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Article

A Further Extension for Ramanujan's Beta Integral and Applications

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Abstract: In 1915, Ramanujan stated the following formula $\int_0^\infty t^{x-1}\frac{(-at;q)_\infty}{(-t;q)_\infty}dt = \frac{\pi}{\sin \pi x}\frac{(q^{1-x},a;q)_\infty}{(q,aq^{-x};q)_\infty}$, where $0 < q < 1$, $x > 0$, and $0 < a < q^x$. The above formula is called Ramanujan's beta integral. In this paper, by using q-exponential operator, we further extend Ramanujan's beta integral. As some applications, we obtain some new integral formulas of Ramanujan and also show some new representation with gamma functions and q-gamma functions.

Keywords: q-series; q-exponential operator; q-binomial theorem; q-Gauss formula; q-gamma function; gamma function; Ramanujan's beta integral

MSC: Primary 33D15; Secondary 05A30

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1. Introduction, Preliminaries and Main Results

The gamma function is the most natural extension of the factorial

$$n! = 1 \cdot 2 \cdots n.$$

Euler's original definition is

$$\Gamma(x+1) = \prod_{k=1}^{\infty} \frac{k}{k+x}\left(\frac{k+1}{k}\right)^x. \tag{1}$$

The integral representation of the gamma function is the following form

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad \Re(x) > 0. \tag{2}$$

The q-shifted factorials are defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \tag{3}$$

$$(a;q)_\infty = \lim_{n\to\infty}\prod_{k=0}^{n-1}(1-aq^k) = \prod_{k=0}^{\infty}(1-aq^k), \qquad n \geq 1. \tag{4}$$

Clearly,

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{5}$$

Analogously with $\Gamma(x)$, F. H. Jackson [1] defined $\Gamma_q(x)$ by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad 0 < q < 1. \tag{6}$$

$\Gamma_q(x)$ is called the q-gamma function.

The functional equation for $\Gamma(x)$,

$$\Gamma(x+1) = x\Gamma(x), \tag{7}$$

becomes

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x) \tag{8}$$

for the q-gamma function. In the future, we will always take $0 < q < 1$.

We also adopt the following compact notations for the multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,$$
$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

The basic hypergeometric series, or q-series ${}_r\phi_s$ is usually defined by

$${}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n}{(q, b_1;q)_n (b_2;q)_n \cdots (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n, \tag{9}$$

with $\binom{n}{2} = n(n-1)/2$, where $q \neq 0$, when $r > s+1$. Clearly, we have

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots,, b_r \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n}{(q, b_1, b_2, \ldots, b_r; q)_n} z^n. \tag{10}$$

The usual q-differential operator, or q-derivative operator D_q is defined by (see ([2], p. 177, (2.1)) or [1,3–5]).

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a}, \tag{11}$$
$$D_q^n\{f(a)\} = D_q\left\{D_q^{(n-1)}\{f(a)\}\right\}. \tag{12}$$

The q-shift operator η is (see ([6], p. 112)):

$$\eta\{f(a)\} = f(aq), \tag{13}$$
$$\eta^{-1}\{f(a)\} = f(aq^{-1}). \tag{14}$$

The operator θ is (see [7]):

$$\theta = \eta^{-1} D_q. \tag{15}$$

The q-exponential operator $E(b\theta)$ is defined by (see ([6], p. 112))

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n q^{\binom{n}{2}}}{(q;q)_n}. \tag{16}$$

Recently, Fang further generalized the q-exponential operator $E(b\theta)$ in the following form (see [8], or ([9], p. 1394, Equation (5))):

$$ {}_1\phi_0\left(\begin{matrix} b \\ - \end{matrix}; q, -c\theta\right) = \sum_{n=0}^{\infty} \frac{(b;q)_n(-c\theta)^n}{(q;q)_n} \tag{17}$$

and obtained two q-operator identities as follows:

$$ {}_1\phi_0\left(\begin{matrix} b \\ - \end{matrix}; q, -c\theta\right) \{(as;q)_\infty\} = \frac{(as, bcs;q)_\infty}{(cs;q)_\infty}, \tag{18}$$

$$ {}_1\phi_0\left(\begin{matrix} b \\ - \end{matrix}; q, -c\theta\right) \left\{\frac{(as;q)_\infty}{(a\omega;q)_\infty}\right\} = \frac{(as;q)_\infty}{(a\omega;q)_\infty} {}_2\phi_1\left(\begin{matrix} b, s/\omega \\ q/a\omega \end{matrix}; q, qc/a\right). \tag{19}$$

In 1915, Ramanujan stated the following formula in [10,11]:

$$ \int_0^\infty t^{x-1} \frac{(-at;q)_\infty}{(-t;q)_\infty} dt = \frac{\pi}{\sin \pi x} \frac{(q^{1-x}, a;q)_\infty}{(q, aq^{-x};q)_\infty}, \tag{20}$$

where $0 < q < 1$, $x > 0$, and $0 < a < q^x$. The right-hand side must be interpreted using a limit when x is an integer. The above formula is called Ramanujan's beta integral.

