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Hankel and Toeplitz Determinants for a Subclass of q-Starlike Functions Associated with a General Conic Domain

Hari M. Srivastava [1,2,*], Qazi Zahoor Ahmad [3], Nasir Khan [4], Nazar Khan [3] and Bilal Khan [3]

1 Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada
2 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
3 Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad 22010, Pakistan; zahoorqazi5@gmail.com (Q.Z.A.); nazarmaths@gmail.com (N.K.); bilalmaths789@gmail.com (B.K.)
4 Department of Mathematics, FATA University, Akhorwal (Darra Adam Khel), FR Kohat 26000, Pakistan; dr.nasirkhan@fu.edu.pk
* Correspondence: harimsri@math.uvic.ca

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Abstract: By using a certain general conic domain as well as the quantum (or q-) calculus, here we define and investigate a new subclass of normalized analytic and starlike functions in the open unit disk $\mathbb{U}$. In particular, we find the Hankel determinant and the Toeplitz matrices for this newly-defined class of analytic q-starlike functions. We also highlight some known consequences of our main results.

Keywords: analytic functions; starlike and q-starlike functions; q-derivative operator; q-hypergeometric functions; conic and generalized conic domains; Hankel determinant; Toeplitz matrices

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1. Introduction and Definitions

Let the class of functions, which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

be denoted by $\mathcal{L}(\mathbb{U})$. Also let $\mathcal{A}$ denote the class of all functions f, which are analytic in the open unit disk $\mathbb{U}$ and normalized by

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

Then, clearly, each $f \in \mathcal{A}$ has a Taylor–Maclaurin series representation as follows:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1}$$

Suppose that $\mathcal{S}$ is the subclass of the analytic function class $\mathcal{A}$, which consists of all functions which are also univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be starlike in $\mathbb{U}$ if it satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}^*$ the class of all such starlike functions in $\mathbb{U}$.

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to the function g and write this subordination as follows:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a Schwarz function w which is analytic in $\mathbb{U}$, with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1,$$

such that

$$f(z) = g\big(w(z)\big).$$

In the case when the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see, for example, [1]; see also [2]):

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Next, for a function $f \in \mathcal{A}$ given by (1) and another function $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (z \in \mathbb{U}),$$

the convolution (or the Hadamard product) of f and g is defined here by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{2}$$

Let $\mathcal{P}$ denote the well-known Carathéodory class of functions p, analytic in the open unit disk $\mathbb{U}$, which are normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{3}$$

such that

$$\Re\big(p(z)\big) > 0 \qquad (z \in \mathbb{U}).$$

Following the works of Kanas et al. (see [3,4]; see also [5]), we introduce the conic domain Ω_k $(k \geqq 0)$ as follows:

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}. \tag{4}$$

In fact, subjected to the conic domain Ω_k $(k \geqq 0)$, Kanas and Wiśniowska (see [3,4]; see also [6]) studied the corresponding class k-$\mathcal{ST}$ of k-starlike functions in $\mathbb{U}$ (see Definition 1 below). For fixed k, Ω_k represents the conic region bounded successively by the imaginary axis $(k = 0)$, by a parabola $(k = 1)$, by the right branch of a hyperbola $(0 < k < 1)$, and by an ellipse $(k > 1)$.

For these conic regions, the following functions play the role of extremal functions.

$$p_k(z) = \begin{cases} \dfrac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots & (k=0) \\ 1 + \dfrac{2}{\pi^2}\left[\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2 & (k=1) \\ 1 + \dfrac{2}{1-k^2}\sinh^2\left[\left(\dfrac{2}{\pi}\arccos k\right)\arctan\left(h\sqrt{z}\right)\right] & (0 \leqq k < 1) \\ 1 + \dfrac{1}{k^2-1}\left[1 + \sin\left(\frac{\pi}{2K(\kappa)}\displaystyle\int_0^{\frac{u(z)}{\sqrt{\kappa}}} \dfrac{dt}{\sqrt{(1-t^2)(1-\kappa^2 t^2)}}\right)\right] & (k>1), \end{cases} \tag{5}$$

where

$$u(z) = \frac{z-\sqrt{\kappa}}{1-\sqrt{\kappa z}} \qquad (z \in \mathbb{U}),$$

and $\kappa \in (0,1)$ is so chosen that

$$k = \cosh\left(\frac{\pi K'(\kappa)}{4K(\kappa)}\right).$$

Here $K(\kappa)$ is Legendre's complete elliptic integral of first kind and

$$K'(\kappa) = K\left(\sqrt{1-\kappa^2}\right),$$

that is, $K'(\kappa)$ is the complementary integral of $K(\kappa)$ (see, for example, ([7], p. 326, Equation 9.4 (209))). Indeed, from (5), we have

$$p_k(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots. \tag{6}$$

The class k-$\mathcal{ST}$ is defined as follows.

