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Dini-Type Helicoidal Hypersurfaces with Timelike Axis in Minkowski 4-Space $\mathbb{E}_1^4$

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Abstract: We consider Ulisse Dini-type helicoidal hypersurfaces with timelike axis in Minkowski 4-space $\mathbb{E}_1^4$. Calculating the Gaussian and the mean curvatures of the hypersurfaces, we demonstrate some special symmetries for the curvatures when they are flat and minimal.

Keywords: Minkowski 4-space; Dini-type helicoidal hypersurface; Gauss map; timelike axis

1. Introduction

The concept of finite-type immersion of submanifolds of a Euclidean space has been known in classifying and characterizing Riemannian submanifolds [1]. Chen proposed the problem of classifying these kinds surfaces in the three-dimensional Euclidean space $\mathbb{E}^3$. A Euclidean submanifold is called Chen finite-type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian Δ [1]. Hence, the idea of finite-type can be enlarged to any smooth functions on a submanifold of Euclidean or pseudo-Euclidean spaces.

Takahashi [2] obtained spheres and the minimal surfaces are the unique surfaces in $\mathbb{E}^3$ satisfying the condition $\Delta r = \lambda r$, where r is the position vector, $\lambda \in \mathbb{R}$. Ferrandez, Garay and Lucas [3] showed the surfaces of $\mathbb{E}^3$ providing $\Delta H = AH$. Here H is the mean curvature and $A \in Mat(3,3)$ are either of a right circular cylinder, or of an open piece of sphere, or minimal. Choi and Kim [4] worked the minimal helicoid with pointwise 1-type Gauss map of the first type.

Dillen, Pas, and Verstraelen [5] studied the unique surfaces in $\mathbb{E}^3$ providing $\Delta r = Ar + B$, $A \in Mat(3,3)$, $B \in Mat(3,1)$ are the spheres, the circular cylinder, the minimal surfaces. Senoussi and Bekkar [6] obtained helicoidal surfaces in $\mathbb{E}^3$ by using the fundamental forms I, II and III.

In classical surface geometry, it is well known a pair of the right helicoid and the catenoid is the unique ruled and rotational surface, which is minimal. When we look at ruled (i.e., helicoid) and rotational surfaces, we meet Bour's theorem in [7]. By using a result of Bour [7], Do Carmo and Dajczer [8] worked isometric helicoidal surfaces.

Lawson [9] defined the generalized Laplace-Beltrami operator. Magid, Scharlach and Vrancken [10] studied the affine umbilical surfaces in 4-space. Vlachos [11] introduced hypersurfaces with harmonic mean curvature in $\mathbb{E}^4$. Scharlach [12] gave the affine geometry of surfaces and hypersurfaces in 4-space. Cheng and Wan [13] studied complete hypersurfaces of 4-space with CMC. Arslan, Deszcz and Yaprak [14] obtained Weyl pseudosymmetric hypersurfaces. Arvanitoyeorgos, Kaimakamais and Magid [15] wrote that if the mean curvature vector field of M_1^3 satisfies the equation $\Delta H = \alpha H$ (α a constant), then M_1^3 has constant mean curvature in Minkowski 4-space $\mathbb{E}_1^4$. This equation is a natural generalization of the biharmonic submanifold equation $\Delta H = 0$.

General rotational surfaces in the four-dimensional Euclidean space were originated by Moore [16,17]. Ganchev and Milousheva [18] considered the analogue of these surfaces in $\mathbb{E}_1^4$. Verstraelen, Valrave, and Yaprak [19] studied the minimal translation surfaces in $\mathbb{E}^n$ for arbitrary dimension n. Kim and Turgay [20] studied surfaces with L_1-pointwise 1-type Gauss map in $\mathbb{E}^4$. Moruz and Munteanu [21] considered hypersurfaces defined as the sum of a curve and a surface whose mean curvature vanishes in $\mathbb{E}^4$.

