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Strong Convergence of a New Iterative Algorithm for Split Monotone Variational Inclusion Problems

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Abstract: The main aim of this work is to introduce an implicit general iterative method for approximating a solution of a split variational inclusion problem with a hierarchical optimization problem constraint for a countable family of mappings, which are nonexpansive, in the setting of infinite dimensional Hilbert spaces. Convergence theorem of the sequences generated in our proposed implicit algorithm is obtained under some weak assumptions.

Keywords: split variational inclusion; fixed-point problem; hierarchical optimization problem; nonexpansive mapping; implicit general iterative method

1. Introduction

Let H_1 be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Suppose that C is a nonempty convex and set in H_1, and let P_C be the metric (nearest point) projection from space H_1 onto set C. We use $T : C \to H_1$ to denote a mapping on C. Denote by $\mathrm{Fix}(T)$ the set of fixed points of T, i.e., $\mathrm{Fix}(T) = \{x \in C : x = Tx\}$. We use the notations $\rightharpoonup$ and $\to$ to indicate the weak convergence and the strong convergence, respectively.

Assume that $B : C \to H_1$ is a nonlinear mapping. The classical monotone variational inequality (VI) is to find $x^* \in C$ such that

$$0 \leq \langle Bx^*, x - x^*\rangle, \quad \forall x \in C. \tag{1}$$

We denote by $\mathrm{VI}(C, B)$ the solution set of VI (1). VI (1), which acts as a very powerful and effective research tool, has been applied to study lots of theory problems arising in nonlinear equations, computational mechanics, optimization contact problems in control problems, elasticity, operations research, modern management science, bi-function equilibrium problems in transportation and economics, obstacle, unilateral, moving, etc.; see [1–12] and the references therein.

An operator D is said to be a strongly positive operator on H_1, if there is a constant $\bar{\xi} > 0$ such that

$$\bar{\xi}\|x\|^2 \leq \langle Dx, x\rangle, \quad \forall x \in H_1.$$

Solution methods for Lipchitz mappings, in particular, nonexpansive mappings, have widely been applied to investigate minimization problems of various convex functions. A mapping $M : H_1 \to 2^{H_1}$ is said to be set-valued monotone if for all $x, y \in H_1$, $x' \in Mx$ and $y' \in My$ imply $\langle x - y, x - x'\rangle \geq 0$. Recall that $M : H_1 \to 2^{H_1}$ is a maximal operator if the graph $\mathrm{Gph}(M)$ is not properly contained in the graph of any other monotone operator. As we all know that M is maximal if and only if for $(x, f) \in H_1 \times H_1$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, y') \in \mathrm{Gph}(M)$, we have $x' \in Mx$.

We now assume that set-valued operator $M : H_1 \to 2^{H_1}$ is maximal. We can define a single-valued mapping $J_\lambda^M : H_1 \to H_1$ by

$$J_\lambda^M(x) := (\lambda M + I)^{-1}(x), \quad \forall x \in H_1,$$

is called the resolvent operator associated with mapping M. It deserves mentioning that it is single-valued and Liptchitz.

Let H_2 be another Hilbert space with usual norm ($\|\cdot\|$) and inner product ($\langle \cdot, \cdot \rangle$). Let A be a bounded linear operator from H_1 to A_2. We consider in this paper the following split variational inclusion problem (SVIP): find $x^* \in H_1$ such that

$$0 \in B_1(x^*), \tag{2}$$

and

$$Ax^* = y^* \in H_2 \quad \text{solves} \quad 0 \in B_2(y^*), \tag{3}$$

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are set-valued and maximal monotone. SOLVIP(B_1) stands for the solution set of (2) and SOLVIP(B_2) stands for the solution set of (3), respectively. The solution set of SVIP (2)–(3) will be used and denoted by Γ. From [13], we know that SVIP (2)–(3) is equivalent to approximating $x^* \in H_1$ with $x^* = J_\lambda^{B_1}(x^*)$ such that

$$Ax^* = y^* \in H_2 \quad \text{and} \quad y^* = J_\lambda^{B_2}(y^*),$$

holds for any given $\lambda > 0$. It is remarkable that if $\Gamma \neq \emptyset$, then

$$\langle x - J_\lambda^{B_1}x, J_\lambda^{B_1}x - y \rangle \geq 0, \quad \forall x \in H_1,\ y \in \mathrm{SOLVIP}(B_1),$$

and

$$\langle v - J_\lambda^{B_2}v, J_\lambda^{B_2}v - w \rangle \geq 0, \quad \forall v \in H_2,\ w \in \mathrm{SOLVIP}(B_2).$$

Let $\{S_i\}_{i=1}^\infty$ be a countable family mappings on H_1. We assume that $\{\zeta_i\}_{i=1}^\infty$ is a real sequence in $[0, 1]$. For any $n \geq 1$, we give a W_n mapping by:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = (1-\zeta_n)I + \zeta_n S_n U_{n,n+1}, \\ U_{n,n-1} = (1-\zeta_{n-1})I + \zeta_{n-1} S_{n-1} U_{n,n}, \\ \cdots \\ U_{n,k} = (1-\zeta_k)I + \zeta_k S_k U_{n,k+1}, \\ U_{n,k-1} = (1-\zeta_{k-1})I + \zeta_{k-1} S_{k-1} U_{n,k}, \\ \cdots \\ U_{n,2} = (1-\zeta_2)I + \zeta_2 S_2 U_{n,3}, \\ W_n = U_{n,1} = (1-\zeta_1)I + \zeta_1 S_1 U_{n,2}. \end{cases} \tag{4}$$

