. Since $I$ is a co-filter of $\mathbb{H}$, $\alpha \to \beta \in I$. By Proposition 1(iv), $\beta \leq \alpha \to \beta \in I$. From Proposition 6(i), $\beta \in I$.
$(ii)$ By Proposition 1(iii) and (viii), $\alpha^2 \leq \alpha$ and $\alpha \to \beta \leq \alpha^2 \to \beta = \alpha \to (\alpha \to \beta)$. Since $\alpha \to (\alpha \to \beta) \in I$, by Proposition 6(i), $\alpha \to \beta \in I$.
$(iii)$ Suppose $\alpha, \beta, \gamma \in \mathbb{H}$ such that $((\beta \to (\beta \to \alpha)) \to \gamma)' \in I$ and $\gamma \in I$. Since $I$ is a co-filter of $\mathbb{H}$, $\beta \to (\beta \to \alpha) \in I$, and so by (ii), $\beta \to \alpha \in I$.
$(iv)$ Let $\alpha, \beta \in \mathbb{H}$ such that $(\alpha \to \beta)' \in I$. Then by (HP3), we have

$$\begin{aligned} ((\alpha \to \beta)' \to \beta)' \to (\alpha \to \beta)' &= (\alpha \to \beta) \to ((\alpha \to \beta)' \to \beta)'' && \text{by Propositions 1(viii) and 2(i)} \\ &\geq (\alpha \to \beta) \to ((\alpha \to \beta)' \to \beta) && \text{by (HP3)} \\ &= [(\alpha \to \beta) \odot (\alpha \to \beta)'] \to \beta && \text{by Proposition 2(i)} \\ &= 0 \to \beta \\ &= 1 \end{aligned}$$

Thus, $((\alpha \to \beta)' \to \beta)' \to (\alpha \to \beta)' = 1$, and so $((\alpha \to \beta)' \to \beta)' \leq (\alpha \to \beta)'$. Since $(\alpha \to \beta)' \in I$ and $I$ is a co-filter of $\mathbb{H}$, by Proposition 6(i), $((\alpha \to \beta)' \to \beta)' \in I$. □

If $X \subseteq \mathbb{H}$, then the least co-filter of $\mathbb{H}$ contains $X$ is called the *co-filter generated by* $X$ of $\mathbb{H}$ and we show it by $[X)$.

**Theorem 2.** *If $\mathbb{H}$ has (DNP), then, for any $a \in \mathbb{H}$,*

$$[a) = \{\alpha \in A \mid \exists n \in \mathbb{N} \text{ such that } (a')^n \leq \alpha'\}.$$

**Proof.** Let $B = \{\alpha \in A \mid \exists n \in \mathbb{N}$ such that $(a')^n \leq \alpha'\}$. Since $(a')^n \leq 1 = 0'$, for all $n \in \mathbb{N}$, we have $0 \in B$, and so $B \neq \emptyset$. Now, suppose $\alpha, \beta \in \mathbb{H}$ such that $(\alpha \to \beta)' \in B$ and $\beta \in B$. Then there exist $n, m \in \mathbb{N}$, such that $(a')^n \leq (\alpha \to \beta)''$ and $(a')^m \leq \beta'$. By Proposition 1(viii),

$$(a')^n \odot (a')^m \leq (\alpha \to \beta)'' \odot (a')^m \leq (\alpha \to \beta)'' \odot \beta'$$

By (HP3), we get

$$\begin{aligned} ((\alpha \to \beta)'' \odot \beta') \to \alpha' &= \beta' \to ((\alpha \to \beta)'' \to \alpha') \\ &= \beta' \to (\alpha \to (\alpha \to \beta)''') && \text{by Proposition 2(iii)} \\ &= \beta' \to (\alpha \to (\alpha \to \beta)') \\ &= \alpha \to (\beta' \to (\alpha \to \beta)') \\ &= \alpha \to ((\alpha \to \beta) \to \beta'') \\ &= (\alpha \to \beta) \to (\alpha \to \beta'') && \text{by Propositions 2(i) and 1(viii)} \\ &= 1 \end{aligned}$$

Then $(\alpha \to \beta)'' \odot \beta' \leq \alpha'$, and so $(a')^n \odot (a')^m \leq \alpha'$. Hence, $n + m \in \mathbb{N}$ exists such that $(a')^{n+m} \leq \alpha'$. Therefore, $\alpha \in B$, and so $B$ is a co-filter of $\mathbb{H}$. Also, by Proposition 1(iii), $(a')^n \leq a'$. Thus, $a \in B$ and $B$ is a co-filter of $\mathbb{H}$ which containing $a$. Now, it is enough to prove that $B$ is the least co-filter of $\mathbb{H}$ which containing $a$. Suppose C is a co-filter of $\mathbb{H}$ that contains $a$. We show that $B \subseteq C$. Let $\alpha \in B$. Then there exists $n \in \mathbb{N}$ such that $(a')^n \leq \alpha'$. Thus $(a')^n \to \alpha' = 1$. Since $\mathbb{H}$ has (DNP), by (HP3) and Proposition 2(iv), we get

$$\begin{aligned}
1 &= (a')^n \to \alpha' \\
&= ((a')^{n-1} \odot (a')) \to \alpha' \\
&= (a')^{n-1} \to (a' \to \alpha') \\
&= (a')^{n-1} \to (\alpha \to a'') \\
&= (a')^{n-1} \to (\alpha \to a) \\
&= ((a')^{n-2} \odot (a')) \to (\alpha \to a) \\
&= (a')^{n-2} \to (a' \to (\alpha \to a)) \\
&= (a')^{n-2} \to ((\alpha \to a)' \to a'') \\
&= (a')^{n-2} \to ((\alpha \to a)' \to a)
\end{aligned}$$