Hardy gave the first proof of (20) in [12]. He closed this paper with the evaluation of "another curious integral", which is another important integral. Hardy gave a nice treatment of Ramanujan's method of evaluating integrals of this type in his book on Ramanujan [13]. Rahman and Suslov gave a simple proof of (20) in ([14], pp. 109–110) by Ramanujan's sum formula ${}_1\psi_1$. Askey ([15], p. 349) gave an elementary proof of (20) and obtained the following formula when $a = q^{x+y}$ in (20):

$$ \int_0^\infty t^{x-1} \frac{(-tq^{x+y};q)_\infty}{(-t;q)_\infty} dt = \frac{\Gamma_q(y)\Gamma(x)\Gamma(1-x)}{\Gamma_q(x+y)\Gamma_q(1-x)} \tag{21}$$

in terms of the q-gamma function and the ordinary gamma function. When $q \to 1$, this reduces to

$$ \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(x,y), \tag{22}$$

where $B(x,y)$ denotes the beta function defined by

$$ B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt. \tag{23}$$

Recently, Chen and Liu ([6], p. 123. Equation (7.3)) gave an extension of (20) by the method of the operator as follows:

$$ \int_0^\infty t^{x-1} \frac{(-at, -bt;q)_\infty}{(-t, -abq^{-x-1}t;q)_\infty} dt = \frac{\pi}{\sin \pi x} \frac{(q^{1-x}, a, b;q)_\infty}{(q, aq^{-x}, bq^{-x};q)_\infty}. \tag{24}$$

The aim of the present paper is to further generalize Ramanujan's beta integral by the operator ${}_1\phi_0\left(\begin{matrix} b \\ - \end{matrix}; q, -c\theta\right)$ and to give some new formulas of Ramanujan's beta integral. We also show the connections with gamma functions and q-gamma functions.

We now state our result as follows:

Theorem 1. *If* $0 < q < 1$, $x > 0$, $0 < a, a_1, a_2, \ldots, a_{2r+2} < q^x$, $|qa_2/a| < 1$ *and* $|qa_{2j+2}/a_{2j-1}| < 1$ $(j = 1, 2, \ldots, r)$; $a_1, a_2 \neq a$, $a_{l+2}, a_{l+3} \neq a_l$ $(l = 1, 3, 5, \ldots, 2r-1)$, *then we have*

$$\begin{aligned}
&\int_0^\infty t^{x-1} \frac{(-at, -a_1t, -a_3t, \ldots, -a_{2r+1}t; q)_\infty}{(-t, -a_2t, -a_4t, \ldots, -a_{2r+2}t; q)_\infty} dt \\
&= \frac{\pi}{\sin \pi x} \frac{(q^{1-x}, a; q)_\infty}{(q, aq^{-x}; q)_\infty} \sum_{k=0}^{\infty} \frac{(q^x, a_1/a_2; q)_k (qa_2/a)^k}{(q, q^{1+x}/a; q)_k} \\
&\times \sum_{\substack{k_1+k_2+\cdots+k_r=k \\ 0 \le k_r \le k_{r-1} \le \cdots \le k_2 \le k_1 \le k}} \prod_{j=1}^{r} \frac{(q^{-k_{j-1}}, a_{2j+1}/a_{2j+2}; q)_{k_j} (qa_{2j+2}/a_{2j-1})^{k_j}}{(q, q^{1-k_{j-1}} a_{2j}/a_{2j-1}; q)_{k_j}}.
\end{aligned} \tag{25}$$

2. Proof of Theorem 1

Proof of Theorem 1. Firstly, we write Ramanujan's formula as follows:

$$\int_0^\infty t^{x-1} \frac{(-at; q)_\infty}{(-t; q)_\infty} dt = \frac{\pi}{\sin \pi x} \frac{(q^{1-x}, a; q)_\infty}{(q, aq^{-x}; q)_\infty}. \tag{26}$$