If each S_i is nonexpansive, then W_n is Lipschitz. Indeed, it is also nonexpansive and called a W-mapping defined by $S_n, S_{n-1}, ..., S_1$ and $\zeta_n, \zeta_{n-1}, ..., \zeta_1$. From [14], we know that W_n is a nonexpansive mapping with the relation $\mathrm{Fix}(W_n) = \bigcap_{i=1}^n \mathrm{Fix}(S_i)$, for each $n \geq 1$; for each $x \in H_1$ and for each positive integer k, the $\lim_{n\to\infty} U_{n,k}x$ exists; W is defined by

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in H_1$$

has the nonexpansivity and it satisfies $\mathrm{Fix}(W) = \bigcap_{i=1}^\infty \mathrm{Fix}(S_i)$ (We call a W-mapping generated by $S_1, S_2, ...$ and $\zeta_1, \zeta_2, ...$). Recently, common fixed-point problems, which finds applications in signal process and medical image restoration, have been studied based on mean-valued or projection methods; see [14–19] and references cited therein.

In this present work, we investigate an implicit general iterative method for computing a solution of the SVIP with a hierarchical optimization problem constraint for a countable family of mappings, which will be assumed to have the nonexpansivity, in the framework of real Hilbert spaces. Norm convergence theorems of the sequences generated by our implicit general algorithm are established under some suitable assumptions. Our results extend, unify, develop and improve the corresponding ones in the recent literature.

2. Preliminaries

Now we list some basic notations and facts. H_1 will be assumed to be a real Hilbert space and C will be assume to be a closed nonempty convex subset in H_1. A mapping $F : C \to H_1$ is called a κ-Lipschitzian mapping if there is a number $\kappa > 0$ with $\kappa\|x-y\| \geq \|F(x) - F(y)\|$, $\forall x, y \in C$. In particular, if $\kappa = 1$, then F is said to be a nonexpansive operator. If $\kappa < 1$, then F is said to be a contractive operator. A mapping $F : C \to H_1$ is said to be η-strongly monotone if there exists a number $\eta > 0$ such that $\eta\|x-y\|^2 \leq \langle x-y, Fx-Fy\rangle$, $\forall x, y \in C$. In all Hilbert spaces, we known

$$\lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 = \|\lambda x + (1-\lambda)y\|^2,$$

for all $x, y \in H_1$ and $\lambda \in [0, 1]$.

Fixing $x \in H_1$, we see that there exists a unique nearest point in closed convex set C. We denote it by $P_C x$. $\|x-y\| \geq \|x - P_C x\|$, $\forall y \in C$. The mapping P_C is called the metric or nearest point projection of H_1 onto C. We know that P_C is an nonexpansive operator from space H_1 onto set C. In addition, we know that

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C y - P_C x\|^2, \quad \forall x, y \in H_1. \tag{5}$$

$P_C x$ also enjoys

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{6}$$

for all $x \in H_1$ and $y \in C$. It is not too hard to see that (6) is equivalent to the following relation

$$\|x-y\|^2 - \|x - P_C x\|^2 \geq \|y - P_C x\|^2, \quad \forall x \in H_1, y \in C. \tag{7}$$

It is also not hard to find that every nonexpansive operator $S : H_1 \to H_1$ satisfies the following relation

$$\langle (I-S)x - (I-S)y, Sy - Sx\rangle \leq \frac{1}{2}\|(I-S)x - (I-S)y\|^2, \quad \forall (x, y) \in H_1 \times H_1. \tag{8}$$

In particular, one has

$$\langle (I-S)x, y - Sx\rangle \leq \frac{1}{2}\|(I-S)x\|^2, \quad \forall (x, y) \in H_1 \times \mathrm{Fix}(S). \tag{9}$$

Let $T : H_1 \to H_1$ be a self mapping. It is said to be an averaged operator if it is a combination of the identity operator I and a nonexpansivity operator, that is, $T \equiv (1-\alpha)I + \alpha S$, where $\alpha \in (0, 1)$ and $S : H_1 \to H_1$ is an nonexpansive operator. We mention that the class of averaged mappings are of course nonexpansive. Also, mappings, which are firmly nonexpansive are averaged. Indeed, projections on convex nonempty closed sets and resolvent operators of set-valued monotone operators. Some important properties and relations of averaged mappings are gathered in the following lemma; see e.g., [20–25] and references cited therein.

Lemma 1. *For any given $\lambda > 0$, let the mapping $G : H_1 \to H_1$ be defined as $G := J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)$ where $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Then G is a nonexpansive mapping. If $\Gamma \neq \emptyset$, then $\Gamma = \mathrm{Fix}(G)$.*

Proof. Since $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are mappings enjoys the firm nonexpansivity, they, of course, are averaged. For $L\gamma \in (0,1)$, the mapping $(I + \gamma A^*(J_\lambda^{B_2} - I)A)$ is averaged. So $G := J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)$ is a averaged operator and hence a nonexpansive operator.