By continuing this method, we have

$$1 = ((((\alpha \to a)' \to a)' \to a)' \to \ldots \to a)' \to a$$

Hence,

$$[(((((\alpha \to a)' \to a)' \to a)') \to)\ldots) \to a)' \to a]' = 1' = 0 \in C.$$

Since $C$ is a co-filter of $\mathbb{H}$ and $a \in C$, we obtain,

$$(((((\alpha \to a)' \to a)' \to a)') \to)\ldots) \to a)' \in C.$$

By continuing this method, we can see that $(\alpha \to a)' \in C$. Since $(\alpha \to a)' \in C$, $a \in C$ and $C$ is a co-filter of $\mathbb{H}$, we have $\alpha \in C$. Hence, $B \subseteq C$. Therefore, $B = [a)$. $\square$

**Corollary 2.** *Let $\mathbb{H}$ has (DNP), $X \subseteq \mathbb{H}$ and $a \in \mathbb{H}$. Then the following statements hold:*
*(i)* $[X) = \{\alpha \in A \mid \exists\, n \in \mathbb{N}$ *and* $a_1, \ldots, a_n \in X$ *s.t,* $a_1{}' \odot a_2{}' \odot \ldots \odot a_n{}' \leq \alpha'\}$.
*(ii)* $[I \cup \{a\}) = \{\beta \in A \mid \exists\, n, m \in \mathbb{N}$ *and* $\alpha_1, \ldots, \alpha_m \in I$ *s.t,* $(\alpha_1' \odot \ldots \odot \alpha_m') \odot (a')^n \leq \beta'\}$.

**Proof.** The proof is similar to the proof of Theorem 2. $\square$

**Example 4.** *Let $A$ be the hoop as in Example 3. It is clear that $A$ has (DNP). Since $a' = d$ and $d \leq d = a'$ and $d \leq 1 = 0'$, we get $[a) = \{0, a\}$. Also, since $d' = a$ and $a \leq 1, a, c$, we have $[d) = \{0, b, d\}$.*

**Theorem 3.** *Let $I$ be a co-filter of $\mathbb{H}$. We define the relation $\equiv_I$ on $\mathbb{H}$ as follows,*

$$\alpha \equiv_I \beta \textit{ iff } (\alpha \to \beta)' \in I \textit{ and } (\beta \to \alpha)' \in I, \textit{ for all } \alpha, \beta \in \mathbb{H}.$$

*Then $\equiv_I$ is a congruence relation on $\mathbb{H}$.*

**Proof.** At first, we prove that $\equiv_I$ is an equivalence relation on $\mathbb{H}$. Since, for all $\alpha \in \mathbb{H}$, $\alpha \to \alpha = 1$ and $I$ is a co-filter of $\mathbb{H}$, $(\alpha \to \alpha)' = 0 \in I$. Thus, $\alpha \equiv_I \alpha$, and so $\equiv_I$ is reflexive. It is obvious that $\equiv_I$ is symmetric. For proving transitivity of $\equiv_I$, suppose $\alpha, \beta, \gamma \in \mathbb{H}$ such that $\alpha \equiv_I \beta$ and $\beta \equiv_I \gamma$. Hence, $(\alpha \to \beta)', (\beta \to \alpha)', (\beta \to \gamma)'$ and $(\gamma \to \beta)' \in I$ and by Proposition 1(vii) and (viii),

$$(\alpha \to \gamma)' \leq ((\alpha \to \beta) \odot (\beta \to \gamma))'.$$

By (HP3) and Propositions 1(vii),(viii) and 2(iii), we have,

$$\begin{aligned}(((\alpha\to\beta)\to(\beta\to\gamma)')\to(\alpha\to\beta)')' &= (((\alpha\to\beta)\to(\beta\to\gamma)')\to((\alpha\to\beta)\to 0))' \\ &= ([(\alpha\to\beta)\odot((\alpha\to\beta)\to(\beta\to\gamma)')]\to 0)' \\ &\leq (\beta\to\gamma)''' \\ &= (\beta\to\gamma)'\in I.\end{aligned}$$

Thus, by Proposition 6(i),

$$(((\alpha\to\beta)\to(\beta\to\gamma)')\to(\alpha\to\beta)')'\in I.$$

Since $(\alpha\to\beta)'\in I$ and $I$ is a co-filter of $\mathbb{H}$, $(\alpha\to\beta)\to(\beta\to\gamma)'\in I$. Moreover, $(\alpha\to\gamma)'\leq(\alpha\to\beta)\to(\beta\to\gamma)'$ and $(\alpha\to\beta)\to(\beta\to\gamma)'\in I$, by Proposition 6(i), $(\alpha\to\gamma)'\in I$. By the similar way, $(\gamma\to\alpha)'\in I$. Hence, $\alpha\equiv_I\gamma$. Therefore, $\equiv_I$ is an equivalence relation on $\mathbb{H}$. Now, let $\alpha\equiv_I\beta$, for some $\alpha,\beta\in\mathbb{H}$. Then $(\alpha\to\beta)',(\beta\to\alpha)'\in I$. Thus, by Proposition 1(vi), $\alpha\leq(\alpha\to\gamma)\to\gamma$, for all $\gamma\in\mathbb{H}$. So, by Proposition 1(viii), $\beta\to\alpha\leq\beta\to((\alpha\to\gamma)\to\gamma)$. Then by Proposition 1(viii) and (HP3),