Yoon [22] considered rotational surfaces which has a finite-type Gauss map in $\mathbb{E}^4$. Dursun [23] introduced hypersurfaces of pointwise 1-type Gauss map in Minkowski space. Dursun and Turgay [24] studied minimal, pseudo-umbilical rotational surfaces in $\mathbb{E}^4$. Arslan, Bulca and Milousheva [25] focused pointwise 1-type Gauss map of meridian surfaces in $\mathbb{E}^4$. Aksoyak and Yaylı [26] worked boost-invariant surfaces with pointwise 1-type Gauss map in $\mathbb{E}^4_1$. Also they [27] considered generalized rotational surfaces of pointwise 1-type Gauss map in $\mathbb{E}^4_2$. Güler, Magid and Yaylı [28] defined helicoidal hypersurface with the Laplace-Beltrami operator in $\mathbb{E}^4$. Furthermore, Güler, Hacısalihoğlu and Kim [29] worked rotational hypersurface with the III Laplace-Beltrami operator and the Gauss map in $\mathbb{E}^4$.

There are few works in the literature about Italian Mathematician Ulisse Dini's helicoidal surface [30] in $\mathbb{E}^3$. Moreover, Güler and Kişi [31] introduced helicoidal hypersurfaces of Dini-type with spacelike axis in $\mathbb{E}^4_1$.

In this paper, we study the Ulisse Dini-type helicoidal hypersurface with timelike axis in Minkowski 4-space $\mathbb{E}^4_1$. We give some basic notions of Minkowskian geometry, and define helicoidal hypersurface in Section 2. Moreover, we obtain the Dini-type helicoidal hypersurface timelike axis, and calculate its curvatures in the Section 3. We obtain some special symmetries in the last section.

2. Preliminaries

In this section, we will describe the notation that will be used in the paper, after we give some basic facts and basic definitions.

Let $\mathbb{E}^m_1$ be the Minkowski m-space with the Euclidean metric denoted by

$$\widetilde{g} = \langle\ ,\ \rangle = \sum_{i=1}^{m-1} dx_i^2 - dx_m^2,$$

where $(x_1, x_2, \ldots, x_m)$ is a coordinate system in $\mathbb{E}^m_1$.

Consider an n-dimensional Minkowskian submanifold of the space $\mathbb{E}^m_1$. We denote Levi-Civita connections of $\mathbb{E}^m_1$ and M by $\widetilde{\nabla}$ and ∇, respectively. We will use letters $X,\ Y,\ Z,\ W$ (resp., $\xi,\ \eta$) to show vector fields tangent (resp., normal) to M. The Gauss and the Weingarten formulas are defined by as follows:

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + h(X,Y), \\ \widetilde{\nabla}_X \xi &= -A_\xi(X) + D_X \xi, \end{aligned}$$

where h, D, and A are the second fundamental form, the normal connection and the shape operator of M, respectively.

The shape operator A_ξ is a symmetric endomorphism of the tangent space T_pM at $p \in M$ for each $\xi \in T_p^\perp M$. The second fundamental form and the shape operator are connected by

$$\langle h(X,Y), \xi\rangle = \langle A_\xi X, Y\rangle.$$

The Gauss and Codazzi equations are denoted, respectively, as follows:

$$\langle R(X,Y,)Z,W\rangle = \langle h(Y,Z), h(X,W)\rangle - \langle h(X,Z), h(Y,W)\rangle, \tag{1}$$

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z). \tag{2}$$

Here, $R,\ R^D$ are the curvature tensors related with connections ∇ and D, respectively, and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

2.1. Hypersurfaces of Minkowski Space

Assume that M be an oriented hypersurface in Minkowski space $\mathbb{E}_1^n$, $\mathbf{S}$ its shape operator and x its position vector. We think about a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ occurring of the principal directions of M matching to the principal curvatures k_i for $i = 1, 2, \ldots n$. Let $\{\theta_1, \theta_2, \ldots, \theta\}$ be dual basis of this frame field. Then the first structural equation of Cartan is

$$d\theta_i = \sum_{i=1}^{n} \theta_j \wedge \omega_{ij}, \quad i = 1, 2, \ldots, n.$$