Applying the operator

$${}_1\phi_0 \left(\begin{matrix} a_1 \\ - \end{matrix}; q, -a_2\theta \right)$$

on both sides of (26) with respect to variable a, we obtain

$$\begin{aligned}
&\int_0^\infty t^{x-1} \frac{1}{(-t; q)_\infty} {}_1\phi_0 \left(\begin{matrix} a_1 \\ - \end{matrix}; q, -a_2\theta \right) \{(-at; q)_\infty\} dt \\
&= \frac{\pi}{\sin \pi x} \frac{(q^{1-x}; q)_\infty}{(q; q)_\infty} {}_1\phi_0 \left(\begin{matrix} a_1 \\ - \end{matrix}; q, -a_2\theta \right) \left\{ \frac{(a; q)_\infty}{(aq^{-x}; q)_\infty} \right\}.
\end{aligned} \tag{27}$$

By (18) and (19), we get that

$$\int_0^\infty t^{x-1} \frac{(-at, -a_1a_2t; q)_\infty}{(-t, -a_2t; q)_\infty} dt = \frac{\pi}{\sin \pi x} \frac{(q^{1-x}, a; q)_\infty}{(q, aq^{-x}; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a_1, q^x \\ q^{1+x}/a \end{matrix}; q, qa_2/a \right). \tag{28}$$

By (5), we rewrite the formula (28) in the following form:

$$\int_0^\infty t^{x-1} \frac{(-at; q)_\infty}{(-t, -a_2t; q)_\infty} (-a_1a_2t; q)_\infty dt = \frac{\pi}{\sin \pi x} \frac{(q^{1-x}, a; q)_\infty}{(q, aq^{-x}; q)_\infty} \sum_{k=0}^{\infty} \frac{(q^x; q)_k (qa_2/a)^k}{(q, q^{1+x}/a; q)_k} \frac{(a_1; q)_\infty}{(a_1q^k; q)_\infty}. \tag{29}$$

Next, by applying the operator

$${}_1\phi_0 \left(\begin{matrix} a_3 \\ - \end{matrix}; q, -a_4\theta \right)$$

on both sides of the Equation (29) with respect to variable a_1, we arrive at

$$\begin{aligned}
&\int_0^\infty t^{x-1} \frac{(-at, -a_1a_2t, -a_2a_3a_4t; q)_\infty}{(-t, -a_2t, -a_2a_4t; q)_\infty} dt \\
&= \frac{\pi}{\sin \pi x} \frac{(q^{1-x}, a; q)_\infty}{(q, aq^{-x}; q)_\infty} \sum_{k=0}^{\infty} \frac{(q^x, a_1; q)_k (qa_2/a)^k}{(q, q^{1+x}/a; q)_k} \sum_{k_1=0}^{k} \frac{(q^{-k}, a_3; q)_{k_1} (qa_4/a_1)^{k_1}}{(q, q^{1-k}/a_1; q)_{k_1}}.
\end{aligned} \tag{30}$$

We rewrite (30) in the following form:

$$\int_0^\infty t^{x-1}\frac{(-at,-a_1a_2t;q)_\infty}{(-t,-a_2t,-a_2a_4t;q)_\infty}(-a_2a_3a_4t;q)_\infty dt$$
$$=\frac{\pi}{\sin\pi x}\frac{(q^{1-x},a;q)_\infty}{(q,aq^{-x};q)_\infty}\sum_{k=0}^{\infty}\frac{(q^x,a_1;q)_k(qa_2/a)^k}{(q,q^{1+x}/a;q)_k}\sum_{k_1=0}^{k}\frac{(q^{-k};q)_{k_1}(qa_4/a_1)^{k_1}}{(q,q^{1-k}/a_1;q)_{k_1}}\frac{(a_3;q)_\infty}{(a_3q^{k_1};q)_\infty}. \tag{31}$$

Applying the operator

$$_1\phi_0\left(\begin{matrix}a_5\\ -\end{matrix};q,-a_6\theta\right)$$

on both sides of the Equation (31) with respect to variable a_3, we have

$$\int_0^\infty t^{x-1}\frac{(-at,-a_1a_2t,-a_2a_3a_4t,-a_2a_4a_5a_6t;q)_\infty}{(-t,-a_2t,-a_2a_4t,-a_2a_4a_6t;q)_\infty}dt$$
$$=\frac{\pi}{\sin\pi x}\frac{(q^{1-x},a;q)_\infty}{(q,aq^{-x};q)_\infty}$$
$$\times\sum_{k=0}^{\infty}\frac{(q^x,a_1;q)_k(qa_2/a)^k}{(q,q^{1+x}/a;q)_k}\sum_{k_1=0}^{k}\frac{(q^{-k},a_3;q)_{k_1}(qa_4/a_1)^{k_1}}{(q,q^{1-k}/a_1;q)_{k_1}}\sum_{k_2=0}^{k_1}\frac{(q^{-k_1},a_5;q)_{k_2}(qa_6/a_3)^{k_2}}{(q,q^{1-k_1}/a_3;q)_{k_2}}. \tag{32}$$