$$((\alpha\to\gamma)\to(\beta\to\gamma))'\leq(\beta\to\alpha)'.$$

Since $(\beta\to\alpha)'\in I$ and $I$ is a co-filter of $\mathbb{H}$, by Proposition 6(i), $((\alpha\to\gamma)\to(\beta\to\gamma))'\in I$. By the similar way, $((\beta\to\gamma)\to(\alpha\to\gamma))'\in I$. Hence, $\alpha\to\gamma\equiv_I\beta\to\gamma$. Suppose $\alpha\equiv_I\beta$, for some $\alpha,\beta\in\mathbb{H}$. Then $(\alpha\to\beta)',(\beta\to\alpha)'\in I$ and by Proposition 1(vii) and (HP3), $\alpha\to\beta\leq(\gamma\to\alpha)\to(\gamma\to\beta)$, for all $\gamma\in\mathbb{H}$. Also, by Proposition 1(viii),

$$((\gamma\to\alpha)\to(\gamma\to\beta))'\leq(\alpha\to\beta)'.$$

From $(\alpha\to\beta)'\in I$ and $I$ is a co-filter of $\mathbb{H}$, by Proposition 6(i), $((\gamma\to\alpha)\to(\gamma\to\beta))'\in I$. By the similar way, $((\gamma\to\beta)\to(\gamma\to\alpha))'\in I$. Hence, $\gamma\to\alpha\equiv_I\gamma\to\beta$. Finally, if $\alpha\equiv_I\beta$, for some $\alpha,\beta\in\mathbb{H}$, then $(\alpha\to\beta)',(\beta\to\alpha)'\in I$. From $\alpha\odot\gamma\leq\alpha\odot\gamma$, by Proposition 1(ii),(viii) and (HP3), $\alpha\leq\gamma\to(\alpha\odot\gamma)$, and so

$$\beta\to\alpha\leq\beta\to(\gamma\to(\alpha\odot\gamma))=(\beta\odot\gamma)\to(\alpha\odot\gamma).$$

Then by Proposition 1(viii),

$$((\beta\odot\gamma)\to(\alpha\odot\gamma))'\leq(\beta\to\alpha)'.$$

Since $(\beta\to\alpha)'\in I$ and $I$ is a co-filter of $\mathbb{H}$, by Proposition 6(i), $((\beta\odot\gamma)\to(\alpha\odot\gamma))'\in I$. Similarly, $((\alpha\odot\gamma)\to(\beta\odot\gamma))'\in I$. Hence, $\alpha\odot\gamma\equiv_I\beta\odot\gamma$. Therefore, $\equiv_I$ is a congruence relation on $\mathbb{H}$. □

For any $\alpha\in\mathbb{H}$, $I_\alpha$ will denote the equivalence class of $\alpha$ with respect to $\equiv_I$. It is clear that

$$I_\alpha=\{\beta\in\mathbb{H}\mid\alpha\equiv_I\beta\}=\{\beta\in\mathbb{H}\mid(\alpha\to\beta)'\in I\text{ and }(\beta\to\alpha)'\in I\}.$$

Easily we can see that $I_0=I$ and $I_1=\{\beta\in\mathbb{H}\mid\beta'\in I\}$.

**Theorem 4.** *Let $\mathbb{H}/I=\{I_\alpha\mid\alpha\in\mathbb{H}\}$. Define the operations $\otimes$ and $\rightsquigarrow$ on $\mathbb{H}/I$ as follows:*

$$I_\alpha\otimes I_\beta=I_{\alpha\odot\beta}\text{ and }I_\alpha\rightsquigarrow I_\beta=I_{\alpha\to\beta}.$$

*Then $(\mathbb{H}/I,\otimes,\rightsquigarrow,I_0,I_1)$ is a bounded hoop.*

**Proof.** The proof is straightforward.

**Note.** Let $\alpha, \beta \in \mathbb{H}$. Then the binary relation $'' \leq_I ''$ is defined on $\mathbb{H}/I$ as follows,

$$I_\alpha \leq_I I_\beta \text{ iff } (\alpha \to \beta)' \in I.$$

Then $\leq_I$ is a partially order relation on $\mathbb{H}/I$. Since $(\alpha \to \alpha)' = 0 \in I$, for any $\alpha \in \mathbb{H}$, $I_\alpha \leq_I I_\alpha$. Suppose $I_\alpha \leq_I I_\beta$ and $I_\beta \leq_I I_\alpha$, for any $\alpha, \beta \in \mathbb{H}$. Then $(\alpha \to \beta)' \in I$ and $(\beta \to \alpha)' \in I$. Thus, $\alpha \equiv_I \beta$, and so $I_\alpha = I_\beta$. Now, let $I_\alpha \leq_I I_\beta$ and $I_\beta \leq_I I_\gamma$. Then $(\alpha \to \beta)' \in I$ and $(\beta \to \gamma)' \in I$. By Proposition 1(vii), for $\alpha, \beta, \gamma \in \mathbb{H}$, we have $(\alpha \to \beta) \odot (\beta \to \gamma) \leq \alpha \to \gamma$. Thus, by Proposition 1(viii) and (HP3),

$$(\alpha \to \gamma)' \leq ((\alpha \to \beta) \odot (\beta \to \gamma))',$$

and so

$$(\alpha \to \gamma)' \leq [((\alpha \to \beta) \odot (\beta \to \gamma)) \to 0].$$