Definition 1. *A function $f \in \mathcal{A}$ is said to be in the class k-$\mathcal{ST}$ if and only if*

$$\frac{zf'(z)}{f(z)} \prec p_k(z) \qquad (\forall\, z \in \mathbb{U};\ k \geqq 0).$$

We now recall some basic definitions and concept details of the q-calculus which will be used in this paper (see, for example, ([7], p. 346 et seq.)). Throughout the paper, unless otherwise mentioned, we suppose that $0 < q < 1$ and

$$\mathbb{N} = \{1, 2, 3 \cdots\} = \mathbb{N}_0 \setminus \{0\} \qquad (\mathbb{N}_0 := \{0, 1, 2, \cdots\}).$$

Definition 2. *Let $q \in (0,1)$ and define the q-number $[\lambda]_q$ by*

$$[\lambda]_q = \begin{cases} \dfrac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}) \\ \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases}$$

Definition 3. *Let $q \in (0,1)$ and define the q-factorial $[n]_q!$ by*

$$[n]_q! = \begin{cases} 1 & (n=0) \\ \prod\limits_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases}$$

Definition 4 (see [8,9]). *The q-derivative (or q-difference) operator D_q of a function f defined, in a given subset of $\mathbb{C}$, by*

$$\left(D_q f\right)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)\,z} & (z \neq 0) \\[2ex] f'(0) & (z = 0), \end{cases} \tag{7}$$

provided that $f'(0)$ exists.

From Definition 4, we can observe that

$$\lim_{q \to 1-} \left(D_q f\right)(z) = \lim_{q \to 1-} \frac{f(z) - f(qz)}{(1-q)\,z} = f'(z)$$

for a differentiable function f in a given subset of $\mathbb{C}$. It is also known from (1) and (7) that

$$\left(D_q f\right)(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_n z^{n-1}. \tag{8}$$

Definition 5. *The q-Pochhammer symbol $[\xi]_{n,q}$ $(\xi \in \mathbb{C};\ n \in \mathbb{N}_0)$ is defined as follows:*

$$[\xi]_{n,q} = \frac{\left(q^{\xi}; q\right)_n}{(1-q)^n} = \begin{cases} 1 & (n = 0) \\[2ex] [\xi]_q \, [\xi+1]_q \, [\xi+2]_q \cdots [\xi+n-1]_q & (n \in \mathbb{N}). \end{cases}$$

Moreover, the q-gamma function is defined by the following recurrence relation:

$$\Gamma_q(z+1) = [z]_q \, \Gamma_q(z) \quad \text{and} \quad \Gamma_q(1) = 1.$$

Definition 6 (see [10]). *For $f \in \mathcal{A}$, let the q-Ruscheweyh derivative operator $\mathcal{R}_q^{\lambda}$ be defined, in terms of the Hadamard product (or convolution) given by (2), as follows:*

$$\mathcal{R}_q^{\lambda} f(z) = f(z) * \mathcal{F}_{q,\lambda+1}(z) \qquad (z \in \mathbb{U};\ \lambda > -1),$$

where

$$\mathcal{F}_{q,\lambda+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\lambda+n)}{[n-1]_q! \, \Gamma_q(\lambda+1)} z^n = z + \sum_{n=2}^{\infty} \frac{[\lambda+1]_{q,n-1}}{[n-1]_q!} z^n.$$

We next define a certain q-integral operator by using the same technique as that used by Noor [11].

Definition 7. *For $f \in \mathcal{A}$, let the q-integral operator $\mathcal{F}_{q,\lambda}$ be defined by*

$$\mathcal{F}_{q,\lambda+1}^{-1}(z) * \mathcal{F}_{q,\lambda+1}(z) = z\left(D_q f\right)(z).$$

Then

$$\begin{aligned} \mathcal{I}_q^{\lambda} f(z) &= f(z) * \mathcal{F}_{q,\lambda+1}^{-1}(z) \\ &= z + \sum_{n=2}^{\infty} \psi_{n-1} a_n z^n \qquad (z \in \mathbb{U};\ \lambda > -1), \end{aligned} \tag{9}$$

where

$$\mathcal{F}_{q,\lambda+1}^{-1}(z) = z + \sum_{n=2}^{\infty} \psi_{n-1} z^n$$

and

$$\psi_{n-1} = \frac{[n]_q!\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+n)} = \frac{[n]_q!}{[\lambda+1]_{q,n-1}}.$$

Clearly, we have

$$\mathcal{I}_q^0 f(z) = z(D_q f)(z) \qquad \text{and} \qquad \mathcal{I}_q^1 f(z) = f(z).$$

We note also that, in the limit case when $q \to 1-$, the q-integral operator $\mathcal{F}_{q,\lambda}$ given by Definition 7 would reduce to the integral operator which was studied by Noor [11].

