

Delannoy Numbers and Preferential Arrangements

Kwang-Wu Chen [†] 

Department of Mathematics, University of Taipei, Taipei 10048, Taiwan;
kwchen@uTaipei.edu.tw or kwchen66@gmail.com

[†] Current address: No. 1, Ai-Guo West Road, Taipei 10048, Taiwan.

Received: 25 January 2019; Accepted: 2 March 2019; Published: 6 March 2019



Abstract: A preferential arrangement on $[n] = \{1, 2, \dots, n\}$ is a ranking of the elements of $[n]$ where ties are allowed. The number of preferential arrangements on $[n]$ is denoted by r_n . The Delannoy number $D(m, n)$ is the number of lattice paths from $(0, 0)$ to (m, n) in which only east $(1, 0)$, north $(0, 1)$, and northeast $(1, 1)$ steps are allowed. We establish a symmetric identity among the numbers r_n and $D(p, q)$ by means of algebraic and combinatorial methods.

Keywords: preferential arrangements; Delannoy numbers; harmonic algebras

MSC: 05A18; 05A19; 11M32

1. Introduction

A preferential arrangement on $[n] = \{1, 2, \dots, n\}$ is a ranking of the elements of $[n]$ where ties are allowed. We denote $PA(n)$ as the set of all preferential arrangements on $[n]$. For example,

$$PA(3) = \left\{ \begin{array}{l} 1 > 2 > 3, \quad 1 > 3 > 2, \quad 2 > 1 > 3, \quad 2 > 3 > 1, \quad 3 > 1 > 2, \\ 3 > 2 > 1, \quad 1 = 2 > 3, \quad 3 > 1 = 2, \quad 1 = 3 > 2, \quad 2 > 1 = 3, \\ 2 = 3 > 1, \quad 1 > 2 = 3, \quad 1 = 2 = 3 \end{array} \right\}.$$

The number of $PA(n)$ is denoted by r_n . These numbers r_n are also called the Fubini numbers [1], the ordered Bell numbers [2] (Section 5.2), or surjection number [3] (p. 109). They count the number of weak orderings on a set of n elements. The term “preferential arrangement” was first introduced by Gross [4]. The explicit expression of r_n is usually given by [5] (Equation (13))

$$r_n = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k!, \quad (1)$$

where $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is the Stirling numbers of the second kind. The recursive relation and the generating function

$$r_n = \delta_n + \sum_{k=0}^{n-1} \binom{n}{k} r_k, \quad r(z) = \frac{1}{2 - e^z}$$

were first given by Cayley [6] in 1859. As a preferential arrangement is nothing else than a sequence of non-empty sets, this directly gives the generating function $\text{SEQ}(\text{SET}_{\geq 1}(\mathcal{Z})) = 1/(2 - \exp(z))$, and this also explains the recurrence and the link with Stirling numbers (see e.g., [3] (p. 109)). Accordingly, this number r_n has been given various interpretations and has also been connected to a number of well-known combinatorial sequences [5,7–9].

The Delannoy number $D(m, n)$ is defined for nonnegative integers m and n by [1] (p. 81):

$$D(m, n) = \begin{cases} 1, & \text{if } m \cdot n = 0, \\ D(m-1, n) + D(m-1, n-1) + D(m, n-1), & \text{if } m \cdot n \neq 0. \end{cases}$$

The explicit expression of $D(m, n)$ and the generating function [1] are given by:

$$D(m, n) = \sum_{k=0}^m \binom{m}{k} \binom{m+n-k}{m}, \quad \sum_{m, n \geq 0} D(m, n) x^m y^n = \frac{1}{1-x-y-xy}.$$

The significances of these numbers are explained in [10,11]. Recently, Sun [12] and Liu, Li, & Wang [13] investigated some congruences relations on them. Qi, Čerňanová, Shi, & Guo [14] established several explicit expressions, including determinantal expressions. Moreover, Delannoy numbers are related to some adic dynamical systems [15].

