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Random Coupled Hilfer and Hadamard Fractional Differential Systems in Generalized Banach Spaces

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Abstract: This article deals with some existence and uniqueness result of random solutions for some coupled systems of Hilfer and Hilfer–Hadamard fractional differential equations with random effects. Some applications are made of generalizations of classical random fixed point theorems on generalized Banach spaces.

Keywords: fractional differential systems; mixed Riemann–Liouville integral; mixed Hadamard integral; Hilfer derivative; Hadamard derivative; coupled system; random solution

MSC: 26A33; 34G25

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1. Introduction

Fractional calculus is an extension of the ordinary differentiation and integration to arbitrary non-integer order. In recent years, this theory has become an important object of investigations due to its demonstrated applications in different areas of physics and engineering (see, for example, [1,2] and the references therein). In particular, time fractional differential equations are used when attempting to describe transport processes with long memory. Recently, the study of time fractional ordinary and partial differential equations has received great attention from many researchers, both in theory and in applications; we refer the reader to the monographs of Abbas et al. [3–5], Samko et al. [6], and Kilbas et al. [7], and the papers [8–14] and the references therein. On the other hand, the existence of solutions of initial and boundary value problems for fractional differential equations with the Hilfer fractional derivative have started to draw attention. For the related works, see for example [1,15–20] and the references therein.

Functional differential equations with random effects are differential equations with a stochastic process in their vector field [21–25]. They play a fundamental role in the theory of random dynamical systems.

Consider the following coupled system of Hilfer fractional differential equations:

$$\begin{cases}(D_0^{\alpha_1,\beta_1}u)(t,w)=f_1(t,u(t,w),v(t,w),w)\\(D_0^{\alpha_2,\beta_2}v)(t,w)=f_2(t,u(t,w),v(t,w),w)\end{cases};\ t\in I:=[0,T],\ w\in\Omega, \tag{1}$$

with the following initial conditions:

$$\begin{cases}(I_0^{1-\gamma_1}u)(0,w)=\phi_1(w)\\(I_0^{1-\gamma_2}v)(0,w)=\phi_2(w)\end{cases};\ w\in\Omega, \tag{2}$$

where $T>0$, $\alpha_i\in(0,1)$, $\beta_i\in[0,1]$, $(\Omega,\mathcal{A})$ is a measurable space, $\gamma_i=\alpha_i+\beta_i-\alpha_i\beta_i$, $\phi_i:\Omega\to\mathbb{R}^m$, $f_i: I\times\mathbb{R}^m\times\mathbb{R}^m\times\Omega\to\mathbb{R}^m$; $i=1,2$, are given functions, $I_0^{1-\gamma_i}$ is the left-sided mixed Riemann–Liouville integral of order $1-\gamma_i$, and $D_0^{\alpha_i,\beta_i}$ is the generalized Riemann–Liouville derivative (Hilfer) operator of order α_i and type $\beta_i:\ i=1,2$. Next, we discuss the following coupled system of Hilfer–Hadamard fractional differential equations:

$$\begin{cases}({}^HD_1^{\alpha_1,\beta_1}u)(t,w)=g_1(t,u(t,w),v(t,w),w)\\({}^HD_1^{\alpha_2,\beta_2}v)(t,w)=g_2(t,u(t,w),v(t,w),w)\end{cases};\ t\in[1,T],\ w\in\Omega, \tag{3}$$

with the following initial conditions:

$$\begin{cases}({}^HI_1^{1-\gamma_1}u)(1,w)=\psi_1(w)\\({}^HI_1^{1-\gamma_2}v)(1,w)=\psi_2(w)\end{cases};\ w\in\Omega, \tag{4}$$

where $T>1$, $\alpha_i\in(0,1)$, $\beta_i\in[0,1]$, $\gamma_i=\alpha_i+\beta_i-\alpha_i\beta_i$, $\psi_i:\Omega\to\mathbb{R}^m$, $g_i:[1,T]\times\mathbb{R}^m\times\mathbb{R}^m\times\Omega\to\mathbb{R}^m$; $i=1,2$ are given functions, $\mathbb{R}^m$; $m\in\mathbb{N}^*$, ${}^HI_1^{1-\gamma_i}$ is the left-sided mixed Hadamard integral of order $1-\gamma_i$, and ${}^HD_1^{\alpha_i,\beta_i}$ is the Hilfer–Hadamard fractional derivative of order α_i and type β_i; $i=1,2$.

2. Preliminaries

We denote by C; the Banach space of all continuous functions from I into $\mathbb{R}^m$ with the supremum (uniform) norm $\|\cdot\|_\infty$. As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into $\mathbb{R}^m$. By $L^1(I)$, we denote the space of Lebesgue-integrable functions $v:I\to\mathbb{R}^m$ with the norm:

$$\|v\|_1=\int_0^T\|v(t)\|dt.$$

By $C_\gamma(I)$ and $C_\gamma^1(I)$, we denote the weighted spaces of continuous functions defined by:

$$C_\gamma(I)=\{w:(0,T]\to\mathbb{R}^m: t^{1-\gamma}w(t)\in C\},$$

with the norm:

$$\|w\|_{C_\gamma}:=\sup_{t\in I}\|t^{1-\gamma}w(t)\|,$$

and:

$$C_\gamma^1(I)=\{w\in C:\frac{dw}{dt}\in C_\gamma\},$$

with the norm:

$$\|w\|_{C_\gamma^1}:=\|w\|_\infty+\|w'\|_{C_\gamma}.$$

Furthermore, by $\mathcal{C}:=C_{\gamma_1}\times C_{\gamma_2}$, we denote the product weighted space with the norm:

$$\|(u,v)\|_{\mathcal{C}}=\|u\|_{C_{\gamma_1}}+\|v\|_{C_{\gamma_2}}.$$

Now, we give some definitions and properties of fractional calculus.

