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The Extremal Solution To Conformable Fractional Differential Equations Involving Integral Boundary Condition

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Abstract: In this article, by using the monotone iterative technique coupled with the method of upper and lower solution, we obtain the existence of extremal iteration solutions to conformable fractional differential equations involving Riemann-Stieltjes integral boundary conditions. At the same time, the comparison principle of solving such problems is investigated. Finally, an example is given to illustrate our main results. It should be noted that the conformal fractional derivative is essentially a modified version of the first-order derivative. Our results show that such known results can be translated and stated in the setting of the so-called conformal fractional derivative.

Keywords: fractional differential equations; Riemann-stieltjes integral; monotone iterative method; upper and lower solutions

1. Introduction

Fractional calculus is an excellent tool for the description of the process of mathematical analysis in various areas of finance, physical systems, control systems and mechanics, and so forth [1–5]. Many methods are used to study various fractional differential equations, such as fixed point index theory [6], iterative method [7–9], theory of linear operator [10,11] sequential techniques, and regularization [12], fixed point theorems [13–17], the Mawhin continuation theorem for resonance [18–22], the variational method [23]. The definition of the fractional order derivative used in the aforementioned results is either the Caputo or the Riemann-Liouville fractional order derivative. Recently, Khalil et al. [24] gave a new simple fractional derivative called "the conformable fractional derivative" depending on the familiar limit definition of the derivative of a function and that break with other definitions. This new fractional derivative is called "the conformable fractional derivative" and this new theory is improved by Abdeljawad [25]. However, a conformal fractional derivative is not a fractional derivative, it is simply a first derivative multiplied by an additional simple factor. Therefore, this new definition seems to be a natural extension of the classical derivative. However, it has the advantage of being different from other fractional differentials. Firstly, it can integrate the standard properties of fractional derivatives. It is suitable for many extensions to the classical theorem of calculus, such as the derivative of the product and compound of two functions, the Rolle's and the mean value theorem, conformable integration by parts, fractional power series expansion and many more. Secondly, the conformal fractional derivative of the real function is zero, and the Riemann-Liouville fractional derivative does not satisfy this property. For the two iterative conformal differentials, the semigroup property is not satisfied, and the Caputo differential satisfies this. In particular, for functions that are not differentiable, in conformal sense; however, the function is differentiable. Some functions are not infinitely differentiable at some points; where there is no Taylor power series expansion, in conformal calculus theory, they do exist. This led many people wanting to explore it. Please see [26–34] for recent developments on conformable differentiation. For example, a mean value theorem of the conformable fractional calculus on arbitrary time scales is proved in [33], and whose results reconciled with familiar classical results when the operator T_α is of order $\alpha = 1$ and the time scale coincides with the set of real numbers. In [34], Asawasamrit and Ntoutas introduced a new definition of exponential notations and by employing the method of lower and upper solution combined with the monotone iterative technique, some new conditions for the existence of solutions are presented.

Motivated by the above works, we consider the existence of solutions for the following nonlinear conformable fractional differential equation involving integral boundary condition, using the method of upper and lower solutions and its associated monotone iterative technique

$$\begin{cases} D_\alpha x(t) = f(t, x(t)), \ \ t \in [0,1], \\ x(0) = \int_0^1 x(t) d\mu(t), \end{cases} \tag{1}$$

where $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $\int_0^1 x(t)d\mu(t)$ denotes the Riemann-Stieltjes integral with positive Stieltjes measure of μ, and $D_\alpha f(t)$ stands for the conformable fractional derivative. Based on a comparison result, two monotone iterative sequences are obtained using the upper and lower solutions, and these two sequences approximate the extremal solutions of the given problem. For applications of the method of upper and lower solutions and monotone iterative technique to differential equations and differential systems such as ordinary differential equations [35–37], ordinary differential systems [38], fractional differential equations [39–42], fractional differential systems [43].