By the mathematical induction, iterating $r+1$ times, and applying the operator

$$_1\phi_0\left(\begin{matrix}a_{2r+1}\\ -\end{matrix};q,-a_{2r+2}\theta\right)$$

and noting that (18) and (19), we obtain

$$\int_0^\infty t^{x-1}\frac{(-at,-a_1a_2t,-a_2a_3a_4t,-a_2a_4a_5a_6t,\ldots,-a_2a_4a_6a_8\cdots a_{2r-2}a_{2r}a_{2r+1}a_{2r+2}t;q)_\infty}{(-t,-a_2t,-a_2a_4t,-a_2a_4a_6t,\ldots,-a_2a_4a_6a_8\cdots a_{2r-2}a_{2r}a_{2r+2}t;q)_\infty}dt$$
$$=\frac{\pi}{\sin\pi x}\frac{(q^{1-x},a;q)_\infty}{(q,aq^{-x};q)_\infty}\sum_{k=0}^{\infty}\frac{(q^x,a_1;q)_k(qa_2/a)^k}{(q,q^{1+x}/a;q)_k}\sum_{k_1=0}^{k}\frac{(q^{-k},a_3;q)_{k_1}(qa_4/a_1)^{k_1}}{(q,q^{1-k}/a_1;q)_{k_1}}$$
$$\times\sum_{k_2=0}^{k_1}\frac{(q^{-k_1},a_5;q)_{k_2}(qa_6/a_3)^{k_2}}{(q,q^{1-k_1}/a_3;q)_{k_2}}\cdots\sum_{k_r=0}^{k_{r-1}}\frac{(q^{-k_{r-1}},a_{2r+1};q)_{k_r}(qa_{2r+2}/a_{2r-1})^{k_r}}{(q,q^{1-k_{r-1}}/a_{2r-1};q)_{k_r}}. \tag{33}$$

Letting $a_{2j-1}\longmapsto a_{2j-1}/a_{2j}$ and $a_{2j+2}\longmapsto a_{2j+2}/a_{2j}$ $(j=1,2,\cdots r)$ in (33), we show that

$$\int_0^\infty t^{x-1}\frac{(-at,-a_1t,-a_3t,\ldots,-a_{2r+1}t;q)_\infty}{(-t,-a_2t,-a_4t,\ldots,-a_{2r+2}t;q)_\infty}dt$$
$$=\frac{\pi}{\sin\pi x}\frac{(q^{1-x},a;q)_\infty}{(q,aq^{-x};q)_\infty}\sum_{k=0}^{\infty}\frac{(q^x,a_1/a_2;q)_k(qa_2/a)^k}{(q,q^{1+x}/a;q)_k}$$
$$\times\sum_{\substack{k_1+k_2+\cdots+k_r=k\\0\le k_r\le k_{r-1}\le\cdots\le k_2\le k_1\le k}}\prod_{j=1}^{r}\frac{(q^{-k_{j-1}},a_{2j+1}/a_{2j+2};q)_{k_j}(qa_{2j+2}/a_{2j-1})^{k_j}}{(q,q^{1-k_{j-1}}a_{2j}/a_{2j-1};q)_{k_j}}. \tag{34}$$

The proof of Theorem 1 is complete.

3. Some Applications

In this section, we will obtain the corresponding new integral formulas from (25).

Taking $r=0$ in (25) and defining the empty sum equal to 1, we obtain the following integral formula:

Corollary 1. *For $0 < q < 1$, $x > 0$, $0 < a, a_1, a_2 < q^x$ and $|qa_2/a| < 1$; $a_1, a_2 \neq a$, we have*

$$\int_0^\infty t^{x-1} \frac{(-at, -a_1 t; q)_\infty}{(-t, -a_2 t; q)_\infty} dt = \frac{\pi}{\sin \pi x} \frac{(q^{1-x}, a; q)_\infty}{(q, aq^{-x}; q)_\infty} {}_2\phi_1 \left(\begin{matrix} q^x, a_1/a_2 \\ q^{1+x}/a \end{matrix} ; q, qa_2/a \right). \tag{35}$$