Definition 1. *[4,6,7] The left-sided mixed Riemann–Liouville integral of order $r > 0$ of a function $w \in L^1(I)$ is defined by:*

$$(I_0^r w)(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} w(s)ds; \ for \ a.e. \ t \in I,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function.

Notice that for all $r, r_1, r_2 > 0$ and each $w \in C$, we have $I_0^r w \in C$, and:

$$(I_0^{r_1} I_0^{r_2} w)(t) = (I_0^{r_1+r_2} w)(t); \ for \ a.e. \ t \in I.$$

Definition 2. *[4,6,7] The Riemann–Liouville fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I)$ is defined by:*

$$\begin{aligned} (D_0^r w)(t) &= \left(\frac{d}{dt} I_0^{1-r} w \right)(t) \\ &= \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^{-r} w(s)ds; \ for \ a.e. \ t \in I. \end{aligned}$$

Let $r \in (0,1]$, $\gamma \in [0,1)$ and $w \in C_{1-\gamma}(I)$. Then, the following expression leads to the left inverse operator as follows.

$$(D_0^r I_0^r w)(t) = w(t); \ for \ all \ t \in (0,T].$$

Moreover, if $I_0^{1-r} w \in C^1_{1-\gamma}(I)$, then the following composition is proven in [6]:

$$(I_0^r D_0^r w)(t) = w(t) - \frac{(I_0^{1-r} w)(0^+)}{\Gamma(r)} t^{r-1}; \ for \ all \ t \in (0,T].$$

Definition 3. *[4,6,7] The Caputo fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I)$ is defined by:*

$$\begin{aligned} ({}^c D_0^r w)(t) &= \left(I_0^{1-r} \frac{d}{dt} w \right)(t) \\ &= \frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} \frac{d}{ds} w(s)ds; \ for \ a.e. \ t \in I. \end{aligned}$$

In [1], R.Hilfer studied applications of a generalized fractional operator having the Riemann–Liouville and the Caputo derivatives as specific cases (see also [17,19]).

Definition 4. *(Hilfer derivative). Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $w \in L^1(I)$, and $I_0^{(1-\alpha)(1-\beta)} w \in AC(I)$. The Hilfer fractional derivative of order α and type β of w is defined as:*

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{(1-\alpha)(1-\beta)} w \right)(t); \ for \ a.e. \ t \in I. \tag{5}$$

Property 1. Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, and $w \in L^1(I)$.

1. The operator $(D_0^{\alpha,\beta} w)(t)$ can be written as:

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{1-\gamma} w \right)(t) = \left(I_0^{\beta(1-\alpha)} D_0^{\gamma} w \right)(t); \ for \ a.e. \ t \in I.$$

Moreover, the parameter γ satisfies:

$$\gamma \in (0,1],\ \gamma \geq \alpha,\ \gamma > \beta,\ 1-\gamma < 1-\beta(1-\alpha).$$

2. The generalization (5) for $\beta = 0$, coincides with the Riemann–Liouville derivative and for $\beta = 1$ with the Caputo derivative.
$$D_0^{\alpha,0} = D_0^{\alpha},\ and\ D_0^{\alpha,1} = {}^{c}D_0^{\alpha}.$$
3. If $D_0^{\beta(1-\alpha)}w$ exists and is in $L^1(I)$, then:

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = (I_0^{\beta(1-\alpha)} D_0^{\beta(1-\alpha)} w)(t);\ for\ a.e.\ t \in I.$$

Furthermore, if $w \in C_\gamma(I)$ and $I_0^{1-\beta(1-\alpha)}w \in C_\gamma^1(I)$, then:

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = w(t);\ for\ a.e.\ t \in I.$$

4. If $D_0^{\gamma}w$ exists and is in $L^1(I)$, then:

$$(I_0^{\alpha} D_0^{\alpha,\beta} w)(t) = (I_0^{\gamma} D_0^{\gamma} w)(t) = w(t) - \frac{I_0^{1-\gamma}(0^+)}{\Gamma(\gamma)} t^{\gamma-1};\ for\ a.e.\ t \in I.$$

Corollary 1. *Let $h \in C_\gamma(I)$. Then, the Cauchy problem:*

$$\begin{cases} (D_0^{\alpha,\beta} u)(t) = h(t);\ t \in I, \\ \\ (I_0^{1-\gamma} u)(t)|_{t=0} = \phi, \end{cases}$$

has the following unique solution:
$$u(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^{\alpha} h)(t).$$

Let $\beta_{\mathbb{R}^m}$ be the σ-algebra of Borel subsets of $\mathbb{R}^m$. A mapping $v : \Omega \to \mathbb{R}^m$ is said to be measurable if for any $B \in \beta_{\mathbb{R}^m}$; one has:
$$v^{-1}(B) = \{w \in \Omega : v(w) \in B\} \subset \mathcal{A}.$$