The following identity can be easily verified:

$$zD_q\left(\mathcal{I}_q^{\lambda+1} f(z)\right) = \left(1 + \frac{[\lambda]_q}{q^\lambda}\right)\mathcal{I}_q^\lambda f(z) - \frac{[\lambda]_q}{q^\lambda}\mathcal{I}_q^{\lambda+1} f(z). \tag{10}$$

When $q \to 1-$, this last identity in (10) implies that

$$z\left(\mathcal{I}^{\lambda+1} f(z)\right)' = (1+\lambda)\mathcal{I}^\lambda f(z) - \lambda\mathcal{I}^{\lambda+1} f(z),$$

which is the well-known recurrence relation for the above-mentioned integral operator which was studied by Noor [11].

In geometric function theory, several subclasses belonging to the class of normalized analytic functions class $\mathcal{A}$ have already been investigated in different aspects. The above-defined q-calculus gives valuable tools that have been extensively used in order to investigate several subclasses of $\mathcal{A}$. Ismail et al. [12] were the first who used the q-derivative operator D_q to study the q-calculus analogous of the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$ (see Definition 8 below). However, a firm footing of the q-calculus in the context of geometric function theory was presented mainly and basic (or q-) hypergeometric functions were first used in geometric function theory in a book chapter by Srivastava (see, for details, ([13], p. 347 et seq.); see also [14]).

Definition 8 (see [12])**.** *A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_q^*$ if*

$$f(0) = f'(0) - 1 = 0 \tag{11}$$

and

$$\left|\frac{z}{f(z)}(D_q f)z - \frac{1}{1-q}\right| \leqq \frac{1}{1-q}. \tag{12}$$

It is readily observed that, as $q \to 1-$, the closed disk:

$$\left|w - \frac{1}{1-q}\right| \leqq \frac{1}{1-q}$$

becomes the right-half plane and the class $\mathcal{S}_q^*$ of q-starlike functions reduces to the familiar class $\mathcal{S}^*$ of normalized starlike functions in $\mathbb{U}$ with respect to the origin $(z = 0)$. Equivalently, by using the principle of subordination between analytic functions, we can rewrite the conditions in (11) and (12) as follows (see [15]):

$$\frac{z}{f(z)}(D_q f)(z) \prec \widehat{p}(z) \qquad \left(\widehat{p}(z) = \frac{1+z}{1-qz}\right). \tag{13}$$

The notation $\mathcal{S}_q^*$ was used by Sahoo and Sharma [16].

Now, making use of the principle of subordination between analytic functions and the above-mentioned q-calculus, we present the following definition.

Definition 9. *A function p is said to be in the class k-$\mathcal{P}_q$ if and only if*

$$p(z) \prec \frac{2p_k(z)}{(1+q)+(1-q)\,p_k(z)},$$

where $p_k(z)$ is defined by (5).

Geometrically, the function $p(z) \in k$-$\mathcal{P}_q$ takes on all values from the domain $\Omega_{k,q}$ $(k \geqq 0)$ which is defined as follows:

$$\Omega_{k,q} = \left\{ w : \Re\left(\frac{(1+q)\,w}{(q-1)\,w+2}\right) > k\left|\frac{(1+q)\,w}{(q-1)\,w+2} - 1\right| \right\}.$$

The domain $\Omega_{k,q}$ represents a generalized conic region.

It can be seen that

$$\lim_{q \to 1-} \Omega_{k,q} = \Omega_k,$$

where Ω_k is the conic domain considered by Kanas and Wiśniowska [3]. Below, we give some basic facts about the class k-$\mathcal{P}_q$.

Remark 1. *First of all, we see that*

$$k\text{-}\mathcal{P}_q \subseteq \mathcal{P}\left[\frac{2k}{2k+1+q}\right],$$

where $\mathcal{P}\left[\frac{2k}{2k+1+q}\right]$ is the well-known class of functions with real part greater than $\frac{2k}{2k+1+q}$. Secondly, we have

$$\lim_{q \to 1-} k\text{-}\mathcal{P}_q = \mathcal{P}(p_k),$$

where $\mathcal{P}(p_k)$ is the well-known function class introduced by Kanas and Wiśniowska [3]. Thirdly, we have

$$\lim_{q \to 1-} 0\text{-}\mathcal{P}_q = \mathcal{P},$$

where $\mathcal{P}$ is the well-known class of analytic functions with positive real part.

Definition 10. *A function f is said to be in the class $\mathcal{ST}(k, \lambda, q)$ if and only if*

$$\frac{z\left(D_q \mathcal{I}_q^\lambda f\right)(z)}{f(z)} \in k\text{-}\mathcal{P}_q \qquad (k \geqq 0;\ \lambda \geqq 0),$$

or, equivalently,

$$\Re\left(\frac{(1+q)\,\frac{z\left(D_q \mathcal{I}_q^\lambda f\right)(z)}{f(z)}}{(q-1)\,\frac{z\left(D_q \mathcal{I}_q^\lambda f\right)(z)}{f(z)}+2}\right) > k\left|\frac{(1+q)\,\frac{z\left(D_q \mathcal{I}_q^\lambda f\right)(z)}{f(z)}}{(q-1)\,\frac{z\left(D_q \mathcal{I}_q^\lambda f\right)(z)}{f(z)}+2} - 1\right|.$$