Here, ω_{ij} demonstrates the connection forms matching to the chosen frame field. We show the Levi-Civita connection of M and $\mathbb{E}_1^n$ by ∇ and $\widetilde{\nabla}$, respectively. Then, from the Codazzi Equation (2) we have

$$\begin{aligned} e_i(k_j) &= \omega_{ij}(e_j)(k_i - k_j), \\ \omega_{ij}(e_l)(k_i - k_j) &= \omega_{il}(e_j)(k_i - k_l) \end{aligned}$$

for distinct $i, j, l = 1, 2, \ldots, n$.

We take $s_j = \sigma_j(k_1, k_2, \ldots, k_n)$, where σ_j is the j-th elementary symmetric function given by

$$\sigma_j(a_1, a_2, \ldots, a_n) = \sum_{1 \le i_1 < i_2 < \ldots, i_j \le n} a_{i_1} a_{i_2} \ldots a_{i_j}.$$

We also use the following notation

$$r_i^j = \sigma_j(k_1, k_2, \ldots, k_{i-1}, k_{i+1}, k_{i+2}, \ldots, k_n).$$

By definition, we have $r_i^0 = 1$ and $s_{n+1} = s_{n+2} = \cdots = 0$.

On the other hand, we will call the function s_k as the k-th mean curvature of M. We would like to note that functions $H = \frac{1}{n} s_1$ and $K = s_n$ are called the mean and the Gauss-Kronecker curvatures of M, respectively. Particularly, M is called j-minimal if $s_j \equiv 0$ on M.

2.2. Helicoidal Hypersurfaces with Timelike Axis in Minkowskian Spaces

In this subsection, we will obtain the helicoidal hypersurfaces with timelike axis in Minkowski 4-space $\mathbb{E}_1^4$. In the rest of this paper, we will identify a vector (a,b,c,d) with its transpose.

Before we proceed, we would like to note that the definition of rotational hypersurfaces in Riemannian space forms were defined in [32]. A rotational hypersurface $M \subset \mathbb{E}_1^n$ generated by a curve C about an axis $\mathbf{r}$ does not meet C is generated by using the orbit of C under those orthogonal transformations of $\mathbb{E}_1^n$ which leave $\mathbf{r}$ pointwise fixed (See [32] remark 2.3).

A curve C rotates about the axis $\mathbf{r}$, and at the same time replaces parallel lines orthogonal to the axis $\mathbf{r}$, so that the speed of replacement is proportional to the speed of rotation. Finally, the resulting hypersurface is called the *helicoidal hypersurface* with axis $\mathbf{r}$.

Consider the particular case $n = 4$ and let C be the curve parametrized by

$$\gamma(u) = (f(u), 0, 0, \varphi(u)),$$

where f and φ are differentiable functions. If $\mathbf{r}$ is the timelike vector $(0, 0, 0, 1)$, then an orthogonal transformation of $\mathbb{E}_1^4$ that leaves $\mathbf{r}$ pointwise fixed has the form $Z(v, w)$ as follows:

$$Z(v,w)=\begin{pmatrix} \cos v\cos w & -\sin v & -\cos v\sin w & 0\\ \cos w\sin v & \cos v & -\sin v\sin w & 0\\ \sin w & 0 & \cos w & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

and the following relations hold:

$$Z^T\varepsilon Z=Z\varepsilon Z^T=\varepsilon,\ Z\mathbf{r}^T=\mathbf{r}^T,\ \det Z=1,\ \varepsilon=diag\,(1,1,1,-1)\,,$$

$v,w\in\mathbb{R}$. Therefore, the parametrization of the rotational hypersurface obtained by a curve C around an axis $\mathbf{r}$ is

$$\mathbf{H}(u,v,w)=Z(v,w)\gamma(u)^T+(av+bw)\,\mathbf{r}^T,$$

where $u\in I$, $v,w\in[0,2\pi]$ and pitches $a,b\in\mathbb{R}\backslash\{0\}$.