Definition 5. *Let $\mathcal{A} \times \beta_{\mathbb{R}^m}$ be the direct product of the σ-algebras $\mathcal{A}$ and $\beta_{\mathbb{R}^m}$ those defined in Ω and $\mathbb{R}^m$, respectively. A mapping $T : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is called jointly measurable if for any $B \in \beta_{\mathbb{R}^m}$, one has:*

$$T^{-1}(B) = \{(w,v) \in \Omega \times E : T(w,v) \in B\} \subset \mathcal{A} \times \beta_{\mathbb{R}^m}.$$

Definition 6. *A function $T : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is called jointly measurable if $T(\cdot, u)$ is measurable for all $u \in \mathbb{R}^m$ and $T(w, \cdot)$ is continuous for all: $w \in \Omega$.*

A random operator is a mapping $T : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ such that $T(w,u)$ is measurable in w for all $u \in \mathbb{R}^m$, and it expressed as $T(w)u = T(w,u)$; we also say that $T(w)$ is a random operator on $\mathbb{R}^m$. The random operator $T(w)$ on E is called continuous (resp. compact, totally bounded, and completely continuous) if $T(w,u)$ is continuous (resp. compact, totally bounded, and completely continuous) in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [26].

Definition 7. *[27] Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y and C be a mapping from Ω into $\mathcal{P}(Y)$. A mapping $T : \{(w,y) : w \in \Omega,\ y \in C(w)\} \to Y$ is called a random operator with stochastic domain C if C*

is measurable (i.e., for all closed $A \subset Y$, $\{w \in \Omega, C(w) \cap A \neq \emptyset\}$ is measurable), and for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega : y \in C(w), T(w, y) \in D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \to Y$ is called a random (stochastic) fixed point of T if for P-almost all $w \in \Omega$, $y(w) \in C(w)$ and $T(w)y(w) = y(w)$, and for all open $D \subset Y$, $\{w \in \Omega : y(w) \in D\}$ is measurable.

Definition 8. *A function $f : I \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ is called random Carathéodory if the following conditions are satisfied:*

(i) The map $(t, w) \to f(x, y, u, w)$ is jointly measurable for all $u \in \mathbb{R}^m$ and
(ii) The map $u \to f(t, u, w)$ is continuous for all $t \in I$ and $w \in \Omega$.

Let $x, , y \in \mathbb{R}^m$ with $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_m)$.

By $x \leq y$, we mean $x_i \leq y_i$; $i = 1, \ldots, m$. Also $|x| = (|x_1|, |x_2|, \ldots, |x_m|)$, $\max(x, y) = (\max(x_1, y_1), \max(x_2, y_2), \ldots, \max(x_m, y_m))$, and $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x_i \in \mathbb{R}_+,\ i = 1, \ldots, m\}$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$; $i = 1, \ldots, m$.

Definition 9. *Let X be a nonempty set. By a vector-valued metric on X, we mean a map $d : X \times X \to \mathbb{R}^m$ with the following properties:*

(i) $d(x, y) \geq 0$ for all $x, y \in X$, and if $d(x, y) = 0$, then $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We call the pair (X, d) a generalized metric space with $d(x, y) := \begin{pmatrix} d_1(x, y) \\ d_2(x, y) \\ \cdot \\ \cdot \\ \cdot \\ d_m(x, y) \end{pmatrix}$.

Notice that d is a generalized metric space on X if and only if d_i; $i = 1, \ldots, m$ are metrics on X. For $r = (r_1, \ldots, r_m) \in \mathbb{R}^m$ and $x_0 \in X$, we will denote by:

$$B_r(x_0) := \{x \in X : d(x_0, x) < r\} = \{x \in X : d_i(x_0, x) < r_i;\ i = 1, \ldots, m\}$$

the open ball centered in x_0 with radius r and:

$$\overline{B}_r(x_0) := \{x \in X : d(x_0, x) \leq r\} = \{x \in X : d_i(x_0, x) \leq r_i;\ i = 1, \ldots, m\}$$

the closed ball centered in x_0 with radius r. We mention that for generalized metric spaces, the notations of open, closed, compact, convex sets, convergence, and Cauchy sequence are similar to those in usual metric spaces.

Definition 10. *[28,29] A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than one. In other words, this means that all the eigenvalues of M are in the open unit disc, i.e., $|\lambda| < 1$; for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$; where I denotes the unit matrix of $M_{m \times m}(\mathbb{R})$.*

Example 1. *The matrix $A \in M_{2\times 2}(\mathbb{R})$ defined by:*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

converges to zero in the following cases:

(1) $b = c = 0,\ a, d > 0$, *and* $\max\{a, d\} < 1$.
(2) $c = 0,\ a, d > 0,\ a + d < 1$, *and* $-1 < b < 0$.
(3) $a + b = c + d = 0,\ a > 1,\ c > 0$, *and* $|a - c| < 1$.

In the sequel, we will make use of the following random fixed point theorems:

Theorem 1. *[23–25] Let $(\Omega, \mathcal{F})$ be a measurable space, X a real separable generalized Banach space, and $F : \Omega \times X \to X$ a continuous random operator, and let $M(w) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matrix such that for every $w \in \Omega$, the matrix $M(w)$ converges to zero and:*

$$d(F(w, x_1), F(w, x_2)) \leq M(w)d(x_1, x_2);\ for\ each\ x_1, x_2 \in X\ and\ w \in \Omega,$$

then there exists a random variable $x : \Omega \to X$ that is the unique random fixed point of F.