Remark 2. *First of all, it is easily seen that*

$$\mathcal{ST}(0, 1, q) = \mathcal{S}_q^*,$$

where $\mathcal{S}_q^*$ is the function class introduced and studied by Ismail et al. [12]. Secondly, we have

$$\lim_{q \to 1-} \mathcal{ST}(k, 1, q) = k\text{-}\mathcal{ST},$$

where k-$\mathcal{ST}$ is a function class introduced and studied by Kanas and Wiśniowska [4]. Finally, we have

$$\lim_{q \to 1-} \mathcal{ST}(0, 1, q) = \mathcal{S}^*,$$

where $\mathcal{S}^*$ is the well-known class of starlike functions in $\mathbb{U}$ with respect to the origin $(z = 0)$.

Remark 3. *Further studies of the new q-starlike function class $\mathcal{ST}(k, \lambda, q)$, as well as of its more consequences, can next be determined and investigated in future papers.*

Let $n \in \mathbb{N}_0$ and $j \in \mathbb{N}$. The following jth Hankel determinant was considered by Noonan and Thomas [17]:

$$\mathcal{H}_j(n) = \begin{vmatrix} a_n & a_{n+1} & \cdot & \cdot & \cdot & a_{n+j-1} \\ a_{n+1} & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{n+j-1} & \cdot & \cdot & \cdot & \cdot & a_{n+2(j-1)} \end{vmatrix},$$

where $a_1 = 1$. In fact, this determinant has been studied by several authors, and sharp upper bounds on $\mathcal{H}_2(2)$ were obtained by several authors (see [18–20]) for various classes of functions. It is well-known that the Fekete–Szegö functional $\left|a_3 - a_2^2\right|$ can be represented in terms of the Hankel determinant as $\mathcal{H}_2(1)$. This functional has been further generalized as $\left|a_3 - \mu a_2^2\right|$ for some real or complex μ. Fekete and Szegö gave sharp estimates of $\left|a_3 - \mu a_2^2\right|$ for μ real and $f \in \mathcal{S}$, the class of normalized univalent functions in $\mathbb{U}$. It is also known that the functional $\left|a_2 a_4 - a_3^2\right|$ is equivalent to $\mathcal{H}_2(2)$ (see [18]). Babalola [21] studied the Hankel determinant $\mathcal{H}_3(1)$ for some subclasses of normalized analytic functions in $\mathbb{U}$. The symmetric Toeplitz determinant $\mathcal{T}_j(n)$ is defined by

$$\mathcal{T}_j(n) = \begin{vmatrix} a_n & a_{n+1} & \cdot & \cdot & \cdot & a_{n+j-1} \\ a_{n+1} & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{n+j-1} & \cdot & \cdot & \cdot & \cdot & a_n \end{vmatrix},$$

so that

$$\mathcal{T}_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \quad \mathcal{T}_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix}, \quad \mathcal{T}_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix},$$

and so on.

For $f \in \mathcal{S}$, the problem of finding the best possible bounds for $||a_{n+1}| - |a_n||$ has a long history (see, for details, [22]). It is a known fact from [22] that

$$\left| |a_{n+1}| - |a_n| \right| < c$$

for a constant c. However, the problem of finding exact values of the constant c for $\mathcal{S}$ and its various subclasses has proved to be difficult. In a very recent investigation, Thomas and Abdul-Halim [23] succeeded in obtaining some sharp estimates for $\mathcal{T}_j(n)$ for the first few values of n and j involving symmetric Toeplitz determinants whose entries are the coefficients a_n of starlike and close-to-convex functions.

In the present investigation, our focus is on the Hankel determinant and the Toeplitz matrices for the function class $\mathcal{ST}(k,\lambda,q)$ given by Definition 10.

2. A Set of Lemmas

In order to prove our main results in this paper, we need each of the following lemmas.

Lemma 1 (see [20])**.** *If the function $p(z)$ given by (3) is in the Carathéodory class $\mathcal{P}$ of analytic functions with positive real part in $\mathbb{U}$, then*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right)$$

and

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - \left|x^2\right|\right)z$$

for some $x, z \in \mathbb{C}$ with $|x| \leqq 1$ and $|z| \leqq 1$.

Lemma 2 (see [24])**.** *Let the function $p(z)$ given by (3) be in the Carathéodory class $\mathcal{P}$ of analytic functions with positive real part in $\mathbb{U}$. Also let $\mu \in \mathbb{C}$. Then*

$$|c_n - \mu c_k c_{n-k}| \leqq 2\max\left(1, |2\mu - 1|\right) \qquad (1 \leqq k \leqq n-1).$$

Lemma 3 (see [22])**.** *Let the function $p(z)$ given by (3) be in the Carathéodory class $\mathcal{P}$ of analytic functions with positive real part in $\mathbb{U}$. Then*

$$|c_n| \leqq 2 \qquad (n \in \mathbb{N}).$$

This last inequality is sharp.

