mathematics

Article

Certain Geometric Properties of Lommel and Hyper-Bessel Functions

Saima Mushtaq [1], Mohsan Raza [1,*] and Muhey U Din [2]

1 Department of Mathematics, Government College University Faisalabad, Faisalabad 38000, Pakistan; saimstar20@yahoo.com
2 Department of Mathematics, Government Post Graduate Islamia College Faisalabad, Faisalabad 38000, Pakistan; muheyudin@yahoo.com
* Correspondence: mohsan976@yahoo.com

Received: 31 December 2018; Accepted: 1 March 2019; Published: 6 March 2019

Abstract: In this article, we are mainly interested in finding the sufficient conditions under which Lommel functions and hyper-Bessel functions are close-to-convex with respect to the certain starlike functions. Strongly starlikeness and convexity of Lommel functions and hyper-Bessel functions are also discussed. Some applications are also the part of our investigation.

Keywords: close-to-convexity; analytic functions; normalized lommel functions; normalized hyper-bessel functions; strongly convexity; strongly starlikeness

Mathematics Subject Classification: 30C45; 33C10

---

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. Let $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$, $\mathcal{K}(\alpha)$, $\widetilde{\mathcal{S}}^*(\alpha)$ and $\widetilde{\mathcal{C}}(\alpha)$ denote the classes of starlike, convex, close-to-convex, strongly starlike and strongly convex functions of order α, respectively, and are defined as:

$$\begin{aligned}
\mathcal{S}^*(\alpha) &= \left\{ f : f \in \mathcal{A} \text{ and } Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in \mathcal{U}, \alpha \in [0,1) \right\}, \\
\mathcal{C}(\alpha) &= \left\{ f : f \in \mathcal{A} \text{ and } Re\left(\frac{(zf'(z))'}{f'(z)}\right) > \alpha,\ z \in \mathcal{U}, \alpha \in [0,1) \right\}, \\
\mathcal{K}(\alpha) &= \left\{ f : f \in \mathcal{A} \text{ and } Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha,\ z \in \mathcal{U}, \alpha \in [0,1),\ g \in \mathcal{S}^*(0) :\equiv \mathcal{S}^* \right\}, \\
\widetilde{\mathcal{S}}^*(\alpha) &= \left\{ f : f \in \mathcal{A} \text{ and } \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2},\ z \in \mathcal{U}, \alpha \in (0,1] \right\},
\end{aligned}$$

and

$$\widetilde{\mathcal{C}}(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2},\ z \in \mathcal{U}, \alpha \in (0,1] \right\}.$$

It is clear that

$$\widetilde{\mathcal{S}^*}(1) = \mathcal{S}^*(0) = \mathcal{S}^*,\ \widetilde{\mathcal{C}}(1) = \mathcal{C}(0) = \mathcal{C} \text{ and } \mathcal{K}(0) = \mathcal{K},$$

where $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ are the classes of starlike, convex and close-to-convex functions, respectively. If f and g are analytic functions, then the function f is said to be subordinate to g, written as $f(z) \prec g(z)$, if there exist a Schwarz function w with $w(0) = 0$ and $|w| < 1$ such that $f(z) = g(w(z))$. Furthermore, if the function g is univalent in $\mathcal{U}$, then we have the following equivalent relation:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}).$$

For two functions f of the form of Equation (1) and g of the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,\ \ (z \in \mathcal{U}).$$

that are analytic in $\mathcal{U}$, we define the convolution of these functions by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,\ \ (z \in \mathcal{U}).$$

Consider the Lommel function of the first kind $\mathcal{L}_{\kappa,\tau}(z)$ is a particular solution of the in-homogeneous Bessel differential equation (see for details, [1,2]):

$$z^2 w''(z) + z w'(z) + (z^2 - \tau^2) w(z) = z^{\kappa+1}, \tag{2}$$

and it can be expressed in terms of hypergeometric series

$$s_{\kappa,\tau}(z) = \frac{z^{\kappa+1}}{(\kappa - \tau + 1)(\kappa + \tau + 1)}\ {}_1F_2\left(1; \frac{\kappa - \tau + 3}{2}, \frac{\kappa + \tau + 3}{2}; -\frac{z^2}{4}\right),$$

where $\kappa \pm \tau$ is a non-negative odd integer. It is observed that Lommel function $s_{\kappa,\tau}$ does not belongs to the class $\mathcal{A}$. Thus, the normalized Lommel function of first kind is defined as:

$$\mathcal{L}_{\kappa,\tau}(z) = (\kappa - \tau + 1)(\kappa + \tau + 1)\, z^{\frac{1-\kappa}{2}} s_{\kappa,\tau}(\sqrt{z}), \tag{3}$$

$$\mathcal{L}_{\kappa,\tau}(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{-1}{4}\right)^n}{(\mathcal{M})_n (\mathcal{N})_n} z^{n+1}. \tag{4}$$