Theorem 2. *[23–25] Let $(\Omega, \mathcal{F})$ be a measurable space, X be a real separable generalized Banach space, and $F : \Omega \times X \to X$ be a completely continuous random operator. Then, either:*

(i) *the random equation $F(w, x) = x$ has a random solution, i.e., there is a measurable function $x : \Omega \to X$ such that $F(w, x(w)) = x(w)$ for all $w \in \Omega$ or*
(ii) *the set $M = \{x : \Omega \to X$ is measurable $:\ \lambda(w)F(w, x) = x\}$ is unbounded for some measurable function $\lambda : \Omega \to X$ with $0 < \lambda(w) < 1$ on Ω.*

Furthermore, we will use the following Gronwall lemma:

Lemma 1. *[23] Let $u : I \to [0, \infty)$ be a real function and $u(\cdot)$ a nonnegative, locally-integrable function on I. Assume that there exist constants $c > 0$ and $r < 1$ such that:*

$$u(t) \leq v(t) + c\int_0^t \frac{u(s)}{(t-s)^r}ds,$$

then, there exists a constant $K := K(r)$ such that:

$$u(t) \leq v(t) + cK\int_0^t \frac{v(s)}{(t-s)^r}ds,$$

for every $t \in I$.

3. Coupled Hilfer Fractional Differential Systems

In this section, we are concerned with the existence and uniqueness results of the system (1) and (2).

Definition 11. *By a solution of the problem (1) and (2), we mean coupled measurable functions $(u, v) \in C_{\gamma_1} \times C_{\gamma_2}$, which satisfy the Equation (1) on I, and the conditions $(I_0^{1-\gamma_1}u)(0^+) = \phi_1$, and $(I_0^{1-\gamma_2}v)(0^+) = \phi_2$.*

The following hypotheses will be used in the sequel.

(H_1) The functions $f_i;\ i = 1, 2$ are Carathéodory.
(H_2) There exist measurable functions $p_i, q_i : \Omega \to (0, \infty);\ i = 1, 2$ such that:

$$\|f_i(t, u_1, v_1) - f_i(t, u_2, v_2)\| \leq p_i(w)\|u_1 - u_2\| + q_i(w)\|v_1 - v_2\|;$$

$for\ a.e.\ t \in I,\ and\ each\ u_i, v_i \in \mathbb{R}^m,\ i = 1, 2.$
(H_3) There exist measurable functions $a_i, b_i : \Omega \to (0, \infty);\ i = 1, 2$ such that:

$$\|f_i(t, u, v)\| \leq a_i(w)\|u\| + b_i(w)\|v\|;\ for\ a.e.\ t \in I,\ and\ each\ u, v \in \mathbb{R}^m.$$

First, we prove an existence and uniqueness result for the coupled system (1)–(2) by using Banach's random fixed point theorem in generalized Banach spaces.

Theorem 3. *Assume that the hypotheses* (H_1) *and* (H_2) *hold. If for every* $w \in \Omega$*, the matrix:*

$$M(w) := \begin{pmatrix} \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)} p_1(w) & \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)} q_1(w) \\ \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)} p_2(w) & \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)} q_2(w) \end{pmatrix}$$

converges to zero, then the coupled system (1) and (2) has a unique random solution.

Proof. Define the operators $N_1 : \mathcal{C} \times \Omega \to C_{\gamma_1}$ and $N_2 : \mathcal{C} \times \Omega \to C_{\gamma_2}$ by:

$$(N_1(u,v))(t,w) = \frac{\phi_1(w)}{\Gamma(\gamma_1)} t^{\gamma_1 - 1} + \int_0^t (t-s)^{\alpha_1 - 1} \frac{f(s,u(s,w),v(s,w),w)}{\Gamma(\alpha_1)} ds, \tag{6}$$

and:

$$(N_2(u,v))(t,w) = \frac{\phi_2(w)}{\Gamma(\gamma_2)} t^{\gamma_2 - 1} + \int_0^t (t-s)^{\alpha_2 - 1} \frac{f(s,u(s,w),v(s,w),w)}{\Gamma(\alpha_2)} ds. \tag{7}$$

Consider the operator $N : \mathcal{C} \times \Omega \to \mathcal{C}$ defined by:

$$(N(u,v))(t,w) = ((N_1(u,v))(t,w), (N_2(u,v))(t,w)). \tag{8}$$

Clearly, the fixed points of the operator N are random solutions of the system (1) and (2).

Let us show that N is a random operator on $\mathcal{C}$. Since $f_i;\ i = 1,2$ are Carathéodory functions, then $w \to f_i(t,u,v,w)$ are measurable maps. We concluded that the maps:

$$w \to (N_1(u,v))(t,w) \text{ and } w \to (N_2(u,v))(t,w),$$

are measurable. As a result, N is a random operator on $\mathcal{C} \times \Omega$ into $\mathcal{C}$. We show that N satisfies all conditions of Theorem 1.