3. Main Results

Throughout this section, unless otherwise mentioned, we suppose that

$$q \in (0,1), \;\; \lambda > -1 \;\; \text{and} \;\; k \in [0,1].$$

Theorem 1. *If the function $f(z)$ given by (1) belongs to the class $\mathcal{ST}(k,\lambda,q)$, where $k \in [0,1]$, then*

$$|a_2| \leqq \frac{(1+q)\,p_1}{2q\psi_1},$$

$$a_3 \leqq \frac{1}{2q\psi_2}\left(p_1 + \left|p_2 - p_1 + \frac{\left(q^2+1\right)p_1^2}{2q}\right|\right)$$

and

$$\begin{aligned} a_4 \leqq \frac{(1+q)}{4\left(q+q^2+q^3\right)\psi_3}\Bigg(&2p_1 + 4\left|p_2 - p_1 + \frac{\left(2+q^2\right)p_1^2}{4q}\right| \\ &+ \left|2p_3 + 2p_1 - 4p_2 - \frac{\left(2\left(1+q^2\right)-q\right)p_1^2}{q} + \frac{\left(4q^2-3q+2\right)}{q}p_1 p_2\right. \\ &\left.+ \frac{\left(q^2+2q-1\right)}{2q^2}\,p_1^3\right|\Bigg), \end{aligned} \tag{14}$$

where p_j $(j = 1,2,3)$ are positive and are the coefficients of the functions $p_k(z)$ defined by (6). Each of the above results is sharp for the function $g(z)$ given by

$$g(z) = \frac{2p_k(z)}{(1+q) + (1-q)\,p_k(z)}.$$

Proof. Let $f(z) \in \mathcal{ST}(k,\lambda,q)$. Then, we have

$$\frac{z\left(D_q f\right)(z)}{f(z)} = \mathfrak{q}(z) \prec S_k(z), \tag{15}$$

where

$$S_k(z) = \frac{2p_k(z)}{(1+q)+(1-q)\,p_k(z)},$$

and the functions $p_k(z)$ are defined by (6).

We now define the function $p(z)$ with $p(0)=1$ and with a positive real part in $\mathbb{U}$ as follows:

$$p(z) = \frac{1+S_k^{-1}(\mathfrak{q}(z))}{1-S_k^{-1}(\mathfrak{q}(z))} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{16}$$

After some simple computation involving (16), we get

$$\mathfrak{q}(z) = S_k\left(\frac{p(z)+1}{p(z)-1}\right).$$

We thus find that

$$\begin{aligned} S_k&\left(\frac{p(z)+1}{p(z)-1}\right) \\ &= 1 + \left(\frac{q+1}{2}\right)\left[\frac{p_1c_1}{2}z + \left\{\frac{p_1c_2}{2} + \left(\frac{p_2}{4} - \frac{p_1}{4} + \left(\frac{(q-1)\,p_1^2}{8}\right)\right)c_1^2\right\}z^2\right. \\ &\quad + \left\{\frac{p_1c_3}{2} + \left(\frac{p_2}{2} - \frac{p_1}{2} + \left(\frac{(q-1)\,p_1^2}{4}\right)\right)c_1c_2\right. \\ &\quad \left.\left. + \left(\frac{p_1}{8} - \frac{p_2}{4} - \frac{(q-1)\,p_1^2}{8} + \frac{p_3}{8} - \frac{(q-1)\,p_1p_2}{8} + \frac{(q-1)^2\,p_1^3}{32}\right)c_1^3\right\}z^3\right] + \cdots. \end{aligned} \tag{17}$$

Now, upon expanding the left-hand side of (15), we have

$$\begin{aligned} \frac{z\left(D_q\mathcal{I}_q^{\lambda}f\right)(z)}{f(z)} &= 1 + q\psi_1a_2z + \left\{\left(q+q^2\right)\psi_2a_3 - q\psi_1^2a_2^2\right\}z^2 \\ &\quad + \left\{\left(q+q^2+q^3\right)\psi_3a_4 - \left(2q+q^2\right)\psi_1\psi_2a_2a_3 + q\psi_1^3a_2^3\right\}z^3 + \cdots. \end{aligned} \tag{18}$$

Finally, by comparing the corresponding coefficients in (17) and (18) along with Lemma 3, we obtain the result asserted by Theorem 1. $\square$