where $\mathcal{M} = \frac{\kappa - \tau + 3}{2}$, $\mathcal{N} = \frac{\kappa + \tau + 3}{2}$ and $(a)_n$ shows the Appell symbol which defined in terms of Eulers gamma functions such that $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)...(a+n-1)$. Clearly, the function $\mathcal{L}_{\kappa,\tau}$ belongs to the class $\mathcal{A}$. To discuss the close-to-convexity of normalized Lommel functions with respect to the certain starlike functions, here we define modified form of the normalized Lommel functions

$$\begin{aligned} \mathbb{L}(z) &= \frac{z}{1+z} * \mathcal{L}_{\kappa,\tau}(z) = \sum_{n=0}^{\infty} \frac{1}{4^n (\mathcal{M})_n (\mathcal{N})_n} z^{n+1} \\ &= z + \sum_{n=1}^{\infty} \frac{1}{4^n (\mathcal{M})_n (\mathcal{N})_n} z^{n+1}. \end{aligned} \tag{5}$$

Next, we consider the hyper-Bessel function in terms of the hypergeometric functions defined below (for details see [3])

$$J_{\beta_c}(z)=\frac{\left(\frac{z}{c+1}\right)^{\beta_1+\beta_2+\ldots+\beta_c}}{\prod\limits_{i=1}^{c}\Gamma\left(\beta_i+1\right)}\,{}_0F_c\left(\overline{(\beta_c+1)};-\left(\frac{z}{c+1}\right)^{c+1}\right), \tag{6}$$

where the notation

$${}_pF_q\left(\begin{pmatrix}(\beta)_p\\(\gamma)_q\end{pmatrix};x\right)=\sum_{n=0}^{\infty}\frac{(\beta_1)_n(\beta_2)_n\cdots(\beta_p)_n}{(\gamma_1)_n(\gamma_2)_n\cdots(\gamma_p)_n}\frac{x^n}{n!}, \tag{7}$$

represents the generalized Hypergeometric functions and β_c represents the array of c parameters $\beta_1,\beta_2,...,\beta_c$. By combining Equations (6) and (7) , we get the following infinite representation of the hyper-Bessel functions

$$J_{\beta_c}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\prod\limits_{i=1}^{c}\Gamma\left(\beta_i+n+1\right)}\left(\frac{z}{c+1}\right)^{n(c+1)+\beta_1+\beta_2+\ldots+\beta_c}. \tag{8}$$

Since the function J_{β_c} is not in class $\mathcal{A}$, the normalized hyper-Bessel function $\mathcal{J}_{\beta_c}$ is defined by

$$\mathcal{J}_{\beta_c}(z)=1+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(n-1)!(c+1)^{(n-1)(c+1)}\prod\limits_{i=1}^{c}(\beta_i+1)_{n-1}}z^{(n-1)(c+1)}. \tag{9}$$

It is observed that the function $\mathcal{J}_{\beta_c}$ defined in Equation (9) does not belong to the class $\mathcal{A}$. Here, we consider the following normalized form of the hyper-Bessel function for our own convenience.

$$\mathcal{H}_{\beta_c}(z)=z\mathcal{J}_{\beta_c}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n)!(c+1)^{n(c+1)}\prod\limits_{i=1}^{c}(\beta_i+1)_n}z^{n(c+1)+1}. \tag{10}$$

To discuss the close-to-convexity of normalized hyper-Bessel functions with respect to the certain starlike functions, here we define modified form of the normalized hyper-Bessel functions

$$\begin{aligned}\mathbb{H}_{\beta_c}(z) &= \frac{z}{1+z}*z\mathcal{J}_{\beta_c}(z)=\sum_{n=0}^{\infty}\frac{1}{(n)!(c+1)^{n(c+1)}\prod\limits_{i=1}^{c}(\beta_i+1)_n}z^{n(c+1)+1}\\ &= z+\sum_{n=2}^{\infty}\frac{z^{(n-1)(c+1)+1}}{(n-1)!(c+1)^{(n-1)(c+1)}\prod\limits_{i=1}^{c}(\beta_i+1)_{n-1}}.\end{aligned} \tag{11}$$

Special functions have great importance in pure and applied mathematics. The wide use of these functions has attracted many researchers to work on the different directions. Recently, many mathematicians study the geometric properties of special functions with different aspects. For details, we refer to [4–9]. Certain conditions for close-to-convexity of some special functions such as Bessel functions, q-Mittag-Leffler functions, Wright functions, and Dini functions have been determined by many mathematicians with different methods (for details, see [4,10–13]). We need the following Lemmas to prove our results.

Lemma 1 ([14]). *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real number such that $a_1 = 1$. Suppose that, $a_1 \geq 8a_2$ and $(n-1)\,a_n \geq (n+1)\,a_{n+1}$, $\forall n \geq 2$. Then, f is close-to-convex with respect to starlike function $\frac{z}{1-z^2}$.*

Lemma 2 ([15]). *Let f have the series representation of the form of $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$. Suppose that*

$$1 \geq 2a_2 \geq ... \geq na_n \geq (n+1)a_{n+1} \geq ... \geq 0 \tag{12}$$

or

$$1 \leq 2a_2 \leq ... \leq na_n \leq (n+1)a_{n+1} \leq ... \leq 2. \tag{13}$$

Then, f is close-to-convex with respect to starlike function $\frac{z}{1-z}$.