Based on some algebraic identities in Hoffman's harmonic algebra, we obtained an interesting identity which connects the numbers r_n and $D(m, n)$:

Main Theorem. For a pair of nonnegative integers n and m , we have:

$$\sum_{p=0}^m \left\{ \begin{matrix} m \\ p \end{matrix} \right\} p! D(p, n) = \sum_{q=0}^n \left[\begin{matrix} n \\ q \end{matrix} \right] \frac{(-1)^{n-q}}{n!} r_{m+q}, \quad (2)$$

where $\left[\begin{matrix} n \\ q \end{matrix} \right]$ and $\left\{ \begin{matrix} m \\ p \end{matrix} \right\}$ are the Stirling numbers of the first kind and the second kind, respectively.

Our paper is organized as follows. In Section 2, we present some algebraic preliminaries on Hoffman's harmonic algebras and provide the basic identity which we will use. In Section 3, we prove the Main Theorem and also establish some more formulas. We introduce combinatorial viewpoints to approach the Main Theorem in the last section.

2. Some Preliminaries on Harmonic Algebras

We summarize the algebraic setup introduced by Hoffman [16,17] as follows. Let us consider the coding of multi-indices $\alpha = (\alpha_1, \dots, \alpha_r)$, where α_i are positive integers and $\alpha_r > 1$, by words (that is, by monomials in non-commutative variables) over the alphabet $X = \{x, y\}$ by the rule

$$\alpha \mapsto x_\alpha = xy^{\alpha_1-1}xy^{\alpha_2-1} \dots xy^{\alpha_r-1}.$$

The weight (or the degree) $|x_\alpha| := |\alpha|$ coincides with the total degree of the monomial x_α , whereas the length (or the depth) $l(x_\alpha) := l(\alpha)$ is the degree with respect to the variable x .

Let $\mathbb{Q}\langle X \rangle = \mathbb{Q}\langle x, y \rangle$ be the \mathbb{Q} -algebra of polynomials in two non-commutative variables, which is graded by the degree (where each of the variables x and y is assumed to be of degree 1); we identify the algebra $\mathbb{Q}\langle X \rangle$ with the graded \mathbb{Q} -vector space \mathfrak{H} spanned by the monomials in the variables x and y ([16]).

We also introduce the graded \mathbb{Q} -vector spaces $\mathfrak{H}^1 = \mathbb{Q}\mathbf{1} \oplus x\mathfrak{H}$ and $\mathfrak{H}^0 = \mathbb{Q}\mathbf{1} \oplus x\mathfrak{H}y$, where $\mathbf{1}$ denotes the unit (the empty word of weight 0 and length 0) of the algebra $\mathbb{Q}\langle X \rangle$. Then, the space \mathfrak{H}^1 can be regarded as the subalgebra of $\mathbb{Q}\langle X \rangle$ generated by the words $z_j = xy^{j-1}$.

Let us define a bilinear product $*$ (the stuffle product or harmonic product) on \mathfrak{H}^1 by the rules

$$1 * w = w * 1 = w, \quad (3)$$

for any word w , and

$$z_j u * z_k v = z_j(u * z_k v) + z_k(z_j u * v) + z_{j+k}(u * v), \quad (4)$$

for any words u, v , any letters $x_i = x$ or y ($i = 1, 2$), and any generators z_j, z_k of the subalgebra \mathfrak{H}^1 , and then extend the above rules to the whole subalgebra \mathfrak{H}^1 by linearity. It is known that each of the above products is commutative and associative. We denote the algebra $(\mathfrak{H}^1, +, *)$ by \mathfrak{H}_*^1 , and we call it a harmonic algebra (or Hoffman's harmonic algebra). For our convenience, we let

$$\underbrace{z_n \cdots z_n}_k = z_n^k, \quad \underbrace{z_n * z_n * \cdots * z_n}_k = z_{*n}^k, \quad \text{and} \quad \binom{k}{\alpha_1, \alpha_2, \dots, \alpha_r} = \frac{k!}{\alpha_1! \alpha_2! \cdots \alpha_r!}.$$

Lemma 1. Let n, k be positive integers. Then:

$$z_{*n}^k = \sum_{r=1}^k \sum_{|\alpha|=k} \binom{k}{\alpha_1, \alpha_2, \dots, \alpha_r} z_{n\alpha_1} z_{n\alpha_2} \cdots z_{n\alpha_r}, \quad (5)$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r)$ is a r -tuple of positive integers.