Clearly, an helicoidal hypersurface with timelike axis written as

$$\mathbf{H}(u,v,w)=\begin{pmatrix} f(u)\cos v\cos w\\ f(u)\sin v\cos w\\ f(u)\sin w\\ \varphi(u)+av+bw\end{pmatrix}. \tag{3}$$

When $w=0$, we have an helicoidal surface with timelike axis in $\mathbb{E}^4_1$.

Now we give some basic elements of the Minkowski 4-space $\mathbb{E}^4_1$. Let $\mathbf{M}=\mathbf{M}(u,v,w)$ be an isometric immersion of a hypersurface M^3 in $\mathbb{E}^4_1$. Using vectors $\overrightarrow{x}=(x_1,x_2,x_3,x_4)$, $\overrightarrow{y}=(y_1,y_2,y_3,y_4)$ and $\overrightarrow{z}=(z_1,z_2,z_3,z_4)$, the Minkowskian inner product and vector product are defined by as follows, respectively,

$$\overrightarrow{x}\cdot\overrightarrow{y}=x_1y_1+x_2y_2+x_3y_3-x_4y_4,$$

$$\begin{aligned}\overrightarrow{x}\times\overrightarrow{y}\times\overrightarrow{z}=(&x_2y_3z_4-x_2y_4z_3-x_3y_2z_4+x_3y_4z_2+x_4y_2z_3-x_4y_3z_2,\\ &-x_1y_3z_4+x_1y_4z_3+x_3y_1z_4-x_3z_1y_4-y_1x_4z_3+x_4y_3z_1,\\ &x_1y_2z_4-x_1y_4z_2-x_2y_1z_4+x_2z_1y_4+y_1x_4z_2-x_4y_2z_1,\\ &x_1y_2z_3-x_1y_3z_2-x_2y_1z_3+x_2y_3z_1+x_3y_1z_2-x_3y_2z_1).\end{aligned}$$

For a hypersurface $\mathbf{M}$ in $\mathbb{E}^4_1$, the first fundamental form matrix is $\mathbf{I}=\left(\ g_{ij}\ \right)_{3\times3}$, and $\det\mathbf{I}=\det\left(\ g_{ij}\ \right)$, and also the second fundamental form matrix is $\mathbf{II}=\left(\ h_{ij}\ \right)_{3\times3}$, and $\det\mathbf{II}=\det\left(\ h_{ij}\ \right)$, where $1\le i,j\le3$, $g_{11}=\mathbf{M}_u\cdot\mathbf{M}_u$, $g_{12}=\mathbf{M}_u\cdot\mathbf{M}_v$,..., $g_{33}=\mathbf{M}_w\cdot\mathbf{M}_w$, and $h_{11}=\mathbf{M}_{uu}\cdot\mathbf{G}$, $h_{12}=\mathbf{M}_{uv}\cdot\mathbf{G}$,..., $h_{33}=\mathbf{M}_{ww}\cdot\mathbf{G}$, and some partial differentials we represent are $\mathbf{M}_u=\frac{\partial\mathbf{M}}{\partial u}$, $\mathbf{M}_{uw}=\frac{\partial^2\mathbf{M}}{\partial u\partial w}$,

$$\mathbf{G}=\frac{\mathbf{M}_u\times\mathbf{M}_v\times\mathbf{M}_w}{\|\mathbf{M}_u\times\mathbf{M}_v\times\mathbf{M}_w\|}$$

is the Gauss map. $\left(\ g_{ij}\ \right)^{-1}\left(\ h_{ij}\ \right)$ gives the matrix of shape operator (i.e., Weingarten map) $\mathbf{S}=\frac{1}{\det\mathbf{I}}\left(\ s_{ij}\ \right)_{3\times3}$. Therefore, we get the Gaussian and the mean curvature formulas, respectively, as follows:

$$K=\det(\mathbf{S})=\frac{\det\mathbf{II}}{\det\mathbf{I}}, \tag{4}$$

and

$$H=\frac{1}{3}tr\,(\mathbf{S})\,. \tag{5}$$

3. Dini-Type Helicoidal Hypersurface with a Timelike Axis

Taking $f(u) = \sin u$ in (3), we define Dini-type helicoidal hypersurface with a timelike axis in $\mathbb{E}^4_1$, as follows:

$$\mathbf{D}(u,v,w) = \begin{pmatrix} \sin u \cos v \cos w \\ \sin u \sin v \cos w \\ \sin u \sin w \\ \varphi(u) + av + bw \end{pmatrix}, \tag{6}$$

where $u \in \mathbb{R}\backslash\{0\}$ and $0 \leq v, w \leq 2\pi$.

Computing the first differentials of (6) depend on u, v, w, we obtain the first quantities as follows:

$$\mathbf{I} = \begin{pmatrix} \cos^2 u - \varphi'^2 & -a\varphi' & -b\varphi' \\ -a\varphi' & \sin^2 u \cos^2 w - a^2 & -ab \\ -b\varphi' & -ab & \sin^2 u - b^2 \end{pmatrix},$$

and have

$$\det \mathbf{I} = -\sin^2 u \left\{ \varphi'^2 \sin^2 u \cos^2 w + \left[\left(b^2 - \sin^2 u \right) \cos^2 w + a^2 \right] \cos^2 u \right\},$$

where $\varphi = \varphi(u)$, $\varphi' = \frac{d\varphi}{du}$.

By using the second differentials depend on u, v, w, we have the second quantities as follows:

$$\mathbf{II} = \begin{pmatrix} -\frac{\sin^2 u \cos w(\varphi'' \cos u + \varphi' \sin u)}{\sqrt{\|\det \mathbf{I}\|}} & \frac{a \sin u \cos^2 u \cos w}{\sqrt{\|\det \mathbf{I}\|}} & \frac{b \sin u \cos^2 u \cos w}{\sqrt{\|\det \mathbf{I}\|}} \\ \frac{a \sin u \cos^2 u \cos w}{\sqrt{\|\det \mathbf{I}\|}} & \frac{\sin^2 u \cos^2 w (b \cos u \sin w - \varphi' \sin u \cos w)}{\sqrt{\|\det \mathbf{I}\|}} & -\frac{a \sin^2 u \cos u \sin w}{\sqrt{\|\det \mathbf{I}\|}} \\ \frac{b \sin u \cos^2 u \cos w}{\sqrt{\|\det \mathbf{I}\|}} & -\frac{a \sin^2 u \cos u \sin w}{\sqrt{\|\det \mathbf{I}\|}} & -\frac{\varphi' \sin^3 u \cos w}{\sqrt{\|\det \mathbf{I}\|}} \end{pmatrix},$$

and we get

$$\det \mathbf{II} = \frac{\begin{pmatrix} -\varphi'^2 \varphi'' \sin^8 u \cos u \cos^5 w + b\varphi' \varphi'' \sin^7 u \cos^2 u \sin w \cos^4 w \\ +a^2 \varphi'' \sin^6 u \cos^3 u \sin^2 w \cos w - \varphi'^3 \sin^9 u \cos^5 w \\ +b\varphi'^2 \sin^8 u \cos u \cos^4 w \sin w \\ + \left(a^2 \left(\cos^2 u \cos^2 w + \sin^2 u \sin^2 w\right) - b^2 \cos^4 u\right) \varphi' \sin^5 u \cos^2 u \cos w \\ -b \left(2a^2 + b^2 \cos^2 w\right) \sin^4 u \cos^5 u \sin w \cos^2 w \end{pmatrix}}{(\det \mathbf{I})^{3/2}}.$$

The Gauss map of a helicoidal hypersurface with a timelike axis is

$$e_{\mathbf{D}} = \frac{1}{\sqrt{\det \mathbf{I}}} \begin{pmatrix} (\varphi' \sin u \cos v \cos^2 w - a \cos u \sin v - b \cos u \cos v \sin w \cos w) \sin u \\ (\varphi' \sin u \sin v \cos^2 w + a \cos u \cos v - b \cos u \sin v \sin w \cos w) \sin u \\ (\varphi' \sin u \sin w + b \cos u \cos w) \sin u \cos w \\ \sin^2 u \cos u \cos w \end{pmatrix}.$$