Remark 1. *If setting $a_2 = aa_1 q^{-x-1}$ in (35), we get*

$$\int_0^\infty t^{x-1} \frac{(-at, -a_1 t; q)_\infty}{(-t, -aa_1 q^{-x-1} t; q)_\infty} dt = \frac{\pi}{\sin \pi x} \frac{(q^{1-x}, a; q)_\infty}{(q, aq^{-x}; q)_\infty} {}_1\phi_0 \left(\begin{matrix} q^x \\ - \end{matrix} ; q, a_1 q^{-x} \right). \tag{36}$$

Applying the q-binomial theorem (see ([16], p. 8. (1.3.2)))

$${}_1\phi_0 \left(\begin{matrix} a \\ - \end{matrix} ; q, z \right) = \frac{(az; q)_\infty}{(z; q)_\infty} \tag{37}$$

in (36), we obtain

$$\int_0^\infty t^{x-1} \frac{(-at, -a_1 t; q)_\infty}{(-t, -aa_1 q^{-x-1} t; q)_\infty} dt = \frac{\pi}{\sin \pi x} \frac{(q^{1-x}, a, a_1; q)_\infty}{(q, aq^{-x}, a_1 q^{-x}; q)_\infty}. \tag{38}$$

Setting $a_1 = b$ in (38), we obtain (24) immediately. Hence, we say that the formula (35) is an extension of result (24) of Chen and Liu.

Corollary 2. *For $0 < q < 1$, $x > 0$, $0 < a, b < q^x$; $a \neq b$, we have*

$$\int_0^\infty t^{x-1} \frac{(-at; q)_\infty}{(-bt; q)_\infty} dt = \frac{\pi}{\sin \pi x} \frac{(q^{1-x}, a, q/a, bq^{1+x}/a; q)_\infty}{(q, aq^{-x}, qb/a, q^{1+x}/a; q)_\infty}. \tag{39}$$

Proof. Setting $a_1 \to 1, a_2 \to b$ in (35) and applying q-Gauss sum formula ([16], p. 14, Equation (1.5.1)):

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; q, c/ab \right) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \qquad |c/ab| < 1,$$

we obtain (39).

Remark 2. *If letting $b \to 1$ in (39), we obtain (20) immediately. Hence, the formula (39) is also an extension of (20):*

Taking $r = 1$ in (25), we have

Corollary 3. *For $0 < q < 1$, $x > 0$, $0 < a, a_1, a_2, a_3, a_4 < q^x$, $|qa_2/a| < 1$ and $|qa_4/a_1| < 1$; $a_1, a_2 \neq a$, $a_3, a_4 \neq a_1$, we have*

$$\begin{aligned} &\int_0^\infty t^{x-1} \frac{(-at, -a_1 t, -a_3 t; q)_\infty}{(-t, -a_2 t, -a_4 t; q)_\infty} dt \\ &= \frac{\pi}{\sin \pi x} \frac{(q^{1-x}, a; q)_\infty}{(q, aq^{-x}; q)_\infty} \sum_{k=0}^\infty \frac{(q^x, a_1/a_2; q)_k (qa_2/a)^k}{(q, q^{1+x}/a; q)_k} {}_2\phi_1 \left(\begin{matrix} q^{-k}, a_3/a_4 \\ a_2 q^{1-k}/a_1 \end{matrix} ; q, qa_4/a_1 \right). \end{aligned} \tag{40}$$

Theorem 2. *If* $0 < q < 1$, $0 < a, a_1, a_2, \ldots, a_{2r+2} < q^n$ $(n = 1, 2, \cdots)$, $|qa_2/a| < 1$ *and* $|qa_{2j+2}/a_{2j-1}| < 1$, $(j = 1, 2, \ldots, r)$; $a_1, a_2 \neq a$, $a_{l+2}, a_{l+3} \neq a_l$ $(l = 1, 3, 5, \ldots, 2r-1)$, *then we have*

$$\begin{aligned}
&\int_0^\infty t^{n-1} \frac{(-at, -a_1t, -a_3t, \ldots, -a_{2r+1}t; q)_\infty}{(-t, -a_2t, -a_4t, \ldots, -a_{2r+2}t; q)_\infty} dt \\
&= \frac{(-1)^{n-1}(q;q)_n q^n \log q}{(a^{-1}q;q)_n a^n (1-q^n)} \sum_{k=0}^{\infty} \frac{(q^n, a_1/a_2; q)_k (qa_2/a)^k}{(q, q^{1+n}/a; q)_k} \\
&\times \sum_{\substack{k_1+k_2+\cdots+k_r=k \\ 0 \le k_r \le k_{r-1} \le \cdots \le k_2 \le k_1 \le k}} \prod_{j=1}^{r} \frac{(q^{-k_{j-1}}, a_{2j+1}/a_{2j+2}; q)_{k_j} (qa_{2j+2}/a_{2j-1})^{k_j}}{(q, q^{1-k_{j-1}} a_{2j}/a_{2j-1}; q)_{k_j}}.
\end{aligned} \tag{41}$$