Next, let us show that if $\Gamma \neq \emptyset$ then $\Gamma = \mathrm{Fix}(G)$. Indeed, it is clear that $\Gamma \subseteq \mathrm{Fix}(G)$. Conversely, we take $p \in \mathrm{Fix}(G)$ and $q \in \Gamma$ arbitrarily. Then $J_\lambda^{B_1}(p + \gamma A^*(J_\lambda^{B_2} - I)Ap) = p$. Hence,

$$\langle (p + \gamma A^*(J_\lambda^{B_2} - I)Ap) - p, p - u\rangle \geq 0, \quad \forall u \in \mathrm{SOLVIP}(B_1),$$

which immediately yields

$$\langle J_\lambda^{B_2}Ap - Ap, Ap - Au\rangle \geq 0, \quad \forall u \in \mathrm{SOLVIP}(B_1).$$

One has

$$\langle J_\lambda^{B_2}Ap - Ap, v - J_\lambda^{B_2}Ap\rangle \geq 0, \quad \forall v \in \mathrm{SOLVIP}(B_2).$$

Using the last two inequalities, we obtain

$$\langle J_\lambda^{B_2}Ap - Ap, v - J_\lambda^{B_2}Ap + Ap - Au\rangle \geq 0, \quad \forall u \in \mathrm{SOLVIP}(B_1),\ v \in \mathrm{SOLVIP}(B_2),$$

which immediately leads to

$$\langle J_\lambda^{B_2}Ap - Ap, v - Au\rangle \geq \|J_\lambda^{B_2}Ap - Ap\|^2, \quad \forall u \in \mathrm{SOLVIP}(B_1),\ v \in \mathrm{SOLVIP}(B_2). \tag{10}$$

Taking into account $q \in \Gamma$, one knows that $q \in \mathrm{SOLVIP}(B_1)$ and $Aq \in \mathrm{SOLVIP}(B_2)$. So it follows from (10) that $J_\lambda^{B_2}Ap = Ap$, i.e., $Ap \in \mathrm{SOLVIP}(B_2)$. Also, from $p \in \mathrm{Fix}(G)$ we get

$$p = J_\lambda^{B_1}(p + \gamma A^*(J_\lambda^{B_2} - I)Ap) = J_\lambda^{B_1}p.$$

Hence, $p \in \mathrm{SOLVIP}(B_1)$. Consequently, $p \in \Gamma$. This completes the proof. □

Lemma 2. *[26], Let $\{S_i\}_{i=1}^\infty$ be a countable family on a real Hilbert space H_1 with the restriction $\bigcap_{i=1}^\infty \mathrm{Fix}(S_i) \neq \emptyset$. $\{\zeta_i\}_{i=1}^\infty$ will be assumed to be a sequence in $(0,l]$ for some $l \in (0,1]$. If C is any bounded set in H_1 and each S_i is the self nonexpansivity, then $\lim_{n\to\infty} \sup_{x\in C} Wx - W_nx = 0$.*

Through the rest of this paper, $\{\zeta_i\}_{i=1}^\infty$ will be assumed to be in $(0,l]$ for some $l \in (0,1)$.

Lemma 3. *[27], Assume that both $\{x_n\}$ and $\{z_n\}$ are bounded real sequences in infinite dimensional space either Banach or Hilbert. We support that $\{\beta_n\}$ is a sequence with the restriction that it is bounded away from $[0,1]$, that is, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. We assume $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n\ \forall n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Hence, $\lim_{n\to\infty}\|x_n - z_n\| = 0$.*

Lemma 4. *[28], Let C be a closed nonempty convex set in a real Hilbert space H_1, and let $B : C \to H_1$ be a monotone and hemicontinuous mapping. We the following:*

(i) $\mathrm{VI}(C,B) = \{x^* \in C : \langle By, y - x^*\rangle \geq 0,\ \forall y \in C\}$;

(ii) $\mathrm{VI}(C,B) = \mathrm{Fix}(P_C(I - \lambda B))$ *for all* $\lambda > 0$;

(iii) $\mathrm{VI}(C,B)$ *is singleton, if B is strongly monotone and Lipschitz continuous.*

Lemma 5. *[29], All Hilbert spaces satisfies the well known Opial condition: the inequality $\liminf_{n\to\infty}\|x_n - y\| \geq \liminf_{n\to\infty}\|x_n - x\|$ holds for every $y \neq x$ and for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$.*

Lemma 6. *[30], Assume that S is a nonexpansive self-mapping on a closed convex nonempty set C in H_1. If S is fixed-point free, then $I - S$ is demi-closed at zero, i.e., if $\{x_n\}$ is a sequence in C weakly converging to some x in the set and the sequence $\{(S-I)x_n\}$ converges strongly to zero, then $(S-I)x = 0$, where I stands for the identity operator.*

Lemma 7. *[31], Assume that $\{a_n\}$ be a real iterative sequence with the conditions $a_{n+1} - (1-\lambda_n)a_n \le \lambda_n\gamma_n$, $\forall n \ge 0$, where $\{\lambda_n\}$ and $\{\gamma_n\}$ are real sequences with the restrictionis $\{\lambda_n\} \subset [0,1]$ and $\sum_{n=0}^{\infty}\lambda_n = \infty$, $\limsup_{n\to\infty}\gamma_n \le 0$. Then $\lim_{n\to\infty} a_n = 0$.*