Lemma 3 ([16]). *Let $M(z)$ be convex and univalent in the open unit disc with condition $M(0) = 1$. Let $F(z)$ be analytic in the open unit disc with condition $F(0) = 1$ and $F \prec M$ in the open unit disc. Then, $\forall\, n \in \mathbb{N} \cup \{0\}$, and we obtain*

$$(n+1)z^{-1-n}\int_0^z t^n F(t)dt \prec (n+1)z^{-1-n}\int_0^z t^n M(t)dt.$$

2. Close to Convexity of Modified Lommel Functions

In this section we discuss some conditions under which the modified Lommel functions and modified hyper-Bessel functions are assured to be close-to-convex with respect to the functions

$$\frac{z}{1-z^2} \text{ and } \frac{z}{1-z}.$$

Theorem 1. *Let $\kappa, \tau \in \mathbb{R}^+ := (0, \infty)$ and $\kappa \geq \tau$. Then, $\mathbb{L}_{\kappa,\tau}$ defined in Equation (5) is close-to-convex with respect to starlike function $\frac{z}{1-z^2}$.*

Proof. Consider

$$\mathbb{L}_{\kappa,\tau}(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

where

$$a_n = \frac{1}{4^{n-1}\,(\mathcal{M})_{n-1}\,(\mathcal{N})_{n-1}}, \quad (n \in \mathbb{N}\backslash\,\{1\})$$

with $\mathcal{M} = (\kappa - \tau + 3)/2$ and $\mathcal{N} = (\kappa + \tau + 3)/2$. Note that $\mathcal{M} > 0$ and $\mathcal{N} > 0$ by the hypothesis. It is enough to prove that a_n satisfies the hypothesis of Lemma 1. Clearly, $a_1 \geq 8a_2$ since $\mathcal{M} > 0$ and $\mathcal{N} > 0$. For $n \in \mathbb{N}\backslash\,\{1\}$, consider the following inequality

$$(n-1)a_n \geq (n+1)a_{n+1}. \tag{14}$$

Then, Equation (14) is equivalent to

$$4(n-1)\,(\mathcal{M}+n-1)\,(\mathcal{N}+n-1) - (n+1) \geq 0. \tag{15}$$

Since $\mathcal{M} > 0$ and $\mathcal{N} > 0$, the inequality in Equation (15) holds for all $n \geq 2$. Hence, $\{a_n\}_{n=1}^{\infty}$ satisfies the hypothesis of Lemma 1 which completes the proof of Theorem 1. □

Theorem 2. *Let $\kappa, \tau \in \mathbb{R}^+ := (0, \infty)$ and $\kappa \geq \tau$. Then, $\mathbb{L}_{\kappa,\tau}$ defined in Equation (5) is close-to-convex with respect to starlike function $\frac{z}{1-z}$.*

Proof. Consider

$$\mathbb{L}_{\kappa,\tau}(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

where

$$a_n = \frac{1}{4^{n-1} (\mathcal{M})_{n-1} (\mathcal{N})_{n-1}} \; (n \in \mathbb{N} \backslash \{1\}).$$

with $\mathcal{M} = (\kappa - \tau + 3)/2$ and $\mathcal{N} = (\kappa + \tau + 3)/2$. Note that $\mathcal{M} > 0$ and $\mathcal{N} > 0$ by the hypothesis. It is enough to prove that a_n satisfies the hypothesis of Lemma 2. For $n \in \mathbb{N} \backslash \{1\}$, consider the following inequality

$$n a_n \geq (n+1) a_{n+1}. \tag{16}$$

Then, Equation (16) is equivalent to

$$4n (\mathcal{M}+n-1)(\mathcal{N}+n-1) - (n+1) \geq 0.$$

Since $\mathcal{M} > 0$ and $\mathcal{N} > 0$, the inequality in Equation (16) holds for all $n \geq 2$. Hence, $\{a_n\}_{n=1}^{\infty}$ satisfies the hypothesis of Lemma 2, which completes the proof of Theorem 2. □

3. Close to Convexity of Modified Hyper-Bessel Functions

Theorem 3. *Let $i \in \{1, 2, 3, ..., c\}$, $\beta_i > -1$ and $\zeta\eta \geq 3/2$, where*

$$\zeta = (c+1)^{c+1} \text{ and } \eta = \prod_{i=1}^{c} (\beta_i + 1).$$

Then, $\mathbb{H}_{\beta_c}$ defined in Equation (11) is close-to-convex with respect to starlike function $\frac{z}{1-z^2}$.