For any $w \in \Omega$ and each $(u_1,v_1),(u_2,v_2) \in \mathcal{C}$, and $t \in I$, we have:

$$\begin{aligned}
& \|t^{1-\gamma_1}(N_1(u_1,v_1))(t,w) - t^{1-\gamma_1}(N_1(u_2,v_2))(t,w)\| \\
\leq\ & \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} \|f_1(s,u_1(s,w),v_1(s,w),w) - f_1(s,u_2(s,w),v_2(s,w),w)\| ds \\
\leq\ & \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} (p_1(w)\|u_1(s,w) - v_1(s,w)\| \\
+\ & q_1(w)\|u_2(s,w) - v_2(s,w)\|) ds \\
\leq\ & \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} (p_1(w) s^{1-\gamma_1}\|u_1(s,w) - v_1(s,w)\| \\
+\ & q_1(w) s^{1-\gamma_1}\|u_2(s,w) - v_2(s,w)\|) ds \\
\leq\ & \frac{p_1(w)\|u_1(\cdot,w) - v_1(\cdot,w)\|_{C_{\gamma_1}} + q_1(w)\|u_2(\cdot,w) - v_2(\cdot,w)\|_{C_{\gamma_2}}}{\Gamma(\alpha_1)} \\
\times\ & \int_0^t (t-s)^{\alpha_1-1} ds \\
\leq\ & \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)} (p_1(w)\|u_1(\cdot,w) - v_1(\cdot,w)\|_{C_{\gamma_1}} + q_1(w)\|u_2(\cdot,w) - v_2(\cdot,w)\|_{C_{\gamma_2}}).
\end{aligned}$$

Then,

$$\|(N_1(u_1,v_1))(\cdot,w) - (N_1(u_2,v_2))(\cdot,w)\|_{C_{\gamma_1}}$$

$$\leq \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)}(p_1(w)\|u_1(\cdot,w)-v_1(\cdot,w)\|_{C_{\gamma_1}}+q_1(w)\|u_2(\cdot,w)-v_2(\cdot,w)\|_{C_{\gamma_2}}).$$

Furthermore, for any $w \in \Omega$ and each $(u_1, v_1), (u_2, v_2) \in \mathcal{C}$, and $t \in I$, we get:

$$\begin{aligned} &\|(N_2(u_1,v_1))(\cdot,w)-(N_2(u_2,v_2))(\cdot,w)\|_{C_{\gamma_2}} \\ \leq\ & \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)}(p_2(w)\|u_1(\cdot,w)-v_1(\cdot,w)\|_{C_{\gamma_1}}+q_2(w)\|u_2(\cdot,w)-v_2(\cdot,w)\|_{C_{\gamma_2}}). \end{aligned}$$

Thus,

$$d((N(u_1,v_1))(\cdot,w),(N(u_2,v_2))(\cdot,w)) \leq M(w)d((u_1(\cdot,w),v_1(\cdot,w)),(u_2(\cdot,w),v_2(\cdot,w))),$$

where:

$$d((u_1(\cdot,w),v_1(\cdot,w)),(u_2(\cdot,w),v_2(\cdot,w))) = \begin{pmatrix} \|u_1(\cdot,w)-v_1(\cdot,w)\|_{C_{\gamma_1}} \\ \|u_2(\cdot,w)-v_2(\cdot,w)\|_{C_{\gamma_2}} \end{pmatrix}.$$

Since for every $w \in \Omega$, the matrix $M(w)$ converges to zero, then Theorem 1 implies that the operator N has a unique fixed point, which is a random solution of system (1) and (2). □

Now, we prove an existence result for the coupled system (1) and (2) by using the random nonlinear alternative of the Leray–Schauder type in generalized Banach space.

Theorem 4. *Assume that the hypotheses (H_1) and (H_3) hold. Then, the coupled system (1) and (2) has at least one random solution.*

Proof. We show that the operator $N : \mathcal{C} \times \Omega \to \mathcal{C}$ defined in (8) satisfies all conditions of Theorem 2. The proof will be given in four steps.

Step 1. $N(\cdot,\cdot,w)$ *is continuous.*

Let $(u_n, v_n)_n$ be a sequence such that $(u_n, v_n) \to (u, v) \in \mathcal{C}$ as $n \to \infty$. For any $w \in \Omega$ and each $t \in I$, we have:

$$\begin{aligned} &\|t^{1-\gamma_1}(N_1(u_n,v_n))(t,w)-t^{1-\gamma_1}(N_1(u,v))(t,w)\| \\ \leq\ & \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\|f_1(s,u_n(s,w),v_n(s,w),w)-f_1(s,u(s,w),v(s,w),w)\|ds \\ \leq\ & \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}s^{1-\gamma_1}\|f_1(s,u_n(s,w),v_n(s,w),w)-f_1(s,u(s,w),v(s,w),w)\|ds \\ \leq\ & \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)}\|f_1(\cdot,u_n(\cdot,w),v_n(\cdot,w),w)-f_1(\cdot,u(\cdot,w),v(\cdot,w),w)\|_{C_{\gamma_1}}. \end{aligned}$$

Since f_1 is Carathéodory, we have:

$$\|(N_1(u_n,v_n))(\cdot,w)-(N_1(u,v))(\cdot,w)\|_{C_{\gamma_1}} \to 0 \ as\ n \to \infty.$$

On the other hand, for any $w \in \Omega$ and each $t \in I$, we obtain:

$$\begin{aligned} &\|t^{1-\gamma_2}(N_2(u_n,v_n))(t,w)-t^{1-\gamma_2}(N_2(u,v))(t,w)\| \\ \leq\ & \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)}\|f_2(\cdot,u_n(\cdot,w),v_n(\cdot,w),w)-f_2(\cdot,u(\cdot,w),v(\cdot,w),w)\|_{C_{\gamma_2}}. \end{aligned}$$

Furthermore, from the fact that f_2 is Carathéodory, we get:

$$\|(N_2(u_n,v_n))(\cdot,w)-(N_2(u,v))(\cdot,w)\|_{C_{\gamma_2}} \to 0 \ as\ n \to \infty.$$

Hence, $N(\cdot,\cdot,w)$ is continuous.