3. Main Results

Theorem 1. *Let $A : H_1 \to H_2$, where H_1 and H_2 are two different Hilbert spaces, be a linearly bounded operator. Suppose that $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are maximal monotone mappings. Let $f : H_1 \to H_1$ be a contraction mapping with contractive coefficient $\alpha \in (0,1)$ and let the linearly bounded operator $D : H_1 \to H_1$ be strongly positive with coefficient $\bar{\xi} > 0$ and $0 < \xi < \frac{\bar{\xi}}{\alpha}$. Let the mapping $G : H_1 \to H_1$ be defined as $G := J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)$, where $\lambda > 0$, $\gamma \in (0, \frac{1}{L})$, L be the spectral radius of A^*A and A^* is the adjoint operator of A. Assume that $\Omega := (\bigcap_{i=1}^{\infty}\mathrm{Fix}(S_i)) \cap \Gamma \neq \emptyset$. For an arbitrary $x_1 \in H_1$, we define $\{x_n\}$ and $\{y_n\}$ by*

$$\begin{cases} y_n = G((1-\gamma_n)W_n y_n + \gamma_n x_n), \\ x_{n+1} = \alpha_n \xi f(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n D]y_n, \quad \forall n \ge 1, \end{cases} \tag{11}$$

where $\{W_n\}$ is defined in (4), and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real number sequences in $(0,1)$. Suppose the parameter sequences satisfy the following three restrictions:

(C1) $\{\beta_n\}_{n=1}^{\infty} \subset [a,b]$ for some $a, b \in (0,1)$;

(C2) $\alpha_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C3) $1 > \limsup_{n\to\infty}\gamma_n \ge \liminf_{n\to\infty}\gamma_n > 0$ and $\lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0$.

Then $\{x_n\}$ converges to a point $z \in \Omega$ in norm and z is a solution to

$$\langle (D - \xi f)z, z - p\rangle \le 0, \quad \forall p \in \Omega,$$

that is, $P_\Omega(z - Dz + \xi f(z)) = z$.

Proof. First of all, taking into account that $\alpha_n \to 0$ as $n \to \infty$ and $1 > \limsup_{n\to\infty}\gamma_n \ge \liminf_{n\to\infty}\gamma_n > 0$, we can suppose $\{\alpha_n(\bar{\xi} - \xi\alpha)\} \subset (0,1)$ and $\{\gamma_n\} \subset [c,d] \subset (0,1)$ for some $c, d \in (0,1)$. Please note that the mapping $G : H_1 \to H_1$ is defined as $G := J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)$, where $\lambda > 0$, $L\gamma \in (0,1)$, L is the radius of the operator A^*A. By virtue of Lemma 1, we get that G is is nonexpansivity. It is easy to see that there exists an element $y_n \in H_1$, which is unique, such that

$$y_n = G(\gamma_n x_n + (1-\gamma_n)W_n y_n). \tag{12}$$

Define a mapping F_n by

$$F_n x = G(\gamma_n x_n + (1-\gamma_n)W_n x), \quad \forall x \in H_1.$$

Since each $W_n : H_1 \to H_1$ is a nonexpansive mapping, we deduce that, all $x, y \in H_1$,

$$\begin{aligned} \|F_n x - F_n y\| &= \|G(\gamma_n x_n + (1-\gamma_n)W_n x) - G(\gamma_n x_n + (1-\gamma_n)W_n y)\| \\ &\le (1-\gamma_n)\|W_n y - W_n x\| \le (1-\gamma_n)\|x - y\|. \end{aligned}$$

Also, from $\{\gamma_n\} \subset [c, d] \subset (0, 1)$, we get $1 > 1 - \gamma_n > 0$ for all $n \geq 1$. Thus, F_n is a contraction operator. This shows that there exists an element $y_n \in C$, satisfying (12). Indeed, y_n is also unique. So, it can be readily seen that the general implicit iterative scheme (11) can be rewritten as

$$\begin{cases} u_n = \gamma_n x_n + (1 - \gamma_n) W_n y_n, \\ y_n = J_\lambda^{B_1}(\gamma A^*(J_\lambda^{B_2} - I)Au_n + u_n), \\ x_{n+1} = \alpha_n \xi f(x_n) + [(1 - \beta_n)I - \alpha_n D]y_n + \beta_n x_n, \quad \forall n \geq 1. \end{cases} \tag{13}$$

Next, we divide the rest of our proofs into some steps to prove this theorem.

Step 1. We prove that $\{x_n\}, \{y_n\}, \{u_n\}, \{f(x_n)\}$ and $\{W_n y_n\}$ are bounded sequence in H_1. By arbitrarily taking an element $p \in \Omega = \bigcap_{n=1}^\infty \mathrm{Fix}(S_n) \cap \Gamma$, we get $p = J_\lambda^{B_1} p$, $Ap = J_\lambda^{B_2}(Ap)$ and $W_n p = p \; \forall n \geq 1$. Since each $W_n : H_1 \to H_1$ is a nonexpansive operator, it follows that

$$\begin{aligned} \|u_n - p\| &\leq \gamma_n \|x_n - p\| + (1 - \gamma_n)\|p - W_n y_n\| \\ &\leq \gamma_n \|x_n - p\| + (1 - \gamma_n)\|p - y_n\|. \end{aligned} \tag{14}$$