Thus, $(\alpha \to \gamma)' \leq (\alpha \to \beta) \to ((\beta \to \gamma) \to 0)$. Moreover, by Proposition 1(ii),

$$\alpha \to \beta \leq (\alpha \to \gamma)' \to (\beta \to \gamma)'.$$

Hence, by Proposition 1(viii), we obtain,

$$((\alpha \to \gamma)' \to (\beta \to \gamma)')' \leq (\alpha \to \beta)' \in I.$$

Since $(\alpha \to \beta)' \in I$ and $I$ is a co-filter of $\mathbb{H}$, by Proposition 6(i), $((\alpha \to \gamma)' \to (\beta \to \gamma)')' \in I$. Also, $(\beta \to \gamma)' \in I$, then $(\alpha \to \gamma)' \in I$. Hence, $I_\alpha \leq I_\gamma$. Therefore, $\leq_I$ is a partially order relation on $\mathbb{H}/I$. For proving $(\mathbb{H}/I, \otimes, \rightsquigarrow, I_0, I_1)$ is a bounded hoop, we have $I_\alpha = I_\beta$ and $I_\gamma = I_\delta$ iff $\alpha \equiv_I \beta$ and $\gamma \equiv_I \delta$. Since $\equiv_I$ is a congruence relation on $\mathbb{H}$, so all operations are well-defined. Thus, by routine calculations, we can see that $(\mathbb{H}/I, \otimes, I_1)$ is a commutative monoid and (HP2) holds. Let $I_\alpha, I_\beta, I_\gamma \in \mathbb{H}/I$, for any $\alpha, \beta, \gamma \in \mathbb{H}$. Since $\mathbb{H}$ is a hoop, by (HP3) and (HP4) we have,

$$(I_\alpha \otimes I_\beta) \rightsquigarrow I_\gamma = I_{(\alpha \odot \beta)} \rightsquigarrow I_\gamma = I_{(\alpha \odot \beta) \to \gamma} = I_{\alpha \to (\beta \to \gamma)} = I_\alpha \rightsquigarrow (I_{\beta \to \gamma}) = I_\alpha \rightsquigarrow (I_\beta \rightsquigarrow I_\gamma).$$

Also, for any $I_\alpha, I_\beta \in \mathbb{H}/I$, we get

$$I_\alpha \otimes (I_\alpha \rightsquigarrow I_\beta) = I_\alpha \otimes (I_{\alpha \to \beta}) = I_{\alpha \odot (\alpha \to \beta)} = I_{\beta \odot (\beta \to \alpha)} = I_\beta \otimes (I_{\beta \to \alpha}) = I_\beta \otimes (I_\beta \rightsquigarrow I_\alpha).$$

Therefore, $(\mathbb{H}/I, \otimes, \rightsquigarrow, I_0, I_1)$ is a bounded hoop. □

**Example 5.** *Let $A$ be the hoop as in Example 3. Then $I = \{0, b, d\}$ is a co-filter of $A$. Thus, by routine calculations, we can see that $[b] = [d] = [0] = \{0, b, d\}$ and $[a] = [c] = [1] = \{a, c, 1\}$. Hence, $\frac{A}{\equiv_I} = \{[0], [1]\}$. Therefore, $\frac{A}{\equiv_I}$ is a bounded hoop.*

**Example 6.** *Let $A$ be the hoop as in Example 2. We can see that $A$ does not have (DNP) property, in general. So by Proposition 5 and Example 2, filter and co-filter are different notions. Then $A$ is a co-filter of $A$ and the quotient is $\frac{A}{A} = \{[1]\}$ that is a hoop algebra. But $F = \{b, 1\}$ is a filter of $A$ and the quotient $\frac{A}{\equiv_F} = \{[0], [a], [1]\}$ that is a hoop with (DNP).*

## 4. Some Applications of Co-Filters

In this section, we try to investigate under which conditions the quotient structure of this congruence relation will be Brouwerian semilattice, Heyting algebra, Wajsberg hoop, Hilbert algebra and BL-algebra.

**Definition 4** ([11]). *A Brouwerian lattice is an algebra* $(\mathbb{H}, \wedge, \vee, \rightarrow, ')$ *with the lattice infimum* $(\wedge)$ *and the lattice supremum* $(\vee)$ *in which two operations "* $'$ *" and "*$\rightarrow$*" are defined by* $\alpha' = \alpha \rightarrow 0$ *and*

$$\alpha \wedge \beta \leq \gamma \textit{ iff } \alpha \leq \beta \rightarrow \gamma$$

*respectively.*

**Theorem 5.** *Let $I$ be a co-filter of $\mathbb{H}$ and for all $\alpha \in \mathbb{H}$, $\alpha^2 = \alpha$. Then $\mathbb{H}/I$ is a Brouwerian semilattice.*