Step 2. $N(\cdot,\cdot,w)$ maps bounded sets into bounded sets in $\mathcal{C}$.

Let $R > 0$, and set:

$$B_R := \{(\mu,\nu) \in \mathcal{C} : \|\mu\|_{C_{\gamma_1}} \le R, \|\nu\|_{C_{\gamma_2}} \le R\}.$$

For any $w \in \Omega$ and each $(u,v) \in B_R$ and $t \in I$, we have:

$$\begin{aligned}
\|t^{1-\gamma_1}(N_1(u,v))(t,w)\| &\le \frac{\|\phi_1(w)\|}{\Gamma(\gamma_1)} + \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\|f_1(s,u(s,w),v(s,w),w)\|ds \\
&\le \frac{\|\phi_1(w)\|}{\Gamma(\gamma_1)} \\
&\quad + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}s^{1-\gamma_1}(a_1(w)\|u(s,w\| + b_1(w)\|v(s,w\|)ds \\
&\le \frac{\|\phi_1(w)\|}{\Gamma(\gamma_1)} + \frac{R}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}s^{1-\gamma_1}(a_1(w)+b_1(w))ds \\
&\le \frac{\|\phi_1(w)\|}{\Gamma(\gamma_1)} + \frac{(a_1(w)+b_1(w)T^{\alpha_1}}{\Gamma(1+\alpha_1)} \\
&:= \ell_1.
\end{aligned}$$

Thus,

$$\|(N_1(u,v))(\cdot,w)\|_{C_{\gamma_1}} \le \ell_1.$$

Furthermore, for any $w \in \Omega$ and each $(u,v) \in B_R$ and $t \in I$, we get:

$$\begin{aligned}
\|(N_2(u,v))(\cdot,w)\|_{C_{\gamma_2}} &\le \frac{\|\phi_2(w)\|}{\Gamma(\gamma_2)} + \frac{(a_2(w)+b_2(w)T^{\alpha_2}}{\Gamma(1+\alpha_)} \\
&:= \ell_2.
\end{aligned}$$

Hence,

$$\|(N(u,v))(\cdot,w)\|_{\mathcal{C}} \le (\ell_1,\ell_2) := \ell.$$

Step 3. $N(\cdot,\cdot,w)$ maps bounded sets into equicontinuous sets in $\mathcal{C}$.

Let B_R be the ball defined in Step 2. For each $t_1, t_2 \in I$ with $t_1 \le t_2$ and any $(u,v) \in B_R$ and $w \in \Omega$, we have:

$$\begin{aligned}
& \|t_1^{1-\gamma_1}(N_1(u,v))(t_1,w) - t_2^{1-\gamma_1}(N_1(u,v))(t_2,w)\| \\
\le\ & \frac{t_2^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_{t-1}^{t_2}(t_2-s)^{\alpha_1-1}\|f_1(s,u(s,w),v(s,w),w)\|ds \\
\le\ & \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)}(t_2-t_1)^{\alpha_1}(a_1(w)\|u(\cdot,w)\|_{C_{\gamma_1}} + b_1(w)\|v(\cdot,w)\|_{C_{\gamma_2}}) \\
\le\ & \frac{RT^{\alpha_1}(a_1(w)+b_1(w))}{\Gamma(1+\alpha_1)}(t_2-t_1)^{\alpha_1} \\
& \to 0 \ as\ t_1 \to t_2.
\end{aligned}$$

Furthermore, we get:

$$\begin{aligned}
& \|t_1^{1-\gamma_2}(N_2(u,v))(t_1,w) - t_2^{1-\gamma_2}(N_2(u,v))(t_2,w)\| \\
\le\ & \frac{RT^{\alpha_1 2}(a_2(w)+b_2(w))}{\Gamma(1+\alpha_2)}(t_2-t_1)^{\alpha_2}
\end{aligned}$$

$$\to 0 \text{ as } t_1 \to t_2.$$

As a consequence of Steps 1–3, with the Arzela–Ascoli theorem, we conclude that $N(\cdot,\cdot,w)$ maps B_R into a precompact set in $\mathcal{C}$.

Step 4. The set $E(w)$ consisting of $(u(\cdot,w),v(\cdot,w)) \in \mathcal{C}$ such that $(u(\cdot,w),v(\cdot,w)) = \lambda(w)(N((u,v))(\cdot,w)$ for some measurable function $\lambda : \Omega \to (0,1)$ is bounded in $\mathcal{C}$.

Let $(u(\cdot,w),v(\cdot,w)) \in \mathcal{C}$ such that $(u(\cdot,w),v(\cdot,w)) = \lambda(w)(N((u,v))(\cdot,w)$. Then, $u(\cdot,w) = \lambda(w)(N_1((u,v))(\cdot,w)$ and $v(\cdot,w) = \lambda(w)(N_2((u,v))(\cdot,w)$. Thus, for any $w \in \Omega$ and each $t \in I$, we have:

$$\begin{aligned}
\|t^{1-\gamma_1}u(t,w)\| &\leq \frac{\|\phi_1(w)\|}{\Gamma(\gamma_1)} + \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\|f_1(s,u(s,w),v(s,w),w)\|ds \\
&\leq \frac{\|\phi_1(w)\|}{\Gamma(\gamma_1)} \\
&\quad + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}s^{1-\gamma_1}(a_1(w)\|u(s,w\| + b_1(w)\|v(s,w\|)ds.
\end{aligned}$$