Note that

$$\begin{aligned} &\|y_n - p\|^2 \\ &\leq \|u_n + \gamma A^*(J_\lambda^{B_2} - I)Au_n - p\|^2 \\ &\leq \|u_n - p\|^2 + \gamma^2 \langle (J_\lambda^{B_2} - I)Au_n, AA^*(J_\lambda^{B_2} - I)Au_n \rangle + 2\gamma \langle u_n - p, A^*(J_\lambda^{B_2} - I)Au_n \rangle. \end{aligned} \tag{15}$$

Please note that

$$\begin{aligned} L\gamma^2 \|(J_\lambda^{B_2} - I)Au_n\|^2 &= L\gamma^2 \langle (J_\lambda^{B_2} - I)Au_n, (J_\lambda^{B_2} - I)Au_n \rangle \\ &\geq \gamma^2 \langle (J_\lambda^{B_2} - I)Au_n, AA^*(J_\lambda^{B_2} - I)Au_n \rangle. \end{aligned} \tag{16}$$

By considering item $2\gamma \langle u_n - p, A^*(J_\lambda^{B_2} - I)Au_n \rangle$ and by using (9), we have

$$\begin{aligned} &2\gamma \langle u_n - p, A^*(J_\lambda^{B_2} - I)Au_n \rangle = 2\gamma \langle A(u_n - p), (J_\lambda^{B_2} - I)Au_n \rangle \\ &= 2\gamma \{ \langle Ap - J_\lambda^{B_2} Au_n, Au_n - J_\lambda^{B_2} Au_n \rangle - \|(J_\lambda^{B_2} - I)Au_n\|^2 \} \\ &\leq 2\gamma \{ \tfrac{1}{2} \|(J_\lambda^{B_2} - I)Au_n\|^2 - \|(J_\lambda^{B_2} - I)Au_n\|^2 \} \\ &= -\gamma \|(J_\lambda^{B_2} - I)Au_n\|^2. \end{aligned} \tag{17}$$

Using inequalities (15), (16) and (17), we obtain

$$\begin{aligned} \|y_n - p\|^2 &\leq \|u_n - p\|^2 + L\gamma^2 \|(J_\lambda^{B_2} - I)Au_n\|^2 - \gamma \|(J_\lambda^{B_2} - I)Au_n\|^2 \\ &= \|u_n - p\|^2 + \gamma(L\gamma - 1)\|(J_\lambda^{B_2} - I)Au_n\|^2. \end{aligned} \tag{18}$$

From $(L\gamma) \in (0, 1)$, we get

$$\|y_n - p\| \leq \|u_n - p\|, \quad \forall n \geq 1. \tag{19}$$

Substituting (19) for (14), we have

$$\|u_n - p\| \leq (1 - \gamma_n)\|u_n - p\| + \gamma_n \|x_n - p\|,$$

which combining (19) yields that

$$\|y_n - p\| \leq \|u_n - p\| \leq \|x_n - p\|, \quad \forall n \geq 1. \tag{20}$$

Thanks to the two restrictions (C1) and (C2), we can suppose that $\alpha_n \leq (1 - \beta_n)\|D\|^{-1}$, $\forall n \geq 1$. Since D is linearly strongly positive bounded, we can easily get that

$$1 - \beta_n - \alpha_n \bar{\xi} \geq \|(1 - \beta_n)I - \alpha_n D\|. \tag{21}$$

In view of (13), (20) and (21), one has that

$$\begin{aligned}\|x_{n+1}-p\| &\leq \alpha_n\xi\|f(x_n)-f(p)\|+\alpha_n\|\xi f(p)-Dp\|+\beta_n\|x_n-p\| \\ &\quad +\|[(1-\beta_n)I-\alpha_n D](y_n-p)\| \\ &\leq \alpha_n\xi\alpha\|x_n-p\|+\alpha_n\|\xi f(p)-Dp\|+\beta_n\|x_n-p\|+(1-\beta_n-\alpha_n\bar{\xi})\|x_n-p\| \\ &\leq \max\{\|p-x_1\|,\tfrac{\|\xi f(p)-Dp\|}{\bar{\xi}-\xi\alpha}\}, \quad \forall n\geq 1.\end{aligned}$$

It immediately yields that $\{x_n\}$ is a bounded sequence in H_1. Indeed, $\{y_n\},\{u_n\},\{f(x_n)\},\{W_n y_n\}$ and $\{Dy_n)\}$ (due to (20) and the Lipschitz continuity of $W_n,\ D$ and f) are bounded sequences. From this, we fix a bounded subset $C\subset H_1$ with the restriction

$$u_n, x_n, y_n \in C, \quad \forall n\geq 1. \tag{22}$$

Step 2. We aim that $\|x_{n+1}-x_n\|\to 0$ and $\|y_{n+1}-y_n\|\to 0$ as $n\to\infty$. Indeed, we set

$$x_{n+1}=\beta_n x_n+(1-\beta_n)v_n, \quad \forall n\geq 1. \tag{23}$$

This shows that

$$\begin{aligned} v_n &= \tfrac{1}{1-\beta_n}\{\alpha_n\xi f(x_n)+\beta_n x_n+y_n-\beta_n y_n-\alpha_n Dy_n\}-\tfrac{\beta_n}{1-\beta_n}x_n \\ &= \tfrac{\alpha_n}{1-\beta_n}(\xi f(x_n)-Dy_n)+y_n. \end{aligned} \tag{24}$$