Proof. By (4), we easily obtain

$$(q^{1-x};q)_\infty = (1-q^{1-x})(1-q^{2-x}) \cdots (1-q^{n-1-x})(1-q^{n-x})(1-q^{n+1-x})(1-q^{n+2-x}) \cdots, \tag{42}$$

$$(aq^{-x};q)_\infty = (1-aq^{-x})(1-aq^{1-x})(1-aq^{2-x}) \cdots (1-aq^{n-1-x})(1-aq^{n-x})(1-aq^{n+1-x})(1-aq^{n+2-x}) \cdots. \tag{43}$$

Noting (42), (43) and using the L'Hospital rule, via some simple calculation, we get

$$\begin{aligned}
&\lim_{x \to n} \frac{\pi}{\sin \pi x} \frac{(q^{1-x}, a; q)_\infty}{(q, aq^{-x}; q)_\infty} \\
&= \frac{(a;q)_\infty}{(q;q)_\infty} \lim_{x \to n} \frac{\pi}{\sin \pi x} \frac{(q^{1-x};q)_\infty}{(aq^{-x};q)_\infty} \\
&= \frac{(a;q)_\infty}{(q;q)_\infty} \frac{(-1)^{n-1} q^{-1-2-\cdots-(n-1)} (1-q)(1-q^2)\cdots(1-q^{n-1})(1-q)(1-q^2)\cdots}{(-1)^n q^{-1-2-\cdots-n}(a-q)(a-q^2)\cdots(a-q^n)(1-a)(1-aq)(1-aq^2)\cdots} \lim_{x \to n} \frac{\pi(1-q^{n-x})}{\sin \pi x} \\
&= \frac{(a;q)_\infty}{(q;q)_\infty} \frac{-(1-q)(1-q^2)\cdots(1-q^{n-1})(q;q)_\infty}{q^{-n}(a-q)(a-q^2)\cdots(a-q^n)(a;q)_\infty} \frac{\log q}{\cos n\pi} \\
&= \frac{(-1)^{n-1}(q;q)_n q^n \log q}{(a^{-1}q;q)_n a^n (1-q^n)}.
\end{aligned}$$

Letting $x \longrightarrow n$ on both sides of (25) and using the above limit, we obtain (41) immediately.

Taking $r = 0$ in (41) and defining the empty sum equal to 1, we obtain the following integral formula:

Corollary 4. *For* $0 < q < 1$, $0 < a, a_1, a_2 < q^n$ $(n = 1, 2, \cdots)$ *and* $|qa_2/a| < 1$; $a_1, a_2 \neq a$, *we have*

$$\int_0^\infty t^{n-1} \frac{(-at, -a_1t; q)_\infty}{(-t, -a_2t; q)_\infty} dt = \frac{(-1)^{n-1}(q;q)_n q^n \log q}{(a^{-1}q;q)_n a^n (1-q^n)} \sum_{k=0}^{\infty} \frac{(q^n, a_1/a_2; q)_k (qa_2/a)^k}{(q, q^{1+n}/a; q)_k}. \tag{44}$$

Remark 3. *Taking* $a_1 = a_2$ *in* (44), *we deduce*

$$\int_0^\infty t^{n-1} \frac{(-at;q)_\infty}{(-t;q)_\infty} dt = \frac{(-1)^{n-1}(q;q)_n q^n \log q}{(a^{-1}q;q)_n a^n (1-q^n)} = \frac{-(q;q)_{n-1} q^n \log q}{(q-a)(q^2-a)\cdots(q^n-a)}, \tag{45}$$

which is exactly Askey's result in ([15], p. 349, (2.9)).