Proof. We use an induction on k to prove Equation (5).

$$\begin{aligned} z_{*n}^k &= z_{*n}^{k-1} * z_n \\ &= \sum_{r=1}^{k-1} \sum_{|\alpha|=k-1} \binom{k-1}{\alpha_1, \alpha_2, \dots, \alpha_r} z_{n\alpha_1} z_{n\alpha_2} \cdots z_{n\alpha_r} * z_n. \end{aligned}$$

The above identity follows from the induction hypothesis. Since

$$\begin{aligned} &z_{n\alpha_1} \cdots z_{n\alpha_r} * z_n \\ &= z_n z_{n\alpha_1} \cdots z_{n\alpha_r} + z_{n\alpha_1} z_n z_{n\alpha_2} \cdots z_{n\alpha_r} + \cdots + z_{n\alpha_1} \cdots z_{n\alpha_r} z_n \\ &\quad + z_{n(\alpha_1+1)} z_{n\alpha_2} \cdots z_{n\alpha_r} + z_{n\alpha_1} z_{n(\alpha_2+1)} z_{n\alpha_3} \cdots z_{n\alpha_r} + \cdots + z_{n\alpha_1} \cdots z_{n\alpha_{r-1}} z_{n(\alpha_r+1)}, \end{aligned}$$

we have:

$$\begin{aligned} z_{*n}^k &= \sum_{r=1}^{k-1} \sum_{|\alpha|=k-1} \binom{k-1}{\alpha_1, \alpha_2, \dots, \alpha_r} (z_n z_{n\alpha_1} \cdots z_{n\alpha_r} + \cdots + z_{n\alpha_1} \cdots z_{n\alpha_r} z_n \\ &\quad + (z_{n(\alpha_1+1)} z_{n\alpha_2} \cdots z_{n\alpha_r} + \cdots + z_{n\alpha_1} \cdots z_{n\alpha_{r-1}} z_{n(\alpha_r+1)})) . \end{aligned}$$

The former summand has $r+1$ instances of z_i and the latter summand has r instances of z_i in each summation. We rewrite the summation such that each summand has the same instances of z_i .

$$\begin{aligned} z_{*n}^k &= z_{nk} + \sum_{r=1}^{k-2} \left(\sum_{|\alpha|=k-1} \binom{k-1}{\alpha_1, \dots, \alpha_r} (z_n z_{n\alpha_1} \cdots z_{n\alpha_r} + \cdots + z_{n\alpha_1} \cdots z_{n\alpha_r} z_n) \right. \\ &\quad \left. + \sum_{|\alpha|=k-1} \binom{k-1}{\alpha_1, \dots, \alpha_{r+1}} (z_{n(\alpha_1+1)} \cdots z_{n\alpha_{r+1}} + \cdots + z_{n\alpha_1} \cdots z_{n(\alpha_{r+1}+1)}) \right) + k! z_n^k. \end{aligned}$$

We simplify the summation in the above identity as follows.

$$\begin{aligned} & \sum_{r=1}^{k-2} \left(\sum_{|\alpha|=k} \left(\sum_{i=1}^{r+1} \frac{(k-1)!}{\alpha_1! \cdots \alpha_{i-1}! (\alpha_i - 1)! \alpha_{i+1}! \cdots \alpha_{r+1}!} \right) z_{n\alpha_1} \cdots z_{n\alpha_{r+1}} \right) \\ &= \sum_{r=1}^{k-2} \left(\sum_{|\alpha|=k} \binom{k-1}{\alpha_1, \dots, \alpha_{r+1}} (\alpha_1 + \cdots + \alpha_{r+1}) z_{n\alpha_1} \cdots z_{n\alpha_{r+1}} \right) \\ &= \sum_{r=1}^{k-2} \left(\sum_{|\alpha|=k} \binom{k}{\alpha_1, \dots, \alpha_{r+1}} z_{n\alpha_1} \cdots z_{n\alpha_{r+1}} \right). \end{aligned}$$

Combining the first term z_{nk} and the last term $k!z_n^k$, we get our conclusion. \square