Theorem 2. *If the function $f(z)$ given by (1) belongs to the class $\mathcal{ST}(k,\lambda,q)$, then*

$$\begin{aligned} \mathcal{T}_3(2) \leqq &\left[\left(\frac{1+q}{2q\psi_1}\right)p_1^2 + \left(\frac{1+q}{4\left(q+q^2+q^3\right)\psi_3}\right)\left[\Omega_1+\Omega_2\right]\right] \\ &\cdot\left[4\left(\frac{(1+q)^2}{16q^2\psi_1^2}\right)p_1^2 + 16\left|\Omega_3\right| + \frac{p_1^2}{4q^2\psi_2^2} + 2\Omega_5p_1^2\left|2 - \frac{\Omega_4}{\Omega_5p_1^2}\right|\right], \end{aligned}$$

where

$$\Omega_1 = 2p_1 + 4\left|p_2 - p_1 + \frac{(2+q^2)}{4q}p_1^2\right|,$$

$$\Omega_2 = \left|2p_3 + 2p_1 - 4p_2 - \left(2\left(1+q^2\right) - q\right)p_1^2 + \left(\frac{4q^2-3q+2}{q}\right)p_1p_2 + \left(\frac{q^2+q+1}{2q^2}p_1^3\right)\right|,$$

$$\Omega_3 = \frac{1}{2q^2\psi_2^2}\left(\frac{p_2}{4} - \frac{p_1}{4} + \frac{(q^2+1)\,p_1^2}{8q}\right)^2 - \Omega_5\cdot\left[\frac{p_3}{4} + \frac{p_1}{4} - \frac{p_2}{2} - \frac{\left[2\left(1+q^2\right)-q\right]p_1^2}{8q} + \frac{4q^2-3q+2}{8q}p_1p_2 + \left(\frac{q^2+2q-1}{16q^2}\right)p_1^3\right],$$

$$\Omega_4 = \frac{p_1}{2q^2\psi_2^2}\left(\frac{p_2}{4} - \frac{p_1}{4} + \frac{(q^2+1)\,p_1^2}{8q}\right) - \Omega_5 p_1\left(p_2 - p_1 + \frac{(2+q^2)\,p_1^2}{4q}\right),$$

$$\Omega_5 = \frac{(1+q)^2}{16q^2\left(1+q+q^2\right)\psi_1\psi_3}$$

and p_j $(j = 1, 2)$ *are positive and are the coefficients of the functions* $p_k(z)$ *defined by* (6).

Proof. Upon comparing the corresponding coefficients in (17) and (18), we find that

$$a_2 = \frac{(1+q)\,p_1c_1}{4q\psi_1}, \tag{19}$$

$$a_3 = \frac{1}{2q\psi_2}\left[\frac{p_1c_2}{2} + \left(\frac{p_2}{4} - \frac{p_1}{4} + \frac{(q^2+1)\,p_1^2}{8q}\right)c_1^2\right], \tag{20}$$

$$a_4 = \frac{(1+q)}{4\left(q+q^2+q^3\right)\psi_3}\left[p_1c_3 + \left(p_2 - p_1 + \frac{(2+q^2)\,p_1^2}{4q}\right)c_1c_2 + \left(\frac{p_3}{4} + \frac{p_1}{4} - \frac{p_2}{2} - \frac{\left(2\left(1+q^2\right)-q\right)p_1^2}{8q} + \frac{(4q^2-3q+2)}{8q}p_1p_2 + \frac{(q^2+2q-1)}{16q^2}p_1^3\right)c_1^3\right]. \tag{21}$$

By a simple computation, $\mathcal{T}_3(2)$ can be written as follows:

$$\mathcal{T}_3(2) = (a_2 - a_4)\left(a_2^2 - 2a_3^2 + a_2a_4\right).$$

Now, if $f \in \mathcal{ST}(k, \lambda, q)$, then it is clearly seen that

$$\begin{aligned} |a_2 - a_4| &\leqq |a_2| + |a_4| \\ &\leqq \left(\frac{1+q}{2q\psi_1}\right)p_1^2 + \left(\frac{1+q}{4\left(q+q^2+q^3\right)\psi_3}\right)(\Omega_1 + \Omega_2). \end{aligned}$$

We need to maximize $\left|a_2^2 - 2a_3^2 + a_2 a_4\right|$ for a function $f \in \mathcal{ST}(k, \lambda, q)$. So, by writing a_2, a_3, and a_4 in terms of c_1, c_2, and c_3, with the help of (19)–(21), we get

$$\begin{aligned} &\left|a_2^2 - 2a_3^2 + a_2 a_4\right| \\ &\quad = \left| \left(\frac{(1+q)^2}{16q^2\psi_1^2} \right) p_1^2 c_1^2 - \Omega_3 c_1^4 - \Omega_4 c_1^2 c_2 - \frac{p_1^2}{8q^2\psi_2^2} c_2^2 + \Omega_5 p_1^2 c_1 c_3 \right|. \end{aligned} \tag{22}$$

Finally, by applying the trigonometric inequalities, Lemmas 2 and 3 along with (22), we obtain the result asserted by Theorem 2. □

As an application of Theorem 2, we first set $\psi_{n-1} = 1$ and $k = 0$ and then let $q \to 1-$. We thus arrive at the following known result.