Taking $r = 1$ in (41), we get

Corollary 5. *For* $0 < q < 1$, $0 < a, a_1, a_2, a_3, a_4 < q^n$ $(n = 1, 2, \cdots)$, $|qa_2/a| < 1$ *and* $|qa_4/a_1| < 1$; $a_1, a_2 \neq a$, $a_3, a_4 \neq a_1$, *we have*

$$\int_0^\infty t^{n-1}\frac{(-at,-a_1t,-a_3t;q)_\infty}{(-t,-a_2t,-a_4t;q)_\infty}dt$$
$$=\frac{(q;q)_n q^n \log q}{(a-q)(a-q^2)\cdots(a-q^n)(1-q^n)}\sum_{k=0}^{\infty}\frac{(q^n,a_1/a_2;q)_k(qa_2/a)^k}{(q,q^{1+n}/a;q)_k}{}_2\phi_1\left(\begin{matrix}q^{-k},a_3/a_4\\ a_2q^{1-k}/a_1\end{matrix};q,qa_4/a_1\right). \tag{46}$$

Theorem 3. *If* $0<q<1$, $0<a,a_1,a_2,\ldots,a_{2r+2}<q$, $|qa_2/a|<1$ *and* $|qa_{2j+2}/a_{2j-1}|<1$ $(j=1,2,\ldots,r)$; $a_1,a_2\neq a$, $a_{l+2},a_{l+3}\neq a_l$ $(l=1,3,5,\ldots,2r-1)$, *then we have*

$$\int_0^\infty\frac{(-at,-a_1t,-a_3t,\ldots,-a_{2r+1}t;q)_\infty}{(-t,-a_2t,-a_4t,\ldots,-a_{2r+2}t;q)_\infty}dt$$
$$=\frac{q\log q}{a-q}\sum_{k=0}^{\infty}\frac{(a_1/a_2;q)_k(qa_2/a)^k}{(q^2/a;q)_k}$$
$$\times\sum_{\substack{k_1+k_2+\cdots+k_r=k\\0\le k_r\le k_{r-1}\le\cdots\le k_2\le k_1\le k}}\prod_{j=1}^{r}\frac{(q^{-k_{j-1}},a_{2j+1}/a_{2j+2};q)_{k_j}(qa_{2j+2}/a_{2j-1})^{k_j}}{(q,q^{1-k_{j-1}}a_{2j}/a_{2j-1};q)_{k_j}}. \tag{47}$$

Proof. Taking $n=1$ in (41), we obtain (47) immediately.

Taking $r=0$ in (47) and defining the empty sum equal to 1, we obtain the following integral formula:

Corollary 6. *For* $0<q<1$, $0<a,a_1,a_2<q$ *and* $|qa_2/a|<1$; $a_1,a_2\neq a$, *we have*

$$\int_0^\infty\frac{(-at,-a_1t;q)_\infty}{(-t,-a_2t;q)_\infty}dt=\frac{q\log q}{a-q}{}_2\phi_1\left(\begin{matrix}q,a_1/a_2\\ q^2/a\end{matrix};q,qa_2/a\right). \tag{48}$$

Taking $r=1$ in (47), we deduce

Corollary 7. *For* $0<q<1$, $0<a,a_1,a_2,a_3,a_4<q$, $|qa_2/a|<1$ *and* $|qa_4/a_1|<1$; $a_1,a_2\neq a$, $a_3,a_4\neq a_1$, *we have*

$$\int_0^\infty\frac{(-at,-a_1t,-a_3t;q)_\infty}{(-t,-a_2t,-a_4t;q)_\infty}dt=\frac{q\log q}{a-q}\sum_{k=0}^{\infty}\frac{(a_1/a_2;q)_k(qa_2/a)^k}{(q^2/a;q)_k}{}_2\phi_1\left(\begin{matrix}q^{-k},a_3/a_4\\ a_2q^{1-k}/a_1\end{matrix};q,qa_4/a_1\right). \tag{49}$$

4. Connections with the q-Gamma Function

In this section, we give the corresponding formulas with the q-gamma function from (25).

Theorem 4. *If* $0<q<1$, $x>0$, $0<a_1,a_2,\ldots,a_{2r+2}<q^x$, $|a_2/q^{x+y-1}|<1$ *and* $|qa_{2j+2}/a_{2j-1}|<1$, $(j=1,2,\ldots,r)$; $a_1,a_2\neq q^{x+y}$, $a_{l+2},a_{l+3}\neq a_l$ $(l=1,3,5,\ldots,2r-1)$, *then we have*