We define a rational linear map: $\zeta : \mathfrak{H}_*^0 \rightarrow \mathbb{R}$ by $\zeta(\mathbf{1}) = 1$ and

$$\zeta(z_{\alpha_1} z_{\alpha_2} \cdots z_{\alpha_r}) = \zeta(\alpha_1, \alpha_2, \dots, \alpha_r) := \sum_{1 \leq k_1 < k_2 < \cdots < k_r} k_1^{-\alpha_1} k_2^{-\alpha_2} \cdots k_r^{-\alpha_r},$$

where $\zeta(\alpha_1, \alpha_2, \dots, \alpha_r)$ is the multiple zeta value (MZV) [18–20]. Since $z_{\alpha_1} z_{\alpha_2} \cdots z_{\alpha_r} \in \mathfrak{H}_*^0$, we have $\alpha_r \geq 2$. Thus, this guarantees that $\zeta(\alpha_1, \dots, \alpha_r)$ is well-defined. Then, this map is an algebra homomorphism [21]: $\zeta(w_1 * w_2) = \zeta(w_1)\zeta(w_2)$. If we apply this map to the result of the above lemma, then we have the following identity:

$$\zeta(n)^k = \sum_{r=1}^k \sum_{\substack{|\alpha|=k \\ \alpha_i \geq 1}} \binom{k}{\alpha_1, \dots, \alpha_r} \zeta(n\alpha_1, \dots, n\alpha_r),$$

where n, k are positive integers and $n \geq 2$. This result was recently proved in [22] (Theorem 1.3) by another method. In fact, this result can be also obtained from [21] (Proposition 3) and [23] (Proposition 4), and it is also true for $n = 1$, provided one uses shuffle-regularized MZVs.

It is worth noting that \mathfrak{H}_*^1 is isomorphic to the algebra QSym of quasi-symmetric functions, and QSym has the algebra Sym of symmetric functions as a subalgebra [24]. The well-known identity

$$m_1^k = \sum_{\lambda \vdash k} \binom{k}{\lambda} m_\lambda$$

of symmetric functions (which follows from the multinomial theorem), where m_λ is the monomial symmetric function corresponding to the partition λ . We define a monomorphism $\text{Sym} \rightarrow \mathfrak{H}_*^1$ sending m_1 to z_n . We could use this map to send the above identity to Equation (5).

We use the result in Lemma 1 to get some relations between Delannoy numbers and preferential arrangements in the next section.

3. Preferential Arrangements and Delannoy Numbers

The definition of the stuffle product $*$ indicates that the stuffle product of two multiple zeta values of depth m and n will produce $D(m, n)$ numbers of multiple zeta values ([25]). We will give another proof using a combinatorial approach in the next section.

The Delannoy number $D(m, n)$ can be viewed as the number of lattice paths from $(0, 0)$ to (m, n) in which only east $(1, 0)$, north $(0, 1)$, and northeast $(1, 1)$ steps are allowed. The lattice paths described here are called Delannoy paths which give an alternative characterization of the stuffle product.

By counting the number of multiple zeta values in Equation (5) produced from the stuffle product, we obtain the following interesting identity which connects r_n and $D(m, n)$.

Proposition 1. For a pair of nonnegative integers n and k , we have

$$r_{n+k} = \sum_{p=0}^k \sum_{q=0}^n \left\{ \begin{matrix} k \\ p \end{matrix} \right\} \left\{ \begin{matrix} n \\ q \end{matrix} \right\} p!q!D(p, q). \quad (6)$$

where $\left\{ \begin{matrix} m \\ p \end{matrix} \right\}$ are the Stirling numbers of the second kind.

Proof. There are

$$\sum_{r=1}^k \sum_{\substack{|\alpha|=k \\ \alpha_i \geq 1}} \binom{k}{\alpha_1, \alpha_2, \dots, \alpha_r}$$

terms in the right-hand side of Equation (5). Since $\sum_{\substack{|\alpha|=k \\ \alpha_i \geq 0}} \binom{k}{\alpha_1, \dots, \alpha_r} = r^k$, we have

$$\sum_{\substack{|\alpha|=k \\ \alpha_i \geq 1}} \binom{k}{\alpha} = \sum_{\substack{|\alpha|=k \\ \alpha_i \geq 1}} \binom{k}{\alpha_1, \alpha_2, \dots, \alpha_r} = \sum_{j=1}^r (-1)^{r-j} \binom{r}{j} j^k$$