Proof. Consider

$$\mathbb{H}_{\beta_c}(z) = z + \sum_{n=2}^{\infty} a_n z^{(n-1)(c+1)+1},$$

where $a_n = \frac{1}{(n-1)!(c+1)^{(n-1)(c+1)} \prod_{i=1}^{c} (\beta_i+1)_{n-1}}$, $\forall n \geq 2$. It is enough to prove that a_n satisfies the hypothesis of Lemma 1. Clearly, $a_1 = 1$ and $a_1 \geq 8a_2$ for $\zeta\eta + 8 > 0$. To complete the proof, we find the condition under which $(n-1)a_n - (n+1)a_{n+1} \geq 0$. Thus, take

$$\begin{aligned} (n-1)a_n &\geq (n+1)a_{n+1} \\ \frac{(n-1)}{(n-1)!(c+1)^{(n-1)(c+1)} \prod\limits_{i=1}^{c} (\beta_i+1)_{n-1}} &\geq \frac{(n+1)}{(n)!(c+1)^{n(c+1)} \prod\limits_{i=1}^{c} (\beta_i+1)_n}. \end{aligned}$$

This implies that

$$\eta (c+1)^{c+1} \geq \frac{(n+1)}{n(n-1)}, \quad \forall n \geq 2.$$

One can easily observe that $(n-1)a_n - (n+1)a_{n+1} \geq 0$ for $\zeta \geq \frac{(n+1)}{n(n-1)\eta}$, $\forall n \geq 2$. This is true when $\zeta\eta \geq 3/2$, for all $n \geq 2$. Hence, $\{a_n\}_{n=1}^{\infty}$ satisfies the hypothesis of Lemma 1, which completes the proof. □

Theorem 4. *Let $\mathbb{H}_{\beta_c}$ defined in Equation (11) satisfy the following condition:*

$$(c+1)^{c+1}\eta \geq 3/4, \ \forall n \geq 2,$$

where η is defined in Theorem 3. Then, $\mathbb{H}_{\beta_c}$ is close-to-convex with respect to starlike function $\frac{z}{1-z}$.

Proof. Let

$$\begin{aligned}\mathbb{H}_{\beta_c}(z) &= z + \sum_{n=2}^{\infty} \frac{1}{(n-1)!(c+1)^{(n-1)(c+1)} \prod_{i=1}^{c} (\beta_i+1)_{n-1}} z^{(n-1)(c+1)+1} \\ &= z + \sum_{n=2}^{\infty} a_n z^n,\end{aligned}$$

where $a_1 = 1$ and for $n \geq 2$,

$$a_n = \frac{1}{(n-1)!(c+1)^{(n-1)(c+1)} \prod_{i=1}^{c} (\beta_i+1)_{n-1}}.$$

Now,

$$\begin{aligned}na_n - (n+1)a_{n+1} &= \frac{n}{(n-1)!(c+1)^{(n-1)(c+1)} \prod_{i=1}^{c} (\beta_i+1)_{n-1}} \\ &\quad - \frac{(n+1)}{(n)!(c+1)^{n(c+1)} \prod_{i=1}^{c} (\beta_i+1)_{n}}.\end{aligned}$$

To check under which conditions the above expression is positive, consider

$$\begin{aligned}&\frac{n}{(n-1)!(c+1)^{(n-1)(c+1)} \prod_{i=1}^{c} (\beta_i+1)_{n-1}} \\ \geq\ &\frac{(n+1)}{(n)!(c+1)^{n(c+1)} \prod_{i=1}^{c} (\beta_i+1)_{n}}.\end{aligned}$$

This shows that the sequence $\{na_n\}_{n=1}^{\infty}$ is a decreasing sequence if $n^2(c+1)^{c+1}\eta \geq (n+1)$, $\forall n \geq 2$. This condition is satisfied for $(c+1)^{c+1}\eta \geq 3/4$, $\forall n \geq 2$. Thus, from Lemma 2, $\mathbb{H}_{\beta_c}$ is close-to-convex with respect to starlike function $\frac{z}{1-z}$. □

4. Strongly Convexity and Strongly Starlikeness of Lommel Functions

In this section, we are mainly interested in finding some sufficient conditions for the normalized Lommel functions to belong to the classes of strongly convex of order α and strongly starlikeness of order α functions, respectively.

Theorem 5. *Let $\kappa, \tau \in \mathbb{R}$. If $\mathcal{MN} - 2 > 0$, where $\mathcal{M} = \frac{\kappa+\tau+3}{2}$ and $\mathcal{N} = \frac{\kappa-\tau+3}{2}$, then $\mathcal{L}_{\kappa,\tau} \in \widetilde{\mathcal{C}}(\alpha)$, where*

$$\alpha = \frac{2}{\pi} \arcsin\left(\varkappa\sqrt{1-\frac{\varkappa^2}{4}} + \frac{\varkappa}{2}\sqrt{1-\varkappa^2}\right), \tag{17}$$

and $\varkappa = \frac{1}{\mathcal{MN}-1}$.