2. Preliminaries

In this section, we briefly show some necessary definitions and results which will be used in our main results.

Definition 1. *[24] Let $f : [0, +\infty) \to \mathbb{R}$ and $t > 0$. The conformable fractional derivative of order $0 < \alpha \leq 1$ is defined by*

$$D_\alpha f(t) = \lim_{\rho \to 0} \frac{f(t + \rho t^{1-\alpha}) - f(t)}{\rho}$$

for $t > 0$ and the conformable fractional derivative at 0 is defined as $D_\alpha f(0) = \lim_{t \to 0^+} (D_\alpha f)(t)$. If f is differentiable then $D_\alpha f(t) = t^{1-\alpha} f'(t)$.

Definition 2. *[24] Let $\alpha \in (0,1]$. The conformable fractional integral of a function $f : [0, +\infty) \to \mathbb{R}$ of order α is denoted by $I_\alpha f(t)$ and is defined as*

$$I_\alpha f(t) = \int_0^t s^{\alpha-1} f(s) ds.$$

Lemma 1. *[25] Let $f : (0, +\infty) \to \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > 0$ we have*

$$I_\alpha D_\alpha f(t) = f(t) - f(0).$$

Lemma 2. *[24] Let $\alpha \in (0,1]$, $l_1, l_2, q, k \in \mathbb{R}$, and the functions f, h be α-differentiable on $[0, +\infty)$. Then*

(i) $D_\alpha k = 0$ *for all constant functions* $f(t) = k$;
(ii) $D_\alpha(l_1 f + l_2 f) = l_1 D_\alpha f(t) + l_2 D_\alpha h(t)$;
(iii) $D_\alpha t^q = q t^{q-\alpha}$;

(iv) $D_\alpha(fh) = f(t)D_\alpha h(t) + h(t)D_\alpha f(t)$;
(v) $D_\alpha(\frac{f}{h}) = \frac{hD_\alpha f - fD_\alpha h}{h^2}$ *when* $h(t) \neq 0$.

Theorem 1. *[24] (Mean value theorem) Let* $[a,b] \subset [0,+\infty)$, *and let* $f : [0,+\infty) \to \mathbb{R}$. *Suppose that*

(1) f *is continuous on* $[a,b]$;
(2) f *is* α*-differentiable for some* $\alpha \in (0,1]$ *on* $[a,b]$.

Then there exists a constant $\xi \in (a,b)$, *such that* $D_\alpha f(\xi) = \frac{f(b)-f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}$.

Definition 3. *A function* $u \in C([0,1],\mathbb{R})$ *is known as a lower solution of (1), if it satisfies*

$$D_\alpha u(t) \leq f(t,u(t)), t \in [0,1], \tag{2}$$

$$u(0) \leq \int_0^1 u(t)d\mu(t). \tag{3}$$

If inequalities (2), (3) are reversed, then u *is an upper solution of problem (1).*

Next, we present the following existence and uniqueness results for linear equations.

Lemma 3. *Let* $0 < \alpha \leq 1$, $a \in \mathbb{R}$ *and* $M, N \in C([0,1],\mathbb{R})$. *Then linear fractional differential equation involving integral boundary problem:*

$$\begin{cases} D_\alpha u(t) = -M(t)u(t) + N(t), \ \ t \in [0,1], \\ u(0) = \displaystyle\int_0^1 u(t)d\mu(t) + a \end{cases} \tag{4}$$

has a unique solution provided $\triangle_\alpha = 1 - \int_0^1 e^{-\int_0^t s^{\alpha-1}M(s)ds} d\mu(t) \neq 0$.