by the inclusion–exclusion principle. By [26] (Equation (6.19)),

$$\sum_{j=1}^r (-1)^{r-j} \binom{r}{j} j^k = r! \left\{ \begin{matrix} k \\ r \end{matrix} \right\},$$

we can write the above number as

$$\sum_{\substack{|\alpha|=k \\ \alpha_i \geq 1}} \binom{k}{\alpha} = r! \left\{ \begin{matrix} k \\ r \end{matrix} \right\}. \quad (7)$$

For $1 \leq \ell < k$,

$$\begin{aligned} & \sum_{r=1}^k \sum_{|\alpha|=k} \binom{k}{\alpha} z_{n\alpha_1} z_{n\alpha_2} \cdots z_{n\alpha_r} \\ &= z_{*n}^k = z_{*n}^\ell * z_{*n}^{k-\ell} \\ &= \sum_{p=1}^\ell \sum_{|\beta|=\ell} \binom{\ell}{\beta} z_{n\beta_1} z_{n\beta_2} \cdots z_{n\beta_p} * \sum_{q=1}^{k-\ell} \sum_{|\lambda|=k-\ell} \binom{k-\ell}{\lambda} z_{n\lambda_1} z_{n\lambda_2} \cdots z_{n\lambda_q} \\ &= \sum_{p=1}^\ell \sum_{|\beta|=\ell} \sum_{q=1}^{k-\ell} \sum_{|\lambda|=k-\ell} \binom{\ell}{\beta} \binom{k-\ell}{\lambda} z_{n\beta_1} z_{n\beta_2} \cdots z_{n\beta_p} * z_{n\lambda_1} z_{n\lambda_2} \cdots z_{n\lambda_q}. \end{aligned}$$

Since the stuffle product of two MZVs of depth p and q produces $D(p, q)$ numbers of MZVs, we count the numbers of MZVs in the above identity, and then we have:

$$\sum_{r=1}^k \sum_{\substack{|\alpha|=k \\ \alpha_i \geq 1}} \binom{k}{\alpha} = \sum_{p=1}^\ell \sum_{\substack{|\beta|=\ell \\ \beta_i \geq 1}} \sum_{q=1}^{k-\ell} \sum_{\substack{|\lambda|=k-\ell \\ \lambda_i \geq 1}} \binom{\ell}{\beta} \binom{k-\ell}{\lambda} D(p, q).$$

Combining Equations (1) and (7) and the special values of the Stirling numbers of the second kind at zeros, i.e., $\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = 0$ whenever $n > 0$, we conclude the following result:

$$r_{n+k} = \sum_{p=0}^k \sum_{q=0}^n \left\{ \begin{matrix} k \\ p \end{matrix} \right\} \left\{ \begin{matrix} n \\ q \end{matrix} \right\} p!q!D(p, q).$$

This completes our proof. \square

Our Main Theorem is just the Stirling inversion applied to Equation (6).

Main Theorem. For a pair of nonnegative integers n and m , we have:

$$\sum_{p=0}^m \left\{ \begin{matrix} m \\ p \end{matrix} \right\} p! D(p, n) = \sum_{q=0}^n \left[\begin{matrix} n \\ q \end{matrix} \right] \frac{(-1)^{n-q}}{n!} r_{m+q},$$

where $\left[\begin{matrix} n \\ q \end{matrix} \right]$ and $\left\{ \begin{matrix} m \\ p \end{matrix} \right\}$ are the Stirling numbers of the first kind and the second kind, respectively.