Furthermore, we get:

$$\begin{aligned}
\|t^{1-\gamma_2}v(t,w)\| &\leq \frac{\|\phi_2(w)\|}{\Gamma(\gamma_2)} \\
&\quad + \frac{1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}s^{1-\gamma_2}(a_2(w)\|u(s,w\| + b_2(w)\|v(s,w\|)ds.
\end{aligned}$$

Hence, we obtain:

$$\|t^{1-\gamma_1}u(t,w)\| + \|t^{1-\gamma_2}v(t,w)\| \leq a + bc\int_0^t (t-s)^{\alpha-1}(\|s^{1-\gamma_1}u(s,w\| + \|s^{1-\gamma_2}v(s,w\|)ds,$$

where:

$$a := \frac{\|\phi_1(w)\|}{\Gamma(\gamma_1)} + \frac{\|\phi_2(w)\|}{\Gamma(\gamma_2)},\ b := \frac{1}{\Gamma(\alpha_1)} + \frac{1}{\Gamma(\alpha_2)},$$
$$c := \max\{a_1(w)+a_2(w), b_1(w)+b_2(w)\},\ \alpha := \max\{\alpha_1,\alpha_2\}.$$

Lemma 1 implies that there exists $\rho := \rho(\alpha) > 0$ such that:

$$\begin{aligned}
\|t^{1-\gamma_1}u(t,w)\| + \|t^{1-\gamma_2}v(t,w)\| &\leq a + abc\rho\int_0^t (t-s)^{\alpha-1}ds \\
&\leq \frac{a + abc\rho T^\alpha}{\alpha} \\
&= L.
\end{aligned}$$

This gives:

$$\|u(\cdot,w)\|_{C_{\gamma_1}} + \|v(\cdot,w)\|_{C_{\gamma_2}} \leq L.$$

Hence:

$$\|(u(\cdot,w),v(\cdot,w))\|_{\mathcal{C}} \leq L.$$

This shows that the set $E(w)$ is bounded. As a consequence of Steps 1–4 together with Theorem 2, we can conclude that N has at least one fixed point in B_R, which is a solution for the system (1) and (2). $\square$

4. Coupled Hilfer–Hadamard Fractional Differential Systems

Now, we are concerned with the coupled system (3) and (4). Set $C := C([1,T])$, and denote the weighted space of continuous functions defined by:

$$C_{\gamma,\ln}([1,T]) = \{w(t) : (\ln t)^{1-\gamma} w(t) \in C\},$$

with the norm:

$$\|w\|_{C_{\gamma,\ln}} := \sup_{t\in[1,T]} |(\ln t)^{1-r} w(t)|.$$

Furthermore, by $\mathcal{C}_{\gamma_1,\gamma_2,\ln}([1,T]) := C_{\gamma_1,\ln}([1,T]) \times C_{\gamma_2,\ln}([1,T])$, we denote the product weighted space with the norm:

$$\|(u,v)\|_{\mathcal{C}_{\gamma_1,\gamma_2,\ln}([1,T])} = \|u\|_{C_{\gamma_1,\ln}} + \|v\|_{C_{\gamma_2,\ln}}.$$

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [7] for a more detailed analysis.

Definition 12. *[7] (Hadamard fractional integral) The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1([1,T])$ is defined as:*

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

provided the integral exists.

Example 2. *Let $0 < q < 1$. Let $g(x) = \ln x$, $x \in [0,e]$. Then:*

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(2+q)} (\ln x)^{1+q};\ for\ a.e.\ x \in [0,e].$$

Set:

$$\delta = x\frac{d}{dx},\ q > 0,\ n = [q]+1,$$

and:

$$AC_\delta^n := \{u : [1,T] \to E : \delta^{n-1}[u(x)] \in AC(I)\}.$$

Definition 13. *[7] The Hadamard fractional derivative of order $q > 0$ applied to the function $w \in AC_\delta^n$ is defined as:*

$$({}^H D_1^q w)(x) = \delta^n ({}^H I_1^{n-q} w)(x).$$

In particular, if $q \in (0,1]$, then:

$$({}^H D_1^q w)(x) = \delta({}^H I_1^{1-q} w)(x).$$

Example 3. *Let $0 < q < 1$. Let $w(x) = \ln x$, $x \in [0,e]$. Then:*

$$({}^H D_1^q w)(x) = \frac{1}{\Gamma(2-q)} (\ln x)^{1-q},\ for\ a.e.\ x \in [0,e].$$

It has been proven (see, e.g., Kilbas [30], Theorem 4.8) that in the space $L^1(I)$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.:

$$({}^H D_1^q)({}^H I_1^q w)(x) = w(x).$$

From [7], we have:

$$({}^H I_1^q)({}^H D_1^q w)(x) = w(x) - \frac{({}^H I_1^{1-q} w)(1)}{\Gamma(q)}(\ln x)^{q-1}.$$

The Caputo–Hadamard fractional derivative is defined in the following way:

Definition 14. *The Caputo–Hadamard fractional derivative of order $q > 0$ applied to the function $w \in AC_\delta^n$ is defined as:*

$$({}^{Hc} D_1^q w)(x) = ({}^H I_1^{n-q} \delta^n w)(x).$$

In particular, if $q \in (0,1]$, then:

$$({}^{Hc} D_1^q w)(x) = ({}^H I_1^{1-q} \delta w)(x).$$

Definition 15. *Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, $w \in L^1(I)$, and ${}^H I_1^{(1-\alpha)(1-\beta)} w \in AC(I)$. The Hilfer–Hadamard fractional derivative of order α and type β applied to the function w is defined as:*

$$\begin{aligned} ({}^H D_1^{\alpha,\beta} w)(t) &= \left({}^H I_1^{\beta(1-\alpha)} ({}^H D_1^\gamma w)\right)(t) \\ &= \left({}^H I_1^{\beta(1-\alpha)} \delta ({}^H I_1^{1-\gamma} w)\right)(t); \ for\ a.e.\ t \in [1,T]. \end{aligned} \tag{9}$$