Corollary 1 (see [25]). *If the function $f(z)$ given by (1) belongs to the class $\mathcal{S}^*$, then*

$$\mathcal{T}_3(2) \leqq 84.$$

Theorem 3. *If the function $f(z)$ given by (1) belongs to the class $\mathcal{ST}(k, \lambda, q)$, then*

$$\left|a_2 a_4 - a_3^2\right| \leqq \frac{1}{4q^2\psi_2^2}\, p_1^2, \tag{23}$$

where $k \in [0, 1]$ and p_j $(j = 1, 2, 3)$ are positive and are the coefficients of the functions $p_k(z)$ defined by (6).

Proof. Making use of (19)–(21), we find that

$$\begin{aligned} a_2 a_4 - a_3^2 = {} & \frac{A(q)}{16q^2\psi_1\psi_3}\, p_1^2 c_1 c_3 + \left(\frac{A(q)\,\psi_2^2 - \psi_1\psi_3}{16q^2\psi_1\psi_2^2\psi_3}\, p_1 p_2 - \frac{A(q)\,\psi_2^2 - \psi_1\psi_3}{16q^2\psi_1\psi_2^2\psi_3}\, p_1^2 \right. \\ & \left. + \frac{A(q)\left(2+q^2\right)\psi_2^2 - 2\left(1+q^2\right)\psi_1\psi_3}{64q^2\psi_1\psi_3}\, p_1^3 \right) c_1^2 c_2 + \frac{1}{16q^2\psi_2^2}\, p_1^2 c_2^2 \\ & + \left[\frac{A(q)}{64q^2\psi_1\psi_3}\, p_1 p_3 + \left(\frac{A(q)\,\psi_2^2 - \psi_1\psi_3}{64q^2\psi_1\psi_2^2\psi_3} \right) p_1^2 + \left(\frac{\psi_1\psi_3 - A(q)\,\psi_2^2}{32q^2\psi_1\psi_2^2\psi_3} \right) p_1 p_2 \right. \\ & + \left(\frac{2\left(1+q^2\right)\psi_1\psi_3 - \left(2\left(1+q^2\right) - q\right) A(q)\,\psi_2^2}{128q^3\psi_1\psi_2^2\psi_3} \right) p_1^3 \\ & + \left(\frac{A(q)\left(4q^2 - 3q + 2\right)\psi_2^2 - 2\left(1+q^2\right)\psi_1\psi_3}{128q^3\psi_1\psi_2^2\psi_3} \right) p_1^2 p_2 \\ & \left. + \left(\frac{A(q)\left(q^2 + 2q - 1\right)\psi_2^2 - \left(1+q^2\right)^2\psi_1\psi_3}{256q^4\psi_1\psi_2^2\psi_3} \right) p_1^4 - \frac{1}{64q^2\psi_2^2}\, p_2^2 \right] c_1^4, \end{aligned} \tag{24}$$

where

$$A(q) = \frac{(1+q)^2}{1+q+q^2}.$$

We substitute the values of c_2 and c_3 from the above Lemma and, for simplicity, take $Y = 4 - c_1^2$ and $Z = (1 - |x|^2)z$. Without loss of generality, we assume that $c = c_1$ $(0 \leqq c \leqq 2)$, so that

$$\begin{aligned} a_2 a_4 - a_3^2 = & \left[\frac{q(1-q)A(q)\psi_2^2}{128q^2\psi_1\psi_3} p_1^3 + \frac{A(q)}{64q^2\psi_1\psi_3} p_1 p_3 \right. \\ & + \left(\frac{A(q)(4q^2 - 3q + 2)\psi_2^2 - 2(1+q^2)\psi_1\psi_3}{128q^3\psi_1\psi_2^2\psi_3} \right) p_1^2 p_2 \\ & \left. + \left(\frac{A(q)(q^2 + 2q - 1)\psi_2^2 - (1+q^2)^2 \psi_1\psi_3}{256q^4\psi_1\psi_2^2\psi_3} \right) p_1^4 - \frac{1}{64q^2\psi_2^2} p_2^2 \right] c^4 \\ & + \left[\frac{A(q)\psi_2^2 - \psi_1\psi_3}{32q^2\psi_1\psi_2^2\psi_3} p_1 p_2 + \frac{A(q)(2+q^2)\psi_2^2 - 2(1+q^2)\psi_1\psi_3}{128q^2\psi_1\psi_3} p_1^3 \right] c^2 x Y \\ & \cdot \left[-\frac{A(q)}{64q^2\psi_1\psi_3} p_1^2 c^2 Y x^2 - \frac{1}{64q^2\psi_2^2} p_1^2 x^2 Y^2 + \frac{A(q)}{32q^2\psi_1\psi_3} p_1^2 c Y Z \right]. \end{aligned} \tag{25}$$