Hence,

$$\|v_{n+1}-v_n\|\leq \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\|\xi f(x_{n+1})-Dy_{n+1}\|+\tfrac{\alpha_n}{1-\beta_n}\|\xi f(x_n)-Dy_n\|+\|y_{n+1}-y_n\|. \tag{25}$$

By using Lemma 1, we know that $G:=J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)$ is Lipchitz. Indeed, it is nonexpansive. Hence, we obtain from (13) that

$$\|y_{n+1}-y_n\|=\|Gu_{n+1}-Gu_n\|\leq\|u_{n+1}-u_n\|. \tag{26}$$

However, we have that

$$\begin{aligned} &\sup_{x\in C}[\|W_{n+1}x-Wx\|+\|Wx-W_nx\|]+\|u_{n+1}-u_n\| \\ &\geq \|W_{n+1}y_{n+1}-W_ny_{n+1}\|+\|W_ny_{n+1}-W_ny_n\| \\ &\geq \|W_{n+1}y_{n+1}-W_ny_n\|, \end{aligned} \tag{27}$$

where C stands for the bounded subset in H_1 defined by (22). Simple calculations show that

$$\begin{aligned}\|u_{n+1}-u_n\| &\leq (1-\gamma_{n+1})\|W_{n+1}y_{n+1}-W_ny_n\|+\gamma_{n+1}\|x_{n+1}-x_n\| \\ &\quad +|\gamma_{n+1}-\gamma_n|\|x_n-W_ny_n\| \\ &\leq \gamma_{n+1}\|x_{n+1}-x_n\|+(1-\gamma_{n+1})\{\sup_{x\in C}[\|W_{n+1}x-Wx\|+\|Wx-W_nx\|] \\ &\quad +\|u_{n+1}-u_n\|\}+|\gamma_{n+1}-\gamma_n|\|x_n-W_ny_n\|. \end{aligned} \tag{28}$$

So it yields from $\{\gamma_n\}\subset[c,d]$ that

$$\|u_{n+1}-u_n\|\leq\|x_{n+1}-x_n\|+\frac{1}{c}\sup_{x\in C}[\|W_{n+1}x-Wx\|+\|Wx-W_nx\|]+|\gamma_{n+1}-\gamma_n|\frac{\|x_n-W_ny_n\|}{c}. \tag{29}$$

Thus, from (25), (26) and (29) we deduce that

$$\begin{aligned}
&\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\xi f(x_{n+1}) - Dy_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\xi f(x_n) - Dy_n\| \\
&\quad + \frac{1}{c}\sup_{x\in C}[\|W_{n+1}x - Wx\| + \|Wx - W_n x\|] + |\gamma_{n+1} - \gamma_n|\frac{\|x_n - W_n y_n\|}{c} \\
&\geq \|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|.
\end{aligned}$$

Thanks to the three assumptions (C1), (C2), (C3), and Lemma 2,

$$\limsup_{n\to\infty}(\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

From Lemma 3, we thus obtain that

$$\lim_{n\to\infty}\|v_n - x_n\| = 0. \tag{30}$$

This in turn implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{31}$$

This together with (26) and (29), implies that

$$\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|y_{n+1} - y_n\| = 0. \tag{32}$$

Step 3. We aim to prove $\|x_n - u_n\| \to 0,\ \|x_n - y_n\| \to 0,\ \|y_n - W_n y_n\| \to 0$ and $\|x_n - Gx_n\| \to 0$ as $n \to \infty$. Indeed, we set $f_n = \xi f(x_n) - Dy_n$ for all $n \geq 1$. For any $p \in \Omega$, we observe that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|\beta_n(x_n - p) + (1-\beta_n)(y_n - p)\|^2 + 2\langle \alpha_n f_n, x_{n+1} - p\rangle \\
&\leq (1-\beta_n)\|y_n - p\|^2 + \beta_n\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - y_n\|^2 \\
&\quad + 2\alpha_n\|f_n\|\|x_{n+1} - p\| \\
&\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 + 2\alpha_n M^2,
\end{aligned} \tag{33}$$

where $M = \max\{\sup_{n\geq 1}\|f_n\|, \sup_{n\geq 1}\|x_n - p\|\}$. Substituting (18) for (33), we obtain from (20) that $\|p - x_{n+1}\|^2 \leq \|p - x_n\|^2 - \gamma(1-\beta_n)(1-L\gamma)\|(J_\lambda^{B_2} - I)Au_n\|^2 + 2\alpha_n M^2$. Therefore,

$$\lim_{n\to\infty}\|(J_\lambda^{B_2} - I)Au_n\| = 0. \tag{34}$$

From the assumption that $J_\lambda^{B_1}$ is a firmly nonexpansive mapping, we have

$$\begin{aligned}
2\|y_n - p\|^2 &\leq 2\langle y_n - p, u_n + \gamma A^*(J_\lambda^{B_2} - I)Au_n - p\rangle \\
&= \|y_n - p\|^2 + \|u_n - p\|^2 + 2\gamma\langle u_n - p, A^*(J_\lambda^{B_2} - I)Au_n\rangle \\
&\quad + \gamma^2\|A^*(J_\lambda^{B_2} - I)Au_n\|^2 - \|y_n - u_n - \gamma A^*(J_\lambda^{B_2} - I)Au_n\|^2 \\
&\leq \|y_n - p\|^2 + \|u_n - p\|^2 - \gamma\|(J_\lambda^{B_2} - I)Au_n\|^2 + \gamma^2\|A^*(J_\lambda^{B_2} - I)Au_n\|^2 \\
&\quad - \|y_n - u_n - \gamma A^*(J_\lambda^{B_2} - I)Au_n\|^2 \\
&\leq \|y_n - p\|^2 + \|u_n - p\|^2 - \|y_n - u_n\|^2 + 2\gamma\|A(y_n - u_n)\|\|(J_\lambda^{B_2} - I)Au_n\|.
\end{aligned}$$