**Proof.** Let $I$ be a co-filter of $\mathbb{H}$. By Theorem 4, $\mathbb{H}/I$ is a hoop. Thus, by Proposition 1(i), $(\mathbb{H}/I, \leq_I)$ is a meet-semilattice with $I_\alpha \wedge_I I_\beta = I_\alpha \otimes (I_\alpha \rightsquigarrow I_\beta)$, for all $I_\alpha, I_\beta \in \mathbb{H}/I$. Now, we prove that, for all $I_\alpha, I_\beta \in \mathbb{H}/I$,

$$I_\alpha \wedge_I I_\beta \leq I_\gamma \text{ iff } I_\alpha \leq I_\beta \rightsquigarrow I_\gamma.$$

Since $\mathbb{H}/I$ is a hoop, by Proposition 1(iii), $I_\alpha \otimes I_\beta \leq I_\alpha \wedge_I I_\beta \leq I_\gamma$. Thus, $I_\alpha \otimes I_\beta \leq I_\gamma$, and so by Proposition 1(ii), $I_\alpha \leq I_\beta \rightsquigarrow I_\gamma$. Conversely, suppose $I_\alpha \leq I_\beta \rightsquigarrow I_\gamma$, for all $I_\alpha, I_\beta, I_\gamma \in \mathbb{H}/I$. According to definition of $\leq_I$, $(\alpha \rightarrow (\beta \rightarrow \gamma))' \in I$. By Proposition 1(vii), $\beta \rightarrow \gamma \leq (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)$ and by (HP3), $\beta \rightarrow \gamma \leq (\alpha \odot (\alpha \rightarrow \beta)) \rightarrow \gamma$. Also, by Proposition 1(viii) and (HP3), we get

$$\alpha \rightarrow (\beta \rightarrow \gamma) \leq \alpha \rightarrow ((\alpha \odot (\alpha \rightarrow \beta)) \rightarrow \gamma),$$

and so

$$\alpha \rightarrow (\beta \rightarrow \gamma) \leq (\alpha \odot (\alpha \odot (\alpha \rightarrow \beta)) \rightarrow \gamma).$$

Thus,

$$\alpha \rightarrow (\beta \rightarrow \gamma) \leq (((\alpha \odot \alpha) \odot (\alpha \rightarrow \beta)) \rightarrow \gamma).$$

Since for any $\alpha \in \mathbb{H}$, $\alpha^2 = \alpha$, we obtain, $\alpha \rightarrow (\beta \rightarrow \gamma) \leq ((\alpha \odot (\alpha \rightarrow \beta)) \rightarrow \gamma)$ and so,

$$\alpha \rightarrow (\beta \rightarrow \gamma) \leq (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma).$$

Hence, by Proposition 1(viii), we get $((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))' \leq (\alpha \rightarrow (\beta \rightarrow \gamma))'$. Since $I$ is a co-filter of $\mathbb{H}$ and $(\alpha \rightarrow (\beta \rightarrow \gamma))' \in I$, by Proposition 6(i), $((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))' \in I$, so $I_\alpha \rightsquigarrow I_\beta \leq I_\alpha \rightsquigarrow I_\gamma$. Thus, by Proposition 1(ii),(i) and (viii),

$$I_\alpha \wedge_I I_\beta = I_\alpha \otimes (I_\alpha \rightsquigarrow I_\beta) \leq I_\alpha \otimes (I_\alpha \rightsquigarrow I_\gamma) = I_\alpha \wedge I_\gamma \leq I_\gamma.$$

Hence, $I_\alpha \wedge_I I_\beta \leq I_\gamma$. Therefore, $\mathbb{H}/I$ is a Brouwerian semilattice. □

**Example 7.** *Let $\mathbb{H} = \{0, a, b, c, 1\}$ be a set with two operations which are given below:*

| $\rightarrow$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| a | b | 1 | b | 1 | 1 |
| b | a | a | 1 | 1 | 1 |
| c | 0 | a | b | 1 | 1 |
| 1 | 0 | a | b | c | 1 |

| $\odot$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | 0 | a | a |
| b | 0 | 0 | b | b | b |
| c | 0 | a | b | c | c |
| 1 | 0 | a | b | c | 1 |

*Thus, $(\mathbb{H}, \odot, \rightarrow, 0, 1)$ is a hoop and $\alpha^2 = \alpha$, for all $\alpha \in \mathbb{H}$. Then $I = \{0, a\}$ is a co-filter of $\mathbb{H}$, $I_0 = I_a = I$ and $I_b = I_c = I_1 = \{b, c, 1\}$. Hence, by Theorem 5, $\mathbb{H}/I = \{I_0, I_1\}$ is a Brouwerian semilattice.*

**Theorem 6.** *Let $\mathbb{H}$ has (DNP) and $\mathbb{H}/I$ be a Brouwerian semilattice. Then $I$ is a co-filter of $\mathbb{H}$.*

**Proof.** Let $I_\alpha \otimes_I I_\beta = I_\alpha \wedge_I I_\beta$, for all $I_\alpha, I_\beta \in \mathbb{H}/I$. Then $I_\alpha \otimes_I I_\alpha = I_\alpha \wedge_I I_\alpha = I_\alpha$. Thus, $I_{\alpha^2} = I_\alpha$, and so $(\alpha^2 \to \alpha)' \in I$. By Proposition 1(iii), $0 \in I$. Now, suppose $(\alpha \to \beta)'$ and $\beta \in I$, for some $\alpha, \beta \in \mathbb{H}$. Since $I = I_0$, we have $\beta \in I_0$. It means that $(\beta \to 0)' \in I$, and equivalently $I_\beta \leq I_0$. Moreover, $(\alpha \to \beta)' \in I$, then $I_\alpha \leq I_\beta$, and so $I_\alpha \leq I_0$ i.e., $(\alpha \to 0)' \in I$. Hence, $\alpha'' \in I$. Since $\mathbb{H}$ has (DNP), we get $\alpha \in I$. Therefore, $I$ is a co-filter of $\mathbb{H}$. □