Proof. For any pair of sequences, f_n and g_n , if they are related by

$$g_n = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} f_k,$$

then, they have an inversion formula given by [26]

$$f_n = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right] (-1)^{n-k} g_k. \quad (8)$$

We apply this inversion formula to Equation (6) with $g_q = r_{m+q}$, $f_q = \sum_{p=0}^m \left\{ \begin{matrix} m \\ p \end{matrix} \right\} p! q! D(p, q)$. Then, we have:

$$\sum_{p=0}^m \left\{ \begin{matrix} m \\ p \end{matrix} \right\} p! D(p, n) = \sum_{q=0}^n \left[\begin{matrix} n \\ q \end{matrix} \right] \frac{(-1)^{n-q}}{n!} r_{m+q}.$$

This completes our proof. \square

Applying the inversion formula Equation (8) again to the above identity, we have

$$(-1)^{n+m} n! m! D(n, m) = \sum_{p=0}^n \sum_{q=0}^m \left[\begin{matrix} n \\ p \end{matrix} \right] \left[\begin{matrix} m \\ q \end{matrix} \right] (-1)^{p+q} r_{p+q}. \quad (9)$$

If we set $k = 0$ in Equation (6), we get the original formula, Equation (1). Moreover, if we set $k = 1$ in Equation (6) and use the fact $D(1, n) = 2n + 1$, then we have the following identity (see [5] (Equation (29))):

$$r_{n+1} = \sum_{q=0}^n \left\{ \begin{matrix} n \\ q \end{matrix} \right\} q! (2q + 1) = \sum_{q=0}^n \left\{ \begin{matrix} n \\ q \end{matrix} \right\} q! (2(q + 1) - 1) = 2s_n - r_n,$$

where the number s_n is defined by:

$$s_n = \sum_{q=0}^n \left\{ \begin{matrix} n \\ q \end{matrix} \right\} (q + 1)!$$

called "barred preferential arrangements of n elements", introduced by Pippenger [5]. s_n is the number of ways of ranking $\llbracket n \rrbracket$, with ties allowed, and with a "bar" that may be placed above all the elements of $\llbracket n \rrbracket$, between two equivalence classes of tied members, or below all the members. Some relations between s_n and r_n were derived in [5] [Equations (24), (28), and (29)].

4. Combinatorial Approach

In this section, we give a combinatorial approach to prove Proposition 1. First, we connect monomials $z_{k_1} z_{k_2} \cdots z_{k_\ell}$ in \mathfrak{H}_*^1 to a preferential arrangement.

Proposition 2. For positive integers n_1, \dots, n_k , the product $z_{n_1} * z_{n_2} * \cdots * z_{n_k}$ is a sum of r_k monomials, including $\left\{ \begin{matrix} k \\ p \end{matrix} \right\} p!$ monomials of length p for $1 \leq p \leq k$.

Proof. We associate any monomials $z_{k_1} z_{k_2} \cdots z_{k_\ell}$ in the product $z_{n_1} * z_{n_2} * \cdots * z_{n_k}$ to a preferential arrangement on $\llbracket k \rrbracket$ as follows. If the factor z_{n_i} appears before the factor z_{n_j} in a monomial, then it means that $i > j$. If the factor $z_{n_i+n_j}$ appears in a monomial, then it means that $i = j$. For example, the monomial $z_{n_2} z_{n_1+n_4} z_{n_3}$ means that the preferential arrangement is $2 > 1 = 4 > 3$.

From the stuffle product rule in Equation (4) we know that if u, v are any possible words,

$$z_{n_j} u * z_{n_\ell} v = z_{n_j} (u * z_{n_\ell} v) + z_{n_\ell} (z_{n_j} u * v) + z_{n_j+n_\ell} (u * v).$$

The corresponding result gives us all situations of Candidate j and Candidate ℓ if we rank j and ℓ : the term $z_{n_j} (u * z_{n_\ell} v)$ means $j > \ell$, the term $z_{n_\ell} (z_{n_j} u * v)$ means $\ell > j$, and the term $z_{n_j+n_\ell} (u * v)$ means $j = \ell$. By mathematical induction, one can assert the following result: The product of $z_{n_1} * z_{n_2} * \cdots * z_{n_k}$ produces a sum of monomials, with each monomial corresponding to a possible preferential arrangement in $\llbracket k \rrbracket$. That is to say, we can regard $z_{n_1} * z_{n_2} * \cdots * z_{n_k}$ as ranking the elements $1, 2, \dots, k$ where ties are allowed.

Therefore, the product $z_{n_1} * z_{n_2} * \cdots * z_{n_k}$ is a sum of r_k monomials, including $\binom{k}{p} p!$ monomials of length p for $1 \leq p \leq k$ (see Equation (1)). \square

Secondly, we connect a monomial in a stuffle product to a Delannoy path.