Hence, we obtain

$$\|y_n - p\|^2 \leq -\|y_n - u_n\|^2 + \|u_n - p\|^2 + 2\gamma\|A(y_n - u_n)\|\|(J_\lambda^{B_2} - I)Au_n\|. \tag{35}$$

Substituting (35) for (33), one concludes from (20) that

$$\begin{aligned}
(1-\beta_n)\|y_n - u_n\|^2 &\leq \|x_n - p\|\|x_n - x_{n+1}\| + \|x_{n+1} - p\|\|x_n - x_{n+1}\| \\
&\quad + 2\gamma(1-\beta_n)\|A(y_n - u_n)\|\|(J_\lambda^{B_2} - I)Au_n\| + 2\alpha_n M^2.
\end{aligned}$$

(C1), (C2), (31), and (34) send us to

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{36}$$

Also, according to (11) and (19) we have

$$\begin{aligned} \|p - u_n\|^2 &\leq \gamma_n \langle u_n - p, x_n - p\rangle + (1-\gamma_n)\|p - W_n y_n\| \|p - u_n\| \\ &\leq \gamma_n \langle u_n - p, x_n - p\rangle + (1-\gamma_n)\|p - u_n\|^2, \end{aligned}$$

which immediately leads to

$$\|u_n - p\|^2 \leq \frac{1}{2}[\|u_n - p\|^2 - \|x_n - u_n\|^2 + \|x_n - p\|^2].$$

It follows from (19) and (33) that

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &\leq \beta_n \|p - x_n\|^2 + (1-\beta_n)\|p - y_n\|^2 - \beta_n(1-\beta_n)\|x_n - y_n\|^2 + 2\alpha_n \|f_n\| \|p - x_{n+1}\| \\ &\leq \beta_n \|p - x_n\|^2 + (1-\beta_n)\|p - y_n\|^2 - \beta_n(1-\beta_n)\|x_n - y_n\|^2 + 2\alpha_n M^2 \\ &\leq \beta_n \|p - x_n\|^2 + (1-\beta_n)[\|p - x_n\|^2 - \|x_n - u_n\|^2] - \beta_n(1-\beta_n)\|x_n - y_n\|^2 + 2\alpha_n M^2 \\ &= \|x_n - p\|^2 - (1-\beta_n)\|x_n - u_n\|^2 - \beta_n(1-\beta_n)\|x_n - y_n\|^2 + 2\alpha_n M^2. \end{aligned}$$

This implies that

$$\begin{aligned} &(1-\beta_n)\|x_n - u_n\|^2 + \beta_n(1-\beta_n)\|x_n - y_n\|^2 \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n M^2 \\ &\leq \|x_n - p\| \|x_n - x_{n+1}\| + \|x_{n+1} - p\| \|x_n - x_{n+1}\| + 2\alpha_n M^2. \end{aligned}$$

(C1), (C2), and (3.21) send us to

$$\lim_{n\to\infty} \|x_n - u_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{37}$$

Noticing that $\|u_n - x_n\| = (1-\gamma_n)\|W_n y_n - x_n\| \geq (1-d)\|W_n y_n - x_n\|$,

$$\|y_n - W_n y_n\| \leq \|y_n - x_n\| + \|x_n - W_n y_n\|,$$

and

$$\|x_n - Gx_n\| \leq \|x_n - u_n\| + \|u_n - y_n\| + \|Gu_n - Gx_n\| \leq 2\|x_n - u_n\| + \|u_n - y_n\|,$$

we deduce from (36) and (37) that

$$\lim_{n\to\infty} \|x_n - W_n y_n\| = 0, \quad \lim_{n\to\infty} \|y_n - W_n y_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_n - Gx_n\| = 0. \tag{38}$$

Step 4. We aims to $\limsup_{n\to\infty}\langle(\xi f - D)z, x_n - z\rangle \leq 0$, where z denotes the fixed-point of mapping $P_\Omega(I - D + \xi f)$. Indeed, we first show that $\mathrm{VI}(\Omega, D - \xi f)$ consists of one point. As a matter of fact, we note that linear bounded operator D is strongly positive with its coefficient $\bar{\xi} > 0$ and $0 < \xi\alpha < \bar{\xi}$. Then for any $x, y \in H_1$, we have

$$\langle(D - \xi f)x - (D - \xi f)y, x - y\rangle \geq \bar{\xi}\|x - y\|^2 - \xi\alpha\|x - y\|^2 = (\bar{\xi} - \xi\alpha)\|x - y\|^2.$$