Proof. By using the well-known triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|,$$

with the inequalities

$$(n+1)^2 \leq 4^n,\ (\mathcal{M})_n \geq \mathcal{M}^n \quad \forall\, n \in \mathbb{N},$$

we obtain

$$\begin{aligned}
\left|\left(z\mathcal{L}'_{\kappa,\tau}(z)\right)' - 1\right| &\leq \sum_{n=1}^{\infty} \frac{(n+1)^2}{4^n\,(\mathcal{M})_n\,(\mathcal{N})_n} \\
&\leq \frac{1}{\mathcal{M}\mathcal{N}} \sum_{n=1}^{\infty} \left(\frac{1}{\mathcal{M}\mathcal{N}}\right)^{n-1} \\
&= \frac{1}{\mathcal{M}\mathcal{N} - 1} = \varkappa. \qquad (18)
\end{aligned}$$

From Equation (18) , we conclude that

$$\left(z\mathcal{L}'_{\kappa,\tau}(z)\right)' \prec 1 + \varkappa z \quad \Rightarrow \quad \left|\arg\left(z\mathcal{L}'_{\kappa,\tau}(z)\right)'\right| < \arcsin \varkappa. \qquad (19)$$

With the help of Lemma 3, take $n = 0$ with $F(z) = \left(z\mathcal{L}'_{\kappa,\tau}(z)\right)'$ and $M(z) = 1 + \varkappa z$, and we get

$$\frac{z\mathcal{L}'_{\kappa,\tau}(z)}{z} \prec 1 + \frac{\varkappa}{2} z. \qquad (20)$$

This implies that

$$\mathcal{L}'_{\kappa,\tau}(z) \prec 1 + \frac{\varkappa}{2} z.$$

As a result,

$$\left|\arg \mathcal{L}'_{\kappa,\tau}(z)\right| < \arcsin \frac{\varkappa}{2}. \qquad (21)$$

By using Equations (19) and (21) , we obtain

$$\begin{aligned}
\left|\arg\left(\frac{\left(z\mathcal{L}'_{\kappa,\tau}(z)\right)'}{\mathcal{L}'_{\kappa,\tau}(z)}\right)\right| &= \left|\arg\left(z\mathcal{L}'_{\kappa,\tau}(z)\right)' - \arg \mathcal{L}'_{\kappa,\tau}(z)\right| \\
&\leq \left|\arg\left(z\mathcal{L}'_{\kappa,\tau}(z)\right)'\right| + \left|\arg\left(\mathcal{L}'_{\kappa,\tau}(z)\right)\right| \\
&< \arcsin \frac{\varkappa}{2} + \arcsin \varkappa \\
&= \arcsin\left(\varkappa\sqrt{1 - \frac{\varkappa^2}{4}} + \frac{\varkappa}{2}\sqrt{1 - \varkappa^2}\right).
\end{aligned}$$

which implies that $\mathcal{L}_{\kappa,\tau} \in \widetilde{\mathcal{C}}\,(\alpha)$ for $\alpha = \frac{2}{\pi} \arcsin\left(\varkappa\sqrt{1 - \frac{\varkappa^2}{4}} + \frac{\varkappa}{2}\sqrt{1 - \varkappa^2}\right)$. □

Theorem 6. *Let $\kappa, \tau \in \mathbb{R}$. If $\mathcal{M}\mathcal{N} - 2 > 0$, where $\mathcal{M} = \frac{\kappa+\tau+3}{2}$ and $\mathcal{N} = \frac{\kappa-\tau+3}{2}$, then $\mathcal{L}_{\kappa,\tau} \in \widetilde{\mathcal{S}}^*\,(\alpha)$, where*

$$\alpha = \frac{2}{\pi} \arcsin\left(\varkappa\sqrt{1 - \frac{\varkappa^2}{4}} + \frac{\varkappa}{2}\sqrt{1 - \varkappa^2}\right), \qquad (22)$$

and $\varkappa = \frac{1}{2(\mathcal{M}\mathcal{N}-1)}$.

Proof. By using the well-known triangle inequality

$$|z_1+z_2| \leq |z_1| + |z_2|,$$

with the inequalities

$$2(n+1) \leq 4^n,\ (\mathcal{M})_n \geq \mathcal{M}^n,\ \forall\, n \in \mathbb{N},$$

we obtain

$$\begin{aligned} \left|\mathcal{L}'_{\kappa,\tau}(z)-1\right| &\leq \sum_{n=1}^{\infty} \frac{n+1}{4^n\,(\mathcal{M})_n\,(\mathcal{N})_n} \\ &\leq \frac{1}{2\mathcal{M}\mathcal{N}} \sum_{n=1}^{\infty} \left(\frac{1}{\mathcal{M}\mathcal{N}}\right)^{n-1} \\ &= \frac{1}{2\,(\mathcal{M}\mathcal{N}-1)} = \varkappa. \end{aligned} \tag{23}$$

From Equation (23) , we conclude that

$$\mathcal{L}'_{\kappa,\tau} \prec 1+\varkappa z \quad \Rightarrow \quad \left|\arg\left(\mathcal{L}'_{\kappa,\tau}(z)\right)\right| < \arcsin \varkappa. \tag{24}$$