Proof. Multiplying both sides of the first equation of the problem (4) by $e^{\int_0^t s^{\alpha-1}M(s)ds}$ and using Lemma 2, we can get

$$e^{\int_0^t s^{\alpha-1}M(s)ds} D_\alpha u(t) + M(t)u(t)e^{\int_0^t s^{\alpha-1}M(s)ds} = N(t)e^{\int_0^t s^{\alpha-1}M(s)ds}.$$

In other words, by means of the product rule (item (iv) of Lemma 2), the above equality turns to

$$D_\alpha[e^{\int_0^t s^{\alpha-1}M(s)ds}u(t)] = N(t)e^{\int_0^t s^{\alpha-1}M(s)ds}. \tag{5}$$

Applying the conformable fractional integral of order α to both side of (5), we have

$$\begin{aligned} e^{\int_0^t s^{\alpha-1}M(s)ds}u(t) - u(0) &= I_\alpha[N(t)e^{\int_0^t s^{\alpha-1}M(s)ds}] \\ &= \int_0^t s^{\alpha-1}N(s)e^{\int_0^s \tau^{\alpha-1}M(\tau)d\tau}ds. \end{aligned}$$

Then

$$u(t) = e^{-\int_0^t s^{\alpha-1}M(s)ds}[u(0) + \int_0^t s^{\alpha-1}N(s)e^{\int_0^s \tau^{\alpha-1}M(\tau)d\tau}ds]. \tag{6}$$

From the boundary condition of (4), we have

$$\begin{aligned} &\left(1 - \int_0^1 e^{-\int_0^t s^{\alpha-1}M(s)ds}d\mu(t)\right)u(0) \\ = &\int_0^1 e^{-\int_0^t s^{\alpha-1}M(s)ds}\int_0^t s^{\alpha-1}N(s)e^{\int_0^s \tau^{\alpha-1}M(\tau)d\tau}dsd\mu(t) + a. \end{aligned}$$

On account of condition $\triangle_\alpha \neq 0$, then

$$u(0) = \frac{\int_0^1 e^{-\int_0^t s^{\alpha-1}M(s)ds} \int_0^t s^{\alpha-1}N(s)e^{\int_0^s \tau^{\alpha-1}M(\tau)d\tau}dsd\mu(t) + a}{1 - \int_0^1 e^{-\int_0^t s^{\alpha-1}M(s)ds}d\mu(t)},$$

thus problem (4) has a unique solution. The proof is finished. $\square$

In the next Lemma, we discuss comparison results for the linear problem which play a key role in the proof of the main result.

Lemma 4. *Let $0 < \alpha \leq 1$. Suppose that $M, u \in C([0,1], \mathbb{R})$ satisfies*

$$\begin{cases} D_\alpha u(t) \leq -M(t)u(t), & t \in [0,1], \\ u(0) \leq \int_0^1 u(t)d\mu(t). \end{cases}$$

Then $u(t) \leq 0$ on $[0,1]$ provided $\triangle_\alpha > 0$.

Proof. Let $N(t) = D_\alpha u(t) + M(t)u(t)$ and $a = u(0) - \int_0^1 u(t)d\mu(t)$, we know that $N(t) \leq 0$, $a \leq 0$ and

$$\begin{cases} D_\alpha u(t) = -M(t)u(t) + N(t), & t \in [0,1], \\ u(0) = \int_0^1 u(t)d\mu(t) + a \end{cases}$$

Using $\triangle_\alpha > 0$, we have

$$u(0) = \frac{\int_0^1 e^{-\int_0^t s^{\alpha-1}M(s)ds} \int_0^t s^{\alpha-1}N(s)e^{\int_0^s \tau^{\alpha-1}M(\tau)d\tau}dsd\mu(t) + a}{1 - \int_0^1 e^{-\int_0^t s^{\alpha-1}M(s)ds}d\mu(t)} \leq 0.$$

Then by (6), we can conclude that

$$u(t) \leq e^{-\int_0^t s^{\alpha-1}M(s)ds}u(0) \leq 0.$$

The proof is complete. $\square$

3. Main Results

In this section, we prove the existence of extremal solutions for conformable fractional differential equations involving integral boundary condition. For convenience, we list some assumptions.