Upon setting $Z = (1 - |x|^2)z$ and taking the moduli in (25) and using trigonometric inequality, we find that

$$\begin{aligned} \left| a_2 a_4 - a_3^2 \right| \leqq & |\lambda_1| c^4 + |\lambda_2| |x| Y c^2 + \frac{A(q)}{64q^2\psi_1\psi_3} p_1^2 Y |x|^2 c^2 \\ & + \frac{1}{64q^2\psi_2^2} p_1^2 |x|^2 Y^2 + \frac{A(q)}{32q^2\psi_1\psi_3} p_1^2 c^2 Y \left(1 - |x|^2\right) \\ = & \Lambda(c, |x|), \end{aligned} \tag{26}$$

where

$$\begin{aligned} \lambda_1 = & \frac{q(1-q)A(q)\psi_2^2}{128q^2\psi_1\psi_3} p_1^3 + \frac{A(q)}{64q^2\psi_1\psi_3} p_1 p_3 \\ & + \left(\frac{A(q)(4q^2 - 3q + 2)\psi_2^2 - 2(1+q^2)\psi_1\psi_3}{128q^3\psi_1\psi_2^2\psi_3} \right) p_1^2 p_2 \\ & + \left(\frac{A(q)(q^2 + 2q - 1)\psi_2^2 - (1+q^2)^2 \psi_1\psi_3}{256q^4\psi_1\psi_2^2\psi_3} \right) p_1^4 - \frac{1}{64q^2\psi_2^2} p_2^2 \\ \lambda_2 = & \frac{A(q)\psi_2^2 - \psi_1\psi_3}{32q^2\psi_1\psi_2^2\psi_3}; p_1 p_2 + \frac{A(q)(2+q^2)\psi_2^2 - 2(1+q^2)\psi_1\psi_3}{128q^2\psi_1\psi_3} p_1^3. \end{aligned}$$

Now, trivially, we have

$$\Lambda'(|x|) > 0$$

on $[0, 1]$, and so

$$\Lambda(|x|) \leqq \Lambda(1).$$

Hence, by puting $Y = 4 - c_1^2$ and after some simplification, we have

$$\begin{aligned} \left| a_2 a_4 - a_3^2 \right| = & \left(|\lambda_1| - |\lambda_2| + \frac{\psi_1\psi_3 - A(q)\psi_2^2}{64q^2\psi_1\psi_3} p_1^2 \right) c^4 \\ & + \left(4|\lambda_2| + \left(\frac{A(q)\psi_2^2 - \psi_1\psi_3}{16q^2\psi_1\psi_3} p_1^2 \right) \right) c^2 + \frac{1}{4q^2\psi_2^2} p_1^2 \\ = & G(c). \end{aligned} \tag{27}$$

For optimum value of $G(c)$, we consider $G'(c) = 0$, which implies that $c = 0$. So $G(c)$ has a maximum value at $c = 0$. We therefore conclude that the maximum value of $G(c)$ is given by

$$\frac{1}{4q^2\psi_2^2}p_1^2,$$

which occurs at $c = 0$ or

$$c^2 = -\frac{128|\lambda_2|q^2\psi_1\psi_3 + 4A(q)\psi_2^2 - 2\psi_1\psi_3 p_1^2}{\left(64q^2(|\lambda_1| - |\lambda_2|)\psi_1\psi_3 + \psi_1\psi_3 - A(q)\psi_2^2 p_1^2\right)}.$$

This completes the proof of Theorem 3. □

If we put $\psi_{n-1} = 1$ and let $q \to 1-$ in Theorem 3, we have the following known result.

Corollary 2 (see [26]). *If the function $f(z)$ given by* (1) *belongs to the class k-$\mathcal{ST}$, where $k \in [0,1]$, then*

$$\left|a_2 a_4 - a_3^2\right| \leqq \frac{p_1^2}{4}.$$

If we put

$$p_1 = 2 \quad \text{and} \quad \psi_{n-1} = 1,$$

by letting $q \to 1-$ in Theorem 3, we have the following known result.

Corollary 3 (see [18]). *If $f \in \mathcal{S}^*$, then*

$$\left|a_2 a_4 - a_3^2\right| \leqq 1.$$

By letting $k = 1$, $\psi_{n-1} = 1$, $q \to 1-$ and

$$p_1 = \frac{8}{\pi^2}, \quad p_2 = \frac{16}{3\pi^2} \quad \text{and} \quad p_3 = \frac{184}{45\pi^2}$$

in Theorem 3, we have the following known result.

Corollary 4 (see [27]). *If the function $f(z)$ given by* (1) *belong to the class $\mathcal{SP}$, then*

$$\left|a_2 a_4 - a_3^2\right| \leqq \frac{16}{\pi^4}.$$

4. Concluding Remarks and Observations

Motivated significantly by a number of recent works, we have made use of a certain general conic domain and the quantum (or q-) calculus in order to define and investigate a new subclass of normalized analytic functions in the open unit disk $\mathbb{U}$, which we have referred to as q-starlike functions. For this q-starlike function class, we have successfully derived several properties and characteristics. In particular, we have found the Hankel determinant and the Toeplitz matrices for this newly-defined class of q-starlike functions. We also highlight some known consequences of our main results which are stated and proved as theorems and corollaries.

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