**Definition 5** ([11])**.** *A hoop $(\mathbb{H}, \odot, \to, 1)$ is called Wajsberg if, for any $\alpha, \beta \in \mathbb{H}$,*

$$(\alpha \to \beta) \to \beta = (\beta \to \alpha) \to \alpha.$$

**Theorem 7.** *Let $\mathbb{H}$ has (DNP). Then $I$ is a co-filter of $\mathbb{H}$ iff $\mathbb{H}/I$ is a Wajsberg hoop.*

**Proof.** $(\Rightarrow)$ Since $\mathbb{H}$ has (DNP), by Proposition 2(v), $(\alpha \to \beta) \to \beta = (\beta \to \alpha) \to \alpha$, for all $\alpha, \beta \in \mathbb{H}$. Thus,

$$(((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha))' = 0 \in I,$$

and so $(I_\alpha \rightsquigarrow I_\beta) \rightsquigarrow I_\beta \leq (I_\beta \rightsquigarrow I_\alpha) \rightsquigarrow I_\alpha$. By the similar way, $(I_\beta \rightsquigarrow I_\alpha) \rightsquigarrow I_\alpha \leq (I_\alpha \rightsquigarrow I_\beta) \rightsquigarrow I_\beta$. Thus, $(I_\alpha \rightsquigarrow I_\beta) \rightsquigarrow I_\beta = (I_\beta \rightsquigarrow I_\alpha) \rightsquigarrow I_\alpha$, for all $I_\alpha, I_\beta \in \mathbb{H}/I$. Therefore, $\mathbb{H}/I$ is a Wajsberg hoop.
$(\Leftarrow)$ The proof is similar to the proof of Theorem 6. □

**Example 8.** *In Example 1, $\mathbb{H}$ is a hoop with (DNP). Since $I = \{0, b, d\}$ is a co-filter of $\mathbb{H}$, $I_0 = I_b = I_d = I$ and $I_a = I_c = I_1 = \{a, c, 1\}$. Hence, by Theorem 7, $\mathbb{H}/I = \{I_0, I_1\}$ is a Wajsberg hoop.*

**Definition 6** ([11])**.** *A Heyting algebra is an algebra $(A, \vee, \wedge, \to, 1)$, where $(A, \vee, \wedge, 1)$ is a distributive lattice with the greatest element and the binary operation $\to$ on $A$ verifies, for any $x, y, z \in A$,*

$$x \leq y \to z \text{ iff } x \wedge y \leq z.$$

**Theorem 8.** *Let $\mathbb{H}$ has (DNP) and $\alpha^2 = \alpha$, for all $\alpha \in \mathbb{H}$. Then $I$ is a co-filter of $\mathbb{H}$ iff $\mathbb{H}/I$ is a Heyting algebra.*

**Proof.** $(\Rightarrow)$ Since $I$ is a co-filter of $\mathbb{H}$ and $\alpha^2 = \alpha$, for all $\alpha \in \mathbb{H}$, by Theorem 5, $\mathbb{H}/I$ is a Brouwerian semilattice. Moreover, since $\mathbb{H}$ has (DNP), by Theorem 7, $\mathbb{H}/I$ is a Wajsberg hoop. Define $I_\alpha \vee_I I_\beta = (I_\beta \rightsquigarrow I_\alpha) \rightsquigarrow I_\alpha$, for all $I_\alpha, I_\beta \in \mathbb{H}/I$. Then by Propositions 3 and 4, $(\mathbb{H}/I, \wedge_I, \vee_I)$ is a distributive lattice. Therefore, $\mathbb{H}/I$ is a Heyting algebra.
$(\Leftarrow)$ Since $\mathbb{H}/I$ is a Heyting algebra, it is a Brouwerian semilattice. On the other side, $\mathbb{H}$ has (DNP), then by Theorem 6, $I$ is a co-filter of $\mathbb{H}$. □

**Example 9.** *Let $\mathbb{H} = \{0, a, b, 1\}$ be a set with the following Cayley tabels,*

| $\to$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| a | b | 1 | b | 1 |
| b | a | a | 1 | 1 |
| 1 | 0 | a | b | 1 |

| $\odot$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | 0 | a |
| b | 0 | 0 | b | b |
| 1 | 0 | a | b | 1 |

*Then $(\mathbb{H}, \odot, \to, 0, 1)$ is a hoop with (DNP) and for any $\alpha \in \mathbb{H}$, $\alpha^2 = \alpha$. From $I = \{0, b\}$ is a co-filter of $\mathbb{H}$, $I_0 = I_b = I$ and $I_1 = I_a = \{1, a\}$. Then by Theorem 8, $\mathbb{H}/I = \{I, I_1\}$ is a Heyting algebra.*

**Definition 7** ([11])**.** *A Hilbert algebra is a tripe $(A, \to, 1)$ of type $(2, 0)$ such that, for all $x, y, z \in A$, the following three axioms are satisfied,*
$(H1)$ $x \to (y \to x) = 1$.