Hence we knows that monotone operator $D - \xi f$ is strongly and the coefficient satisfies $\bar{\xi} - \xi\alpha > 0$. It is also clear that $D - \xi f$ is Lipschitzian. Therefore, by Lemma 4 (iii) we deduce that $\mathrm{VI}(\Omega, D - \xi f)$ is a single-point set. Say $z \in H_1$, that is, $\mathrm{VI}(\Omega, D - \xi f) = \{z\}$. Also, by Lemma 4 (ii) we have

$z = P_\Omega(z - Dz + \xi f(z))$. Since $\{x_n\}$ is a bounded sequence in H_1, without loss of generality, we may choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\xi f - D)z, x_n - z\rangle = \lim_{i\to\infty}\langle(\xi f - D)z, x_{n_i} - z\rangle. \tag{39}$$

We have proved that sequence $\{x_{n_i}\}$ is bounded, it is not too hard to see its a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converges weakly to w. Let suppose that $x_{n_i} \rightharpoonup w$. From (37), we obtain that $y_{n_i} \rightharpoonup w$.

Next, let us pay our focus to $w \in \bigcap_{i=1}^{\infty}\mathrm{Fix}(S_i) = \mathrm{Fix}(W)$. Supposing on the contrary that, $w \notin \mathrm{Fix}(W)$, i.e., $Ww \neq w$, we see from Lemma 5 that

$$\begin{aligned}\liminf_{i\to\infty}\|y_{n_i} - w\| &< \liminf_{i\to\infty}\|y_{n_i} - Ww\| \\ &\le \liminf_{i\to\infty}\{\|Wy_{n_i} - Ww\| + \|Wy_{n_i} - y_{n_i}\|\} \\ &\le \liminf_{i\to\infty}\{\|Wy_{n_i} - y_{n_i}\| + \|y_{n_i} - w\|\}.\end{aligned} \tag{40}$$

On the other hand, we have

$$\|Wy_n - y_n\| \le \|Wy_n - W_ny_n\| + \|W_ny_n - y_n\| \le \sup_{x\in C}\|Wx - W_nx\| + \|W_ny_n - y_n\|.$$

By using Lemma 3 and (38), we obtain that $\lim_{i\to\infty}\|Wy_n - y_n\| = 0$, which together with (40), yields $\liminf_{i\to\infty}\|y_{n_i} - w\| > \liminf_{i\to\infty}\|y_{n_i} - w\|$. This reaches a contraction, and hence we have $w \in \mathrm{Fix}(W) = \bigcap_{i=1}^{\infty}\mathrm{Fix}(S_i)$. Please $G : H_1 \to H_1$ is a nonexpansive mapping. Since $x_{n_i} \rightharpoonup w$ and $\lim_{n\to\infty}\|x_n - Gx_n\| = 0$ (due to (38)), by Lemma 6, we get that $w \in \mathrm{Fix}(G)$. From Lemma 1, we get that $w \in \Gamma$. Therefore, $w \in \Omega$. Since z is a fixed point of mapping $P_\Omega(I - D + \xi f)$ and $w \in \Omega$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle(\xi f - D)z, x_n - z\rangle &= \lim_{i\to\infty}\langle(\xi f - D)z, x_{n_i} - z\rangle \\ &= \langle(\xi f - D)z, w - z\rangle \\ &= \langle(z - Dz + \xi f(z)) - z, w - z\rangle \le 0.\end{aligned} \tag{41}$$

Step 5. We aim to $x_n \to z$ and $y_n \to z$ as $n \to \infty$. Indeed, by (3.10) and (3.11) we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \alpha_n\langle(\xi f - D)z, x_{n+1} - z\rangle + \beta_n\langle x_n - z, x_{n+1} - z\rangle \\ &\quad + \langle[(1-\beta_n)I - \alpha_nD]y_n - [(1-\beta_n)I - \alpha_nD]z, x_{n+1} - z\rangle \\ &\le \alpha_n\langle(\xi f - D)z, x_{n+1} - z\rangle + \beta_n\langle x_n - z, x_{n+1} - z\rangle \\ &\quad + \|[(1-\beta_n)I - \alpha_nD](y_n - z)\|\|x_{n+1} - z\| \\ &\le \alpha_n\langle(\xi f - D)z, x_{n+1} - z\rangle + \tfrac{1}{2}\beta_n(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\ &\quad + (1 - \beta_n - \alpha_n\bar{\xi})\|y_n - z\|\|x_{n+1} - z\| \\ &\le \alpha_n\langle(\xi f - D)z, x_{n+1} - z\rangle + \tfrac{1}{2}(1 - \alpha_n\bar{\xi})(\|x_n - z\|^2 + \|x_{n+1} - z\|^2).\end{aligned}$$

This immediately implies that

$$\|x_{n+1} - z\|^2 \le (1 - \alpha_n\bar{\xi})\|x_n - z\|^2 + 2\alpha_n\langle(\xi f - D)z, x_{n+1} - z\rangle.$$

By using Lemma 7, we infer that $|x_n - z\| \to 0$ as $n \to \infty$. This completes the proof.
□

4. Conclusions

In this paper, we studied an implicit general iterative method for approximating a solution of a split variational inclusion problem with a hierarchical optimization problem constraint for a countable family of mappings, which are nonexpansive, in the setting of infinite dimensional Hilbert spaces.

Convergence theorem of the sequences generated in our proposed implicit algorithm is obtained without compact assumptions.

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