Proposition 3. If u, v are monomials in \mathfrak{S}^1 of lengths n and m respectively, then $u * v$ is a sum of $D(n, m)$ monomials.

Proof. We write the monomials u and v in \mathfrak{S}^1 as:

$$u = z_{a_1} z_{a_2} \cdots z_{a_n}, \quad v = z_{b_1} z_{b_2} \cdots z_{b_m}.$$

We associate any monomial $z_{k_1} z_{k_2} \cdots z_{k_\ell}$ in the product $u * v$ to a Delannoy path from $(0, 0)$ to (n, m) as follows. If $z_{k_i} = z_{a_j}$, then we move from the standing point to the next point by the direction $(1, 0)$. If $z_{k_i} = z_{b_j}$, then we move from the standing point to the next point by the direction $(0, 1)$. If $z_{k_i} = z_{a_j+b_\ell}$, then we move from the standing point to the next point by the direction $(1, 1)$. The path begins at the point $(0, 0)$, then it follows the directions corresponding to $z_{k_1}, z_{k_2}, \dots, z_{k_\ell}$.

From the stuffle product rule in Equation (4), we know that

$$\begin{aligned} z_{a_1} z_{a_2} \cdots z_{a_n} * z_{b_1} z_{b_2} \cdots z_{b_m} &= z_{a_1} (z_{a_2} \cdots z_{a_n} * z_{b_1} z_{b_2} \cdots z_{b_m}) \\ &+ z_{b_1} (z_{a_1} z_{a_2} \cdots z_{a_n} * z_{b_2} \cdots z_{b_m}) + z_{a_1+b_1} (z_{a_2} \cdots z_{a_n} * z_{b_2} \cdots z_{b_m}). \end{aligned}$$

This indicates that there are three possible directions beginning from the point $(0, 0)$ to the next point. The first term begins with z_{a_1} , i.e., we move to the next point $(1, 0)$; the second term begins with z_{b_1} , i.e., we move to the next point $(0, 1)$; or the third term begins with $z_{a_1+b_1}$, i.e., we move to the next point $(1, 1)$. Since the number of the remaining points are less than $(n+1)(m+1)$, we use the induction hypothesis to conclude the following result.

The product of $z_{a_1} z_{a_2} \cdots z_{a_n} * z_{b_1} z_{b_2} \cdots z_{b_m}$ produces a sum of monomials, where each monomial corresponds to a possible Delannoy path from $(0, 0)$ to (n, m) . Hence, the number of the monomials is $D(n, m)$. \square

Now we give another proof of the Proposition 1. Consider the following product

$$z_{a_1} * z_{a_2} * \cdots * z_{a_n} * z_{b_1} * z_{b_2} * \cdots * z_{b_k}.$$

By Proposition 2 there are r_{n+k} monomials in this product. Also, the factor $z_{a_1} * z_{a_2} * \cdots * z_{a_n}$ has $\binom{n}{p} p!$ monomials of length p for $1 \leq p \leq n$, and the factor $z_{b_1} * z_{b_2} * \cdots * z_{b_k}$ has $\binom{k}{q} q!$ monomials of length q for $1 \leq q \leq k$.

By Proposition 3 the product $u * v$ is a sum of $D(p, q)$ monomials, where the monomial u is from the product $z_{a_1} * z_{a_2} * \cdots * z_{a_n}$ with length p , and the monomial v is from the product $z_{b_1} * z_{b_2} * \cdots * z_{b_k}$ with length q . Thus, we have

$$\sum_{p=0}^k \sum_{q=0}^n \left\{ \begin{matrix} k \\ p \end{matrix} \right\} \left\{ \begin{matrix} n \\ q \end{matrix} \right\} p!q! D(p, q)$$

monomials. By combining these two results, we have the desired identity.

Funding: This research was funded by the Ministry of Science and Technology, Taiwan, R.O.C. grant number MOST 107-2115-M-845-003.

Acknowledgments: The author would like to thank the anonymous referee for some useful comments, especially on the combinatorial viewpoints.

Conflicts of Interest: The author declares no conflict of interest.

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