This new fractional derivative (9) may be viewed as interpolating the Hadamard fractional derivative and the Caputo–Hadamard fractional derivative. Indeed, for $\beta = 0$, this derivative reduces to the Hadamard fractional derivative, and when $\beta = 1$, we recover the Caputo–Hadamard fractional derivative.

$${}^H D_1^{\alpha,0} = {}^H D_1^\alpha, \ and \ {}^H D_1^{\alpha,1} = {}^{Hc} D_1^\alpha.$$

From [31], we conclude the following lemma.

Lemma 2. *Let $g : [1,T] \times E \to E$ be such that $g(\cdot, u(\cdot)) \in C_{\gamma,\ln}([1,T])$ for any $u \in C_{\gamma,\ln}([1,T])$. Then, Problem (3) is equivalent to the following Volterra integral equation:*

$$u(t) = \frac{\phi_0}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^H I_1^\alpha g(\cdot, u(\cdot)))(t).$$

Definition 16. *By a random solution of the coupled system (3) and (4), we mean a coupled measurable function $(u,v) \in C_{\gamma_1,\ln} \times C_{\gamma_2,\ln}$ that satisfies the conditions (4) and Equation (3) on $[1,T]$.*

Now, we give (without proof) similar existence and uniqueness results for the system (3) and (4). Let us introduce the following hypotheses:

(H_1') The functions g_i; $i = 1,2$ are Carathéodory.
(H_2') There exist measurable functions $p_i, q_i : \Omega \to (0,\infty)$; $i = 1,2$ such that:

$$\|g_i(t,u_1,v_1) - g_i(t,u_2,v_2)\| \le p_i(w)\|u_1 - u_2\| + q_i(w)\|v_1 - v_2\|;$$

$for\ a.e.\ t \in [1,T],\ and\ each\ u_i, v_i \in \mathbb{R}^m,\ i = 1,2.$
(H_3') There exist measurable functions $a_i, b_i : \Omega \to (0,\infty)$; $i = 1,2$ such that:

$$\|g_i(t,u,v)\| \le a_i(w)\|u\| + b_i(w)\|v\|; \ for\ a.e.\ t \in [1,T],\ and\ each\ u,v \in \mathbb{R}^m.$$

Theorem 5. *Assume that the hypotheses* (H_1') *and* (H_2') *hold. If for every* $w \in \Omega$*, the matrix:*

$$\begin{pmatrix} \frac{(\ln T)^{\alpha_1}}{\Gamma(1+\alpha_1)} p_1(w) & \frac{(\ln T)^{\alpha_1}}{\Gamma(1+\alpha_1)} q_1(w) \\ \frac{(\ln T)^{\alpha_2}}{\Gamma(1+\alpha_2)} p_2(w) & \frac{(\ln T)^{\alpha_2}}{\Gamma(1+\alpha_2)} q_2(w) \end{pmatrix}$$

converges to zero, then the coupled system (3) and (4) has a unique random solution.

Theorem 6. *Assume that the hypotheses* (H_1') *and* (H_3') *hold. Then, the coupled system (3) and (4) has at at least a random solution.*

5. An Example

We equip the space $\mathbb{R}_-^* := (-\infty, 0)$ with the usual σ-algebra consisting of Lebesgue measurable subsets of $\mathbb{R}_-^*$. Consider the following random coupled Hilfer fractional differential system:

$$\begin{cases} (D_0^{\frac{1}{2},\frac{1}{2}} u)(t,w) = f(t,u(t,w),v(t,w),w); \\ (D_0^{\frac{1}{2},\frac{1}{2}} v)(t) = g(t,u(t,w),v(t,w),w); \\ (I_0^{\frac{1}{4}} u)(0,w) = \cos w, \\ (I_0^{\frac{1}{4}} v_n)(0,w) = \sin w, \end{cases} ;\ w \in \mathbb{R}_-^*,\ t \in [0,1], \tag{10}$$

where:

$$f(t,u,v,w) = \frac{t^{\frac{-1}{4}} w^2 (u(t)+v(t)) \sin t}{64(1+w^2+\sqrt{t})(1+|u|+|v|)};\ t \in [0,1],$$

$$g(t,u,v) = \frac{w^2 (u(t)+v(t)) \cos t}{64(1+|u|+|v|)};\ w \in \mathbb{R}_-^*,\ t \in [0,1].$$

Set $\alpha_i = \beta_i = \frac{1}{2};\ i = 1,2$, then $\gamma_i = \frac{3}{4};\ i = 1,2$. The hypothesis (H_2) is satisfied with:

$$p_1(w) = p_2(w) = q_1(w) = q_2(w) = \frac{w^2}{64(1+w^2)}.$$

Furthermore, if for every $w \in \mathbb{R}_-^*$, the matrix:

$$\frac{w^2}{64(1+w^2)\Gamma(\frac{1}{2})} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

converges to zero, hence, Theorem 3 implies that the system (10) has a unique random solution defined on $[0,1]$.

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