$$\int_0^\infty t^{x-1}\frac{(-tq^{x+y},-a_1t,-a_3t,\ldots,-a_{2r+1}t;q)_\infty}{(-t,-a_2t,-a_4t,\ldots,-a_{2r+2}t;q)_\infty}dt$$
$$=\frac{\Gamma_q(y)\Gamma_q(1-y)\Gamma(x)\Gamma(1-x)}{\Gamma_q(x+y)\Gamma_q(x)\Gamma_q(1-x)}\sum_{k=0}^{\infty}\frac{\Gamma_q(k+x)(a_1/a_2;q)_k}{\Gamma_q(k+1-y)(q;q)_k}(a_2/q^{x+y-1})^k$$
$$\times\sum_{\substack{k_1+k_2+\cdots+k_r=k\\0\le k_r\le k_{r-1}\le\cdots\le k_2\le k_1\le k}}\prod_{j=1}^{r}\frac{(q^{-k_{j-1}},a_{2j+1}/a_{2j+2};q)_{k_j}(qa_{2j+2}/a_{2j-1})^{k_j}}{(q,q^{1-k_{j-1}}a_{2j}/a_{2j-1};q)_{k_j}}. \tag{50}$$

Proof. Taking $a = q^{x+y}$ in (25) and noting that

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \quad \text{and} \quad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x},$$

we easily obtain (50).

Taking $r = 0$ in (50) and defining the empty sum equal to 1, we obtain the following integral formula.

Corollary 8. *For $0 < q < 1$, $x > 0$, $0 < a_1, a_2 < q^x$ and $|a_2/q^{x+y-1}| < 1$; $a_1, a_2 \neq q^{x+y}$, we have*

$$\int_0^\infty t^{x-1}\frac{(-tq^{x+y},-a_1t;q)_\infty}{(-t,-a_2t;q)_\infty}dt = \frac{\Gamma_q(y)\Gamma_q(1-y)\Gamma(x)\Gamma(1-x)}{\Gamma_q(x+y)\Gamma_q(x)\Gamma_q(1-x)}\sum_{k=0}^{\infty}\frac{\Gamma_q(k+x)(a_1/a_2;q)_k}{\Gamma_q(k+1-y)(q;q)_k}(a_2/q^{x+y-1})^k. \tag{51}$$

Taking $a_1 = a_2$ in (51), we deduce the result of Askey as follows:

Corollary 9 ([15], p. 350, Equation (2.10))**.** *For $0 < q < 1$, $x > 0$, we have*

$$\int_0^\infty t^{x-1}\frac{(-tq^{x+y};q)_\infty}{(-t;q)_\infty}dt = \frac{\Gamma_q(y)\Gamma(x)\Gamma(1-x)}{\Gamma_q(x+y)\Gamma_q(1-x)}. \tag{52}$$

Remark 4. *Letting $a \to 0$ and $b \to 1$ in (39), we have*

$$\int_0^\infty \frac{t^{x-1}}{(-t;q)_\infty}dt = \frac{\pi}{\sin \pi x}\frac{(q^{1-x};q)_\infty}{(q;q)_\infty}. \tag{53}$$

Applying

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \quad \text{and} \quad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x},$$

we obtain

$$\int_0^\infty \frac{t^{x-1}}{(-t;q)_\infty}dt = \frac{\Gamma(x)\Gamma(1-x)(1-q)^x}{\Gamma_q(1-x)}, \tag{54}$$

which is exactly the result of Askey (see ([15], p. 353, Equation (4.2))).

5. Conclusions

In this paper, by applying q-exponential operator

$${}_1\phi_0\left(\begin{matrix} b \\ - \end{matrix};q,-c\theta\right) = \sum_{n=0}^{\infty}\frac{(b;q)_n(-c\theta)^n}{(q;q)_n},$$

we further extend the following Ramanujan's beta integral [10]

$$\int_0^\infty t^{x-1}\frac{(-at;q)_\infty}{(-t;q)_\infty}dt = \frac{\pi}{\sin \pi x}\frac{(q^{1-x},a;q)_\infty}{(q,aq^{-x};q)_\infty}.$$

Especially, we obtain two new integral formulas

$$\int_0^\infty t^{x-1}\frac{(-at,-a_1t;q)_\infty}{(-t,-a_2t;q)_\infty}dt = \frac{\pi}{\sin \pi x}\frac{(q^{1-x},a;q)_\infty}{(q,aq^{-x};q)_\infty}{}_2\phi_1\left(\begin{matrix} q^x, a_1/a_2 \\ q^{1+x}/a \end{matrix};q,qa_2/a\right)$$

and

$$\int_0^\infty t^{x-1}\frac{(-at;q)_\infty}{(-bt;q)_\infty}dt=\frac{\pi}{\sin\pi x}\frac{(q^{1-x},a,q/a,bq^{1+x}/a;q)_\infty}{(q,aq^{-x},qb/a,q^{1+x}/a;q)_\infty}.$$

We also show that Ramanujan's beta integral can be represented with q-gamma functions [15].

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