(H_1): $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H_2): Assume that $v_0, w_0 \in E = C[0,1]$ is lower and upper solution of problem (1), and $v_0(t) \leq w_0(t)$.

(H_3): There exists a function $M \in E$ with $\triangle_\alpha > 0$ which satisfies

$$f(t,x) - f(t,\overline{x}) \leq M(t)(\overline{x} - x),$$

for $v_0(t) \leq x \leq \overline{x} \leq w_0(t)$.

Theorem 2. *Assume that (H_1), (H_2), (H_3) hold. Then there exist monotone iterative sequences $\{v_n\}_{n=0}^\infty, \{w_n\}_{n=0}^\infty \subset E$ such that*

$$\lim_{n\to\infty} v_n = v, \quad \lim_{n\to\infty} w_n = w$$

uniformly on $[0,1]$, and v, w are the extremal solutions of problem (1) in the sector $[v_0, w_0] = \{g \in E : v_0(t) \leq g(t) \leq w_0(t),\ 0 \leq t \leq 1\}$.

Proof. For all $v_n, w_n \in E$, let

$$\begin{cases} D_\alpha v_{n+1}(t) = f(t, v_n(t)) - M(t)(v_{n+1}(t) - v_n(t)), \ \ t \in [0,1], \\ D_\alpha w_{n+1}(t) = f(t, w_n(t)) - M(t)(w_{n+1}(t) - w_n(t)), \ \ t \in [0,1], \\ v_{n+1}(0) = \int_0^1 v_{n+1}(t) d\mu(t), \ \ w_{n+1}(0) = \int_0^1 w_{n+1}(t) d\mu(t). \end{cases} \tag{7}$$

Thus, the iterative sequences $\{v_n\}$ and $\{w_n\}$ can be constructed by Lemma 3.

Firstly, we shall prove that

$$v_n \leq v_{n+1} \leq w_{n+1} \leq w_n, \ \ n = 0, 1, 2, \ldots.$$

Let $p = v_0 - v_1$. According to (7) and Definition 3, we have

$$\begin{cases} D_\alpha p(t) = D_\alpha v_0(t) - D_\alpha v_1(t) \leq f(t, v_0(t)) - f(t, v_0(t)) + M(t)(v_1(t) - v_0(t)), \ \ t \in [0,1], \\ p(0) \leq \int_0^1 v_0(t) d\mu(t) - \int_0^1 v_1(t) d\mu(t), \end{cases}$$

i.e.,

$$\begin{cases} D_\alpha p(t) \leq -M(t) p(t), \ \ t \in [0,1], \\ p(0) \leq \int_0^1 p(t) d\mu(t). \end{cases}$$

Therefore, by Lemma 4, we have $v_0(t) \leq v_1(t)$. Similarly, we can prove that $w_1(t) \leq w_0(t)$, $t \in [0,1]$. Now, let $r = v_1 - w_1$, according to (7) and (H_3), we have

$$\begin{cases} D_\alpha r(t) &= f(t, v_0(t)) - f(t, w_0(t)) - M(t)(v_1(t) - v_0(t) - w_1(t) + w_0(t)) \\ &\leq M(t)(w_0(t) - v_0(t)) - M(t)(v_1(t) - v_0(t) - w_1(t) + w_0(t)) \\ &= -M(t) r(t), \\ r(0) &= \int_0^1 r(t) d\mu(t). \end{cases}$$

By Lemma 4, we have $v_1(t) \leq w_1(t)$, $t \in [0,1]$.