With the help of Lemma 3, take $n=0$ with $F(z)=\mathcal{L}'_{\kappa,\tau}(z)$ and $M(z)=1+\varkappa z$, and we get

$$\frac{\mathcal{L}_{\kappa,\tau}}{z} \prec 1+\frac{\varkappa}{2}z. \tag{25}$$

As a result,

$$\left|\arg\left(\frac{\mathcal{L}_{\kappa,\tau}(z)}{z}\right)\right| < \arcsin\frac{\varkappa}{2}. \tag{26}$$

By using Equations (24) and (25) , it implies that $\mathcal{L}_{\kappa,\tau} \in \widetilde{\mathcal{S}}^*(\alpha)$ for $\alpha = \frac{2}{\pi}\arcsin\left(\varkappa\sqrt{1-\frac{\varkappa^2}{4}}+\frac{\varkappa}{2}\sqrt{1-\varkappa^2}\right)$. □

5. Strongly Convexity and Strongly Starlikeness of Hyper-Bessel Functions

Theorem 7. *Let $i \in \{1,2,3,...,c\}$, $\beta_i > -1$ and $\zeta\eta > \frac{1}{2}$, where*

$$\zeta = (c+1)^{c+1} \text{ and } \eta = \prod_{i=1}^{c}(\beta_i+1).$$

Then, $\mathcal{H}_{\beta_c} \in \widetilde{\mathcal{C}}(\alpha)$, where

$$\alpha = \frac{2}{\pi}\arcsin\left(\psi\sqrt{1-\frac{\psi^2}{4}}+\frac{\psi}{2}\sqrt{1-\psi^2}\right), \tag{27}$$

and $\psi = \frac{4\zeta\eta(c+1)^2(2\zeta\eta+1)+(2\zeta\eta-1)\{8\zeta\eta(c+1)+2(2\zeta\eta-1)\}}{(2\zeta\eta-1)^3}$ such that $|\psi| \leq 1$.

Proof. By using the well-known triangle inequality

$$|z_1+z_2| \leq |z_1| + |z_2|,$$

we obtain

$$\begin{aligned}
\left|\left(z\mathcal{H}'_{\beta_c}(z)\right)' - 1\right| &\leq \sum_{n=1}^{\infty} \frac{(nc+n+1)^2}{n!\,(c+1)^{n(c+1)} \prod\limits_{i=1}^{c} (\beta_i+1)_n} \\
&\leq \frac{(c+1)^2}{\zeta\eta} \sum_{n=1}^{\infty} \frac{n^2}{(2\zeta\eta)^{n-1}} + \frac{2\,(c+1)}{\zeta\eta} \sum_{n=1}^{\infty} \frac{n}{(2\zeta\eta)^{n-1}} + \frac{1}{\zeta\eta} \sum_{n=1}^{\infty} \frac{1}{(2\zeta\eta)^{n-1}} \\
&= \frac{(c+1)^2}{\zeta\eta} \frac{1+\frac{1}{2\zeta\eta}}{\left(1-\frac{1}{2\zeta\eta}\right)^3} + \frac{2\,(c+1)}{\zeta\eta} \frac{1}{\left(1-\frac{1}{2\zeta\eta}\right)^2} + \frac{1}{\zeta\eta} \frac{1}{\left(1-\frac{1}{2\zeta\eta}\right)} \\
&= \frac{4\zeta\eta(c+1)^2(2\zeta\eta+1) + (2\zeta\eta-1)\left\{8\zeta\eta(c+1) + 2(2\zeta\eta-1)\right\}}{(2\zeta\eta-1)^3} = \psi. \qquad (28)
\end{aligned}$$

From Equation (28) , we conclude that

$$\left(z\mathcal{H}'_{\beta_c}(z)\right)' \prec 1+\psi z \quad \Rightarrow \quad \left|\arg\left(z\mathcal{H}'_{\beta_c}(z)\right)'\right| < \arcsin\psi. \qquad (29)$$

With the help of Lemma 3, take $n=0$ with $F(z) = \left(z\mathcal{H}'_{\beta_c}(z)\right)'$ and $M(z) = 1+\psi z$, and we get

$$\frac{z\mathcal{H}'_{\beta_c}(z)}{z} \prec 1 + \frac{\psi}{2} z. \qquad (30)$$

This implies that

$$\mathcal{H}'_{\beta_c}(z) \prec 1 + \frac{\psi}{2} z.$$

As a result,

$$\left|\arg \mathcal{H}'_{\beta_c}(z)\right| < \arcsin\frac{\psi}{2}. \qquad (31)$$