$(H2)$ $x \to (y \to z) = (x \to y) \to (x \to z)$.
$(H3)$ *If* $x \to y = y \to x = 1$, *then* $x = y$.
*The Hilbert algebra induces a partial order* $\leq$ *on* $A$, *defined by,* $x \leq y$ *iff* $x \to y = 1$ *and* $1$ *is the greatest element of the induced poset* $(A, \leq)$. *A Hilbert algebra* $A$ *is bounded if there is an element* $0 \in A$ *such that, for any* $x \in A$, $0 \leq x$.

**Lemma 1.** *Let* $\alpha^2 = \alpha$, *for all* $\alpha \in \mathbb{H}$. *Then, for all* $\alpha, \beta, \gamma \in \mathbb{H}$,

$$\alpha \to (\beta \to \gamma) = (\alpha \to \beta) \to (\alpha \to \gamma).$$

**Proof.** Let $\alpha \in \mathbb{H}$ such that $\alpha^2 = \alpha$. Then by Proposition 1(iv), $\beta \leq \alpha \to \beta$, for any $\alpha, \beta \in \mathbb{H}$ and by Proposition 1(viii), $\alpha \to ((\alpha \to \beta) \to \gamma) \leq \alpha \to (\beta \to \gamma)$. Then by (HP3), $(\alpha \to \beta) \to (\alpha \to \gamma) \leq \alpha \to (\beta \to \gamma)$. Conversely, by (HP3), for all $\alpha, \beta, \gamma \in \mathbb{H}$,

$$[\alpha \to (\beta \to \gamma)] \to [(\alpha \to \beta) \to (\alpha \to \gamma)] = [(\alpha \to \beta) \odot \alpha \odot (\alpha \to (\beta \to \gamma))] \to \gamma.$$

By Proposition 1(vii), $\alpha \odot (\alpha \to (\beta \to \gamma)) \leq \beta \to \gamma$. Then by Proposition 1(viii) and (vii),

$$(\alpha \to \beta) \odot \alpha \odot (\alpha \to (\beta \to \gamma)) \leq (\alpha \to \beta) \odot (\beta \to \gamma) \leq \alpha \to \gamma.$$

Thus, $\alpha^2 \odot (\alpha \to \beta) \odot (\alpha \to (\beta \to \gamma)) \leq \gamma$. Since $\alpha^2 = \alpha$, we get $\alpha \odot (\alpha \to \beta) \odot (\alpha \to (\beta \to \gamma)) \leq \gamma$. Hence, by (HP3), $\alpha \to (\beta \to \gamma) \leq (\alpha \to \beta) \to (\alpha \to \gamma)$. $\square$

**Theorem 9.** *Let* $I$ *be a co-filter of* $\mathbb{H}$ *and* $\alpha^2 = \alpha$, *for all* $\alpha \in \mathbb{H}$. *Then* $\mathbb{H}/I$ *is a Hilbert algebra.*

**Proof.** Since $I$ is a co-filter of $\mathbb{H}$, by Theorem 5, $\mathbb{H}/I$ is a hoop. Thus by Proposition 1(iv), it is clear that $I_\alpha \rightsquigarrow (I_\beta \rightsquigarrow I_\alpha) = I_1$, for all $I_\alpha, I_\beta \in \mathbb{H}/I$. Let $I_\alpha, I_\beta \in \mathbb{H}/I$ such that $I_\alpha \rightsquigarrow I_\beta = I_\beta \rightsquigarrow I_\alpha = I_1$. Then $(\alpha \to \beta)' \in I$ and $(\beta \to \alpha)' \in I$ and so $\alpha \equiv_I \beta$. Hence, $I_\alpha = I_\beta$. Moreover, since $\alpha^2 = \alpha$, for all $\alpha \in \mathbb{H}$, by Lemma 1, $\alpha \to (\beta \to \gamma) = (\alpha \to \beta) \to (\alpha \to \gamma)$, for all $\alpha, \beta, \gamma \in \mathbb{H}$, and so

$$[(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))]' = 0 \in I.$$

Thus, by definition of $I_1$,

$$(I_\alpha \rightsquigarrow (I_\beta \rightsquigarrow I_\gamma)) \rightsquigarrow ((I_\alpha \rightsquigarrow I_\beta) \rightsquigarrow (I_\alpha \rightsquigarrow I_\gamma)) = I_1$$

Therefore, $\mathbb{H}/I$ is a Hilbert algebra. $\square$

**Definition 8** ([11]). *A BL-algebra is an algebra* $(A, \vee, \wedge, \odot, \to, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *that, for any* $x, y, z \in A$, *it is satisfying the following axioms:*
$(BL1)$ $(A, \vee, \wedge, 0, 1)$ *is a bounded lattice.*
$(BL2)$ $(A, \odot, 1)$ *is a commutative monoid.*
$(BL3)$ $x \odot y \leq z$ *iff* $x \leq y \to z$.
$(BL4)$ $(x \to y) \vee (y \to x) = 1$.
$(BL5)$ $x \wedge y = x \odot (x \to y)$.