Secondly, we show that v_1, w_1 are lower and upper solutions of (1), respectively.

$$\begin{cases} D_\alpha v_1(t) &= f(t, v_0(t)) - M(t)(v_1(t) - v_0(t)) - f(t, v_1(t)) + f(t, v_1(t)) \\ &\leq M(t)(v_1(t) - v_0(t)) - M(t)(v_1(t) - v_0(t)) + f(t, v_1(t)) \\ &= f(t, v_1(t)), \\ v_1(0) &= \int_0^1 v_1(t) d\mu(t). \end{cases}$$

According to (H_3) and Definition 3, we deduce that v_1 is a lower solution of (1). Similarly, w_1 is a upper solutions of (1). By the above arguments and mathematical induction, it is clear that

$$v_0 \leq \cdots \leq v_n \leq v_{n+1} \leq w_{n+1} \leq w_n \leq \cdots \leq w_0, \ \ n = 0, 1, 2, \ldots. \tag{8}$$

Thirdly, we show that $\lim\limits_{n\to\infty} v_n = v$, $\lim\limits_{n\to\infty} w_n = w$. Hence, we need to conclude that v_n, w_n are uniformly bounded and equicontinuous on $[0,1]$. Obviously, the uniform boundedness of sequences v_n, w_n follows from (8). Thus, there exists $L > 0$ such that

$$|f(t, v_n(t)) - M(t)(v_{n+1}(t) - v_n(t))| \leq L$$

and

$$|f(t, w_n(t)) - M(t)(w_{n+1}(t) - w_n(t))| \leq L.$$

Using Theorem 1, we get

$$\begin{aligned} |v_n(t_1) - v_n(t_2)| &= \frac{1}{\alpha}|D_\alpha v_n(\xi)||t_1^\alpha - t_2^\alpha| \\ &= \frac{1}{\alpha}|f(\xi, v_{n-1}(\xi)) - M(\xi)(v_n(\xi) - v_{n-1}(\xi))||t_1^\alpha - t_2^\alpha|. \end{aligned}$$

Therefore, $\{v_n\}$ are equicontinuous. Similarly, we obtain that $\{w_n\}$ are equicontinuous too. By Arzela-Ascoli Theorems, we conclude that $\{v_n\}$, $\{w_n\}$ have subsequences $\{v_{n_k}\}$, $\{w_{n_k}\}$ such that $\{v_{n_k}\} \to v$, and $\{w_{n_k}\} \to w$ when $k \to \infty$. This together with the monotonicity of sequences $\{v_n\}$ and $\{w_n\}$ implies

$$\lim_{n\to\infty} v_n(t) = v(t), \quad \lim_{n\to\infty} w_n(t) = w(t)$$

uniformly on $[0, 1]$. Please note that the sequence $\{v_n\}$ satisfies

$$\begin{cases} v_n(t) = e^{-\int_0^t s^{\alpha-1}M(s)ds}[v_{n-1}(0) + Rv_{n-1}(t)], \;\; t \in [0, 1], \\ v_n(0) = \int_0^1 v_n(t)d\mu(t), \;\; n = 1, 2, \dots, \end{cases} \tag{9}$$

where

$$Rv_{n-1}(t) = \int_0^t s^{\alpha-1}[f(t, v_{n-1}(s)) + M(s)v_{n-1}(s)]e^{\int_0^s \tau^{\alpha-1}M(\tau)d\tau}ds.$$

Let $n \to \infty$ in (9) . We have

$$\begin{cases} v(t) = e^{-\int_0^t s^{\alpha-1}M(s)ds}[v(0) + Rv(t)], \;\; t \in [0, 1], \\ v(0) = \int_0^1 v(t)d\mu(t). \end{cases}$$

This shows that v is a solution of the nonlinear problem (1). Similarly, we obtain w is a solution of the nonlinear problem (1) too. And

$$v_0(t) \le v(t) \le w(t) \le w_0(t), \;\; t \in [0, 1].$$

Finally, we are going to prove that v, w are minimal and maximal solutions of (1) in the sector $[v_0, w_0]$. In the following, we show this using induction arguments. Suppose that $g(t)$ is any solution of (1) in the $[v_0, w_0]$ that is

$$v_0(t) \le g(t) \le w_0(t), \;\; t \in [0, 1].$$

Assume that $v_n(t) \le g(t) \le w_n(t)$ hold. Let $p(t) = v_{n+1}(t) - g(t)$, we have

$$\begin{cases} D_\alpha p(t) &= D_\alpha v_{n+1}(t) - D_\alpha g(t) \\ &= f(t, v_n(t)) - M(t)(v_{n+1}(t) - v_n(t)) - f(t, g(t)) \\ &\le M(t)(g(t) - v_n(t)) - M(t)(v_{n+1}(t) - v_n(t)) \\ &= -M(t)p(t), \\ p(0) &= \int_0^1 p(t)d\mu(t). \end{cases}$$