By using Equations (29) and (31) , we obtain

$$\begin{aligned}
\left|\arg\left(\frac{\left(z\mathcal{H}'_{\beta_c}(z)\right)'}{\mathcal{H}'_{\beta_c}(z)}\right)\right| &= \left|\arg\left(z\mathcal{H}'_{\beta_c}(z)\right)' - \arg\mathcal{H}'_{\beta_c}(z)\right| \\
&\leq \left|\arg\left(z\mathcal{H}'_{\beta_c}(z)\right)'\right| + \left|\arg\mathcal{H}'_{\beta_c}(z)\right| \\
&< \arcsin\frac{\psi}{2} + \arcsin\psi \\
&= \arcsin\left(\psi\sqrt{1-\frac{\psi^2}{4}} + \frac{\psi}{2}\sqrt{1-\psi^2}\right).
\end{aligned}$$

which implies that $\mathcal{H}_{\beta_c} \in \widetilde{\mathcal{C}}\,(\alpha)$ for $\alpha = \frac{2}{\pi}\arcsin\left(\psi\sqrt{1-\frac{\psi^2}{4}} + \frac{\psi}{2}\sqrt{1-\psi^2}\right)$. □

Theorem 8. *Let $i \in \{1,2,3,...,c\}, \beta_i > -1$ and $\zeta\eta > \frac{1}{2}$, where*

$$\zeta = (c+1)^{c+1} \textit{ and } \eta = \prod_{i=1}^{c} (\beta_i+1).$$

Then, $\mathcal{H}_{\beta_c} \in \widetilde{\mathcal{S}^*}(\alpha)$*, where*

$$\alpha = \frac{2}{\pi}\arcsin\left(\psi\sqrt{1-\frac{\psi^2}{4}} + \frac{\psi}{2}\sqrt{1-\psi^2}\right), \tag{32}$$

and $\psi = \frac{4\zeta\eta(c+2)-2}{(2\zeta\eta-1)^2}$.

Proof. By using the well-known triangle inequality

$$|z_1 + z_2| \le |z_1| + |z_2|,$$

we obtain

$$\begin{aligned}
\left|\mathcal{H}'_{\beta_c}(z) - 1\right| &\le \sum_{n=1}^{\infty} \frac{n(c+1)+1}{n!\,(c+1)^{n(c+1)} \prod_{i=1}^{c} (\beta_i+1)_n} \\
&\le \frac{c+1}{\zeta\eta}\sum_{n=1}^{\infty}\frac{n}{(2\zeta\eta)^{n-1}} + \frac{1}{\zeta\eta}\sum_{n=1}^{\infty}\frac{1}{(2\zeta\eta)^{n-1}} \\
&= \frac{c+1}{\zeta\eta}\frac{1}{\left(1-\frac{1}{2\zeta\eta}\right)^2} + \frac{1}{\zeta\eta}\frac{1}{\left(1-\frac{1}{2\zeta\eta}\right)} \\
&= \frac{4\zeta\eta\,(c+2)-2}{(2\zeta\eta-1)^2} = \psi. \qquad (33)
\end{aligned}$$

From Equation (33), we conclude that

$$\mathcal{H}'_{\beta_c}(z) \prec 1 + \psi z \quad \Rightarrow \quad \left|\arg\left(\mathcal{H}'_{\beta_c}(z)\right)\right| < \arcsin\psi. \tag{34}$$

With the help of Lemma 3, take $n = 0$ with $F(z) = \mathcal{H}'_{\beta_c}(z)$ and $M(z) = 1 + \psi z$, and we get

$$\frac{\mathcal{H}_{\beta_c}(z)}{z} \prec 1 + \frac{\psi}{2}z. \tag{35}$$

As a result,

$$\left|\arg\left(\frac{\mathcal{H}_{\beta_c}(z)}{z}\right)\right| < \arcsin\frac{\psi}{2}. \tag{36}$$

By using Equations (34) and (35), we obtain

$$\begin{aligned}
\left|\arg\left(\frac{z\mathcal{H}'_{\beta_c}(z)}{\mathcal{H}_{\beta_c}(z)}\right)\right| &= \left|\arg\left(\frac{z}{\mathcal{H}_{\beta_c}(z)}\right) - \arg\left(\mathcal{H}'_{\beta_c}(z)\right)\right| \\
&\le \left|\arg\left(\frac{z}{\mathcal{H}_{\beta_c}(z)}\right)\right| + \left|\arg\left(\mathcal{H}'_{\beta_c}(z)\right)\right| \\
&< \arcsin\frac{\psi}{2} + \arcsin\psi \\
&= \arcsin\left(\psi\sqrt{1-\frac{\psi^2}{4}} + \frac{\psi}{2}\sqrt{1-\psi^2}\right).
\end{aligned}$$

which implies that $\mathcal{H}_{\beta_c} \in \widetilde{\mathcal{S}^*}(\alpha)$ for $\alpha = \frac{2}{\pi}\arcsin\left(\psi\sqrt{1-\frac{\psi^2}{4}} + \frac{\psi}{2}\sqrt{1-\psi^2}\right)$. □