**Theorem 10.** *Let* $\mathbb{H}$ *be a* $\vee$*-hoop such that, for all* $\alpha \in \mathbb{H}$, $\alpha^2 = \alpha$ *and* $I$ *be a co-filter of* $\mathbb{H}$. *Then* $\mathbb{H}/I$ *is a BL-algebra.*

**Proof.** Let $\mathbb{H}$ be a $\vee$-hoop. Then $\mathbb{H}/I$ is a $\vee_I$-hoop. Thus, by Proposition 4, $(\mathbb{H}/I, \wedge_I, \vee_I, I_0, I_1)$ is a bounded distributive lattice. Now, we prove that $\mathbb{H}/I$ is a BL-algebra. For this, it is enough to prove that

$(I_\alpha \rightsquigarrow I_\beta) \vee_I (I_\beta \rightsquigarrow I_\alpha) = I_1$, for all $I_\alpha, I_\beta \in \mathbb{H}/I$. Equivalently, we show that $((\alpha \to \beta) \vee (\beta \to \alpha))' \in I$, for all $\alpha, \beta \in \mathbb{H}$. Since for all $\alpha, \beta \in \mathbb{H}$,

$$(\alpha \to \beta), (\beta \to \alpha) \leq (\alpha \to \beta) \vee (\beta \to \alpha),$$

by Proposition 1(viii),

$$((\alpha \to \beta) \vee (\beta \to \alpha))' \leq (\alpha \to \beta)' \wedge (\beta \to \alpha)'.$$

On the other hand, by Proposition 1(iv), $\beta \leq \alpha \to \beta$ and by Proposition 2(ii), $\beta' \leq \beta \to \alpha$, then by Proposition 1(viii), $(\alpha \to \beta)' \leq \beta'$ and $(\beta \to \alpha)' \leq \beta''$. Thus, by Propositions 1(i), 2(i) and $\alpha^2 = \alpha$, for all $\alpha \in \mathbb{H}$, we have

$$\begin{aligned}
((\alpha \to \beta) \vee (\beta \to \alpha))' &\leq (\alpha \to \beta)' \wedge (\beta \to \alpha)' \\
&\leq \beta' \wedge \beta'' \\
&= \beta' \odot (\beta' \to \beta'') \quad \text{by Proposition 1}(i) \\
&= \beta' \odot (\beta' \to (\beta' \to 0)) \quad \text{by (HP3)} \\
&= \beta' \odot ((\beta')^2 \to 0) \quad \text{by } \alpha^2 = \alpha \\
&= \beta' \odot (\beta' \to 0) \quad \text{by Proposition 2}(i) \\
&= \beta' \odot \beta'' \\
&= 0.
\end{aligned}$$

Then $((\alpha \to \beta) \vee (\beta \to \alpha))' = 0 \in I$. Therefore, $\mathbb{H}/I$ is a BL-algebra. □

**Theorem 11.** *Let $\mathbb{H}$ has (DNP) and $\alpha^2 = \alpha$, for all $\alpha \in \mathbb{H}$. Then $I$ is a co-filter of $\mathbb{H}$ iff $\mathbb{H}/I$ is a BL-algebra.*

**Proof.** ($\Rightarrow$) Since $\mathbb{H}$ has (DNP) and $I$ is a co-filter of $\mathbb{H}$, by Theorem 7, $\mathbb{H}/I$ is a Wajsberg hoop. Define $I_\alpha \vee_I I_\beta = (I_\alpha \rightsquigarrow I_\beta) \rightsquigarrow I_\beta$ for all $I_\alpha, I_\beta \in \mathbb{H}/I$. Then by Proposition 3, $\mathbb{H}/I$ is a $\vee_I$-hoop, and so by Proposition 4, $(\mathbb{H}/I, \wedge_I, \vee_I, I_0, I_1)$ is a bounded lattice. On the other side, since $\alpha^2 = \alpha$, for all $\alpha \in \mathbb{H}$, by Theorem 10, $\mathbb{H}/I$ is a BL-algebra.
($\Leftarrow$) Since $\mathbb{H}$ has (DNP) and $\mathbb{H}/I$ is a BL-algebra, $\mathbb{H}/I$ is a distributive lattice. Thus, by Theorem 6, $I$ is a co-filter of $\mathbb{H}$. □

**Remark 1.** *As you see in this section, we investigated the relation among the quotient hoop $\frac{A}{I}$ that is made by a co-filter $I$ with other logical algebras such as Brouwerian semi-lattice, Heyting algebra, Hilbert algebra, Wajsberg hoop and BL-algebra. Clearly these conditions are similar and we know that for example if $A$ has Godel condition ($x^2 = x$) then $\frac{A}{I}$ is Hilbert algebra and by adding (DNP) property to $A$ we obtain that $\frac{A}{I}$ is Heyting algebra.*

## 5. Conclusions and Future Works

We have introduced the notion of co-filter of hoops and a congruence relation on hoop, and then we have constructed the quotient structures by using co-filters. We have considered the relation between filters and co-filters in a hoop with (DNP) property. We have provided conditions for a subset to be a co-filter. We have discussed characterizations of a co-filter. We have studied the relation among this structure and other algebraic structures. Using the notion of co-filters, we have established the quotient Brouwerian semilattice, the quotient Hilbert algebra and the quotient BL-algebra. We have induced a co-filter from a quotient Brouwerian semilattice. In our subsequent research, we will study some kinds of co-filter such as, implicative, ultra and prime one and investigate the relation between them. Also, we will discuss fuzzy co-filters and fuzzy congruence relation by them and study the quotient structure of this fuzzy congruence relation.

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