Finally, we have the Gaussian curvature of a helicoidal hypersurface with a timelike axis as follows:

$$K = \frac{\alpha_1 \varphi'^2 \varphi'' + \alpha_2 \varphi' \varphi'' + \alpha_3 \varphi'' + \alpha_4 \varphi'^3 + \alpha_5 \varphi'^2 + \alpha_6 \varphi' + \alpha_7}{(\det \mathbf{I})^{5/2}},$$

where

$\alpha_1 = -\sin^8 u \cos u \cos^5 w,$
$\alpha_2 = b \sin^7 u \cos^2 u \sin w \cos^4 w,$
$\alpha_3 = a^2 \sin^8 u \cos^3 u \sin^2 w \cos w,$
$\alpha_4 = -\sin^9 u \cos^5 w,$
$\alpha_5 = b \sin^8 u \cos u \sin w \cos^4 w,$

$\alpha_6 = \left(a^2\left(\cos^2 u \cos^2 w + \sin^2 u \sin^2 w\right) - b^2 \cos^4 u\right) \sin^5 u \cos^2 u \cos w, :$
$\alpha_7 = -b(2a^2 + b^2 \cos^2 w) \sin^4 u \cos^5 u \sin w \cos^2 w.$

Then we calculate the mean curvature of a helicoidal hypersurface with a timelike axis as follows:

$$H = \frac{\beta_1 \varphi'' + \beta_2 \varphi'^3 + \beta_3 \varphi'^2 + \beta_4 \varphi' + \beta_5}{3(\det \mathbf{I})^{3/2}},$$

where

$\beta_1 = -\left(\left(b^2 - \sin^2 u\right) \cos^2 w + a^2\right) \sin^4 u \cos u \cos w,$
$\beta_2 = -2 \sin^5 u \cos^3 w,$
$\beta_3 = -b \sin^4 u \cos u \sin w \cos^2 w,$
$\beta_4 = -\left(\left(\left(b^2 + \cos^2 u\right)^2 - 1\right) \cos^2 w + a^2\left(2\cos^2 u + 1\right)\right) \sin^3 u \cos w,$
$\beta_5 = b(\left(b^2 - \sin^2 u\right) \cos^2 w + 2a^2) \sin^2 u \cos^3 u \sin w.$

Therefore, we get the following theorems about flatness and minimality of the hypersurface.

Theorem 1. *Let* $\mathbf{D} : M^3 \longrightarrow \mathbb{E}^4_1$ *be an isometric immersion given by* (6). *Then* M^3 *is flat if and only if*

$$\alpha_1 \varphi'^2 \varphi'' + \alpha_2 \varphi' \varphi'' + \alpha_3 \varphi'' + \alpha_4 \varphi'^3 + \alpha_5 \varphi'^2 + \alpha_6 \varphi' + \alpha_7 = 0. \tag{7}$$

Theorem 2. *Let* $\mathbf{D} : M^3 \longrightarrow \mathbb{E}^4_1$ *be an isometric immersion given by* (6). *Then* M^3 *is minimal if and only if*

$$\beta_1 \varphi'' + \beta_2 \varphi'^3 + \beta_3 \varphi'^2 + \beta_4 \varphi' + \beta_5 = 0. \tag{8}$$

Solving these two equations is an attractive problem.
In the next two propositions, we will use the function

$$\varphi(u) = \cos u + \log\left(\tan \frac{u}{2}\right) \tag{9}$$

as in Dini helicoidal surface used by Ulisse Dini in Euclidean 3-space, and its following derivatives

$$\varphi'(u) = \frac{\tan^2 \frac{u}{2} - 2 \sin u \tan \frac{u}{2} + 1}{2 \tan \frac{u}{2}} \tag{10}$$

and

$$\varphi''(u) = \frac{\tan^4 \frac{u}{2} - 4 \cos u \tan^2 \frac{u}{2} - 1}{4 \tan^2 \frac{u}{2}}. \tag{11}$$