6. Some Applications for Strongly Starlikeness of Lommel Functions

Example 1. *If $\kappa = \frac{1}{2}$ and $\tau = \frac{1}{2}$, then $\mathcal{MN} - 2 > 0$, where $\mathcal{M} = \frac{\kappa+\tau+3}{2}$ and $\mathcal{N} = \frac{\kappa-\tau+3}{2}$. From Equation (22) , we get*

$$\alpha = \frac{2}{\pi} \arcsin\left(\frac{1}{2}\sqrt{\frac{15}{16}} + \frac{1}{4}\sqrt{\frac{3}{4}}\right),$$

and thus from Theorem 6, we have

$$\mathcal{L}_{\frac{1}{2},\frac{1}{2}}(z) = 2 - 2\cos\sqrt{z} \in \widetilde{\mathcal{S}^*}(\alpha).$$

Example 2. *If $\kappa = \frac{3}{2}$ and $\tau = \frac{1}{2}$, then $\mathcal{MN} - 2 > 0$, where $\mathcal{M} = \frac{\kappa+\tau+3}{2}$ and $\mathcal{N} = \frac{\kappa-\tau+3}{2}$. From Equation (22), we get*

$$\alpha = \frac{2}{\pi} \arcsin\left(\frac{1}{4}\sqrt{\frac{63}{64}} + \frac{1}{8}\sqrt{\frac{15}{16}}\right),$$

and thus from Theorem 6, we have

$$\mathcal{L}_{\frac{3}{2},\frac{1}{2}}(z) = \frac{6\sqrt{z} - 6\sin\sqrt{z}}{\sqrt{z}} \in \widetilde{\mathcal{S}^*}(\alpha).$$

Remark 1. *Examples related to strongly convexity can also be obtained.*

Author Contributions: Data curation, M.R. and M.U.D.; Funding acquisition, M.R.; Investigation, S.M.; Methodology, S.M.; Resources, M.R.; Supervision, M.R.; Validation, M.R.; Writing original draft, S.M.; Writing review and editing, M.U.D.

Funding: The work here is partially supported by HEC grant: 5689/Pun- jab/NRPU/R & D/HEC/2016.

Acknowledgments: The authors thank the referees for their valuable suggestions to improve the paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Baricz, Á.; Koumandos, S. Turan type inequalities for some Lommel functions of the first kind. *Proc. Edinburgh Math. Soc.* **2016**, *59*, 569–579. [CrossRef]
2. Watson, G.N. *A Treatise on the Theory of Bessel Functions*, 2nd ed.; Cambridge University Press: Cambridge, UK; New York, NY, USA, 1944.
3. Chaggara, H.; Romdhane, N.B. On the zeros of the hyper-Bessel Function. *Integr. Trans. Spec. Funct.* **2015**, 26, 96–101. [CrossRef]
4. Baricz, Á. Geometric properties of generalized Bessel Functions. *Publ. Math. Debr.* **2008**, *73*, 155–178.
5. Choi, J.H.; Kim, Y.C.; Srivastava, H.M. Convex and starlike generalized HypergeometricFunctions associated with the Hardy spaces. *Com. Var.* **1996**, *31*, 345–355.
6. Miller, S.S.; Mocanu, P.T. Univalence of Gaussian and confluent hypergeometric functions. *Proc. Am. Math. Soc.* **1990**, *110*, 333–342. [CrossRef]
7. Mustafa, N. Geometric Properties of Normalized Wright Functions. *Math. Comput. Appl.* **2016**, *21*, 14. [CrossRef]
8. Orhan, H.; Yağmur, N. Geometric properties of generalized Struve functions. *Ann. Alexandru Ioan Cuza Univ. Math.* **2017**, *63*, 229–244. [CrossRef]
9. Owa, S.; Nunokawa, M.; Saitoh, H.; Srivastava, H.M. Close-to-convexity, starlikeness, and convexity of certain analytic Functions. *Appl. Math. Lett.* **2002**, *15*, 63–69. [CrossRef]
10. Din, M.U.; Raza, M.; Hussain, S.; Darus, M. Certain geometric properties of generalized Dini Functions. *J. Funct. Spaces* **2018**, *2018*, 2684023. [CrossRef]
11. Prajapat, J.K. Certain geometric properties of the Wright Function. *Integr. Trans. Spec. Funct.* **2015**, *26*, 203–212. [CrossRef]

12. Raza, M.; Din, M.U. Close-to-Convexity of q-Mittag-Leffler Functions. *C. R. Acad. Bulg. Sci.* **2018**, *71*, 1581–1591.
13. Raza, M.; Din, M.U.; Malik, S.N. Certain geometric properties of normalized Wright functions. *J. Funct. Spaces* **2016**, *2016*, 1896154. [CrossRef]
14. Mondal, S.R.A. Swaminathan, On the positivity of certain trigonometric sums and their applications. *Comput. Math. Appl.* **2011**, *62*, 3871–3883. [CrossRef]
15. Ozaki, S. On the theory of multivalent functions. *Sci. Rep. Tokyo Bunrika Daigaku* **1935**, *2*, 167–188.
16. Hallenbeck, D.J.; Ruscheweyh, S. Subordination by convex Functions. *Proc. Am. Math. Soc.* **1975**, *52*, 191–195. [CrossRef]