Then, by Lemma 4, we have $v_{n+1}(t) \le g(t)$, $t \in [0, 1]$. By similar method, we can show that $g(t) \le w_{n+1}(t)$, $t \in [0, 1]$. Therefore,

$$v_n \le g \le w_n, \;\; n = 1, 2, \dots.$$

By taking $n \to \infty$ in the above inequalities, we get that $v \le g \le w$. That is v, w are extremal solutions of problem (1) in $[v_0, w_0]$. Thus, the proof is finished. □

Example 1. *Consider the following nonlinear problem:*

$$\begin{cases} D_{\frac{1}{2}}x(t) = -\frac{2}{9}(1+x(t))^3 + 9\sin\frac{x^2(t)}{4}, & t \in [0,1], \\ x(0) = \frac{1}{3}x(\frac{1}{4}) + \frac{1}{6}x(\frac{1}{2}). \end{cases} \tag{10}$$

Let

$$\mu(t) = \begin{cases} 0, & t \in [0, \frac{1}{4}), \\ \frac{1}{3}, & t \in [\frac{1}{4}, \frac{1}{2}), \\ \frac{1}{2}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Obviously, $\alpha = \frac{1}{2}$, $f(t,x) = -\frac{2}{9}(1+x)^3 + 9\sin\frac{x^2}{4}$ *and*

$$\int_0^1 x(t)d\mu(t) = \frac{1}{3}x(\frac{1}{4}) + \frac{1}{6}x(\frac{1}{2}).$$

We can get

$$\int_0^1 d\mu(t) = \frac{1}{2}.$$

Take

$$v_0(t) = -2, \ \ w_0(t) = 0,$$

then,

$$\begin{cases} D_{\frac{1}{2}}v_0(t) = 0 < \frac{2}{9} + 9\sin 1 = f(t, v_0(t)), \\ v_0(0) = -2 < -1 = \int_0^1 v_0(t)d\mu(t), \end{cases}$$

and

$$\begin{cases} D_{\frac{1}{2}}w_0(t) = 0 > -\frac{2}{9} = f(t, w_0(t)), \\ w_0(0) = 0 = \int_0^1 w_0(t)d\mu(t). \end{cases}$$

Then v_0, w_0 *are lower and upper solutions of (10). When* $M(t) = 1$*, it is easy to verify that assumption* (H_3) *holds. In addition,*

$$\int_0^1 e^{-\int_0^t s^{\alpha-1}M(s)ds}d\mu(t) = \int_0^1 e^{-t^{\frac{1}{2}}}d\mu(t) < \int_0^1 d\mu(t) = \frac{1}{2} < 1.$$

By Theorem 2, problem (10) has an extremal iterative solution in $[v_0, w_0]$*.*

4. Conclusions

In this article, on the integral boundary value problem for conformable fractional differential equations, we use the monotone iterative technique to investigate the existence results for extremal solutions for Equation (1). At the same time, two sequences are obtained using the upper and lower solutions, and these two sequences approximate the extremal solutions of nonlinear differential equations. It is clear that the method of using the upper and lower solutions is a very effective method for studying the solvability of conformable fractional differential equations. However, almost all the results derived in the paper are more-or-less straightforward extensions of well-known results from the theory of the first-order differential equations, since the conformal fractional derivative is essentially a modified version of the first-order derivative.

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