Proposition 1. *Let* $\mathbf{D}$ *is Dini-type flat hypersurface with a timelike axis (i.e.* $K = 0$*) in Minkowski 4-space. Using the function* (9) *and its derivatives* (10), (11) *and substituting them into the* (7) *in Theorem 1, we obtain*

$$\sum_{i=0}^{8} A_i \tan^i\left(\frac{u}{2}\right) = 0,$$

where

$A_8 = \alpha_1,$
$A_7 = -4\alpha_1 \sin u + 2\alpha_2 + 2\alpha_4,$
$A_6 = \left(2 + 4\sin^2 u - 4\cos u\right) \alpha_1 - 4\alpha_2 \sin u + 4\alpha_3 - 12\alpha_4 \sin u + 4\alpha_5 u,$
$A_5 = (16 \cos u - 4)\, \alpha_1 \sin u + (-8 \cos u + 2)\, \alpha_2 + \left(24 \sin^2 u + 6\right) \alpha_4 - 16\alpha_5 \sin u + 8\alpha_6,$
$A_4 = \left(-16 \sin^2 u - 8\right) \alpha_1 \cos u + 16\alpha_2 \cos u \sin u - 16\alpha_3 \cos u$
$\quad + \left(-16 \sin^2 u - 24\right) \alpha_4 \sin u + \left(16 \sin^2 u + 8\right) \alpha_5 - 16\alpha_6 \sin u + 16\alpha_7$
$A_3 = (16 \cos u + 4)\, \alpha_1 \sin u + (-8 \cos u - 2)\, \alpha_2 + \left(24 \sin^2 u + 6\right) \alpha_4 - 16\alpha_5 \sin u + 8\alpha_6,$

$$\begin{aligned}
A_2 &= \left(-2-4\sin^2 u-4\cos u\right)\alpha_1+4\alpha_2\sin u-4\alpha_3-12\alpha_4\sin u+4\alpha_5,\\
A_1 &= 4\alpha_1\sin u-2\alpha_2+2\alpha_4,\\
A_0 &= -\alpha_1.
\end{aligned}$$

Proposition 2. *Let* $\mathbf{D}$ *is Dini-type minimal helicoidal hypersurface with a timelike axis (i.e.,* $H=0$*) in Minkowski 4-space. Using the function (9) and its derivatives (10), (11) and substituting them into the (8) in Theorem 2, we get*

$$\sum_{i=0}^{6} B_i \tan^i\left(\frac{u}{2}\right)=0,$$

where

$$\begin{aligned}
B_6 &= \beta_2,\\
B_5 &= 2\beta_1-6\beta_2\sin u+2\beta_3,\\
B_4 &= \left(3+12\sin^2 u\right)\beta_2-8\beta_3\sin u+4\beta_4,\\
B_3 &= -8\beta_1\cos u-\left(12\sin u+8\sin^3 u\right)\beta_2+\left(4+8\sin^2 u\right)\beta_3-8\beta_4\sin u+8\beta_5,\\
B_2 &= \left(3+12\sin^2 u\right)\beta_2-8\beta_3\sin u+4\beta_4,\\
B_1 &= -2\beta_1-6\beta_2\sin u+2\beta_3,\\
B_0 &= \beta_2.
\end{aligned}$$

Corollary 1. *From the Proposition 1, and the Proposition 2, we obtain following special symmetries of* $\mathbf{D}$*, respectively,*

$$A_8\sim A_0,\quad A_7\sim A_1,\quad A_6\sim A_2,\quad A_5\sim A_3,$$

and

$$B_6=B_0,\quad B_5\sim B_1,\quad B_4=B_2,$$

where "$\sim$*" means the* α_i $(i=1,2,...,7)$ *and* β_j $(j=1,2,...,5)$ *term coefficients which ignored signs, respectively, are equal.*

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