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Article

Unique Existence Result of Approximate Solution to Initial Value Problem for Fractional Differential Equation of Variable Order Involving the Derivative Arguments on the Half-Axis

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Abstract: The semigroup properties of the Riemann–Liouville fractional integral have played a key role in dealing with the existence of solutions to differential equations of fractional order. Based on some results of some experts', we know that the Riemann–Liouville variable order fractional integral does not have semigroup property, thus the transform between the variable order fractional integral and derivative is not clear. These judgments bring us extreme difficulties in considering the existence of solutions of variable order fractional differential equations. In this work, we will introduce the concept of approximate solution to an initial value problem for differential equations of variable order involving the derivative argument on half-axis. Then, by our discussion and analysis, we investigate the unique existence of approximate solution to this initial value problem for differential equation of variable order involving the derivative argument on half-axis. Finally, we give examples to illustrate our results.

Keywords: variable order fractional derivative; initial value problem; fractional differential equations; piecewise constant functions; approximate solution

1. Introduction

In this paper, we will observe and study the unique existence of approximate solution to the following initial value problem of variable order

$$\begin{cases} D_{0+}^{p(t)}x(t) = f(t, x, D_{0+}^{q(t)}x), 0 < t < +\infty, \\ x(0) = 0, \end{cases} \tag{1}$$

where $0 < q(t) < p(t) < 1$, $f(t, x, D_{0+}^{q(t)}x)$ are given real functions, and $D_{0+}^{p(t)}$, $D_{0+}^{q(t)}$ denote derivatives of variable order $p(t)$ and $q(t)$ defined by

$$D_{0+}^{p(t)}x(t) = \frac{d}{dt}\int_0^t \frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))}x(s)ds, t > 0. \tag{2}$$

$$D_{0+}^{q(t)}x(t) = \frac{d}{dt}\int_0^t \frac{(t-s)^{-q(t)}}{\Gamma(1-q(t))}x(s)ds, \ \ t > 0,$$

and $\frac{1}{\Gamma(1-p(t))}\int_0^t (t-s)^{-p(t)}x(s)ds$ is integral of variable order $1-p(t)$ for function $x(t)$, for details, please refer to [1].

The operators of variable order, which fall into a more complex category, are the derivatives and integrals whose orders are the functions of certain variables. There are several definitions of variable order fractional integrals and derivatives. The following are several definitions of variable order fractional integrals and derivatives, which can be found in [2]. Let $-\infty < a < b < \infty$.

Definition 1. *Let $p : [a, b] \to (0, +\infty)$, the left Riemann–Liouville fractional integral of order $\alpha(t)$ for function $x(t)$ are defined as the following two types*

$$I_{a+}^{\alpha(t)} x(t) = \int_a^t \frac{(t-s)^{\alpha(t)-1}}{\Gamma(\alpha(t))} x(s) ds, \ \ t > a, \tag{3}$$

$$\boldsymbol{I}_{a+}^{\alpha(t)} x(t) = \int_a^t \frac{(t-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))} x(s) ds, \ \ t > a. \tag{4}$$

Definition 2. *Let $\alpha : [a, b] \to (n-1, n]$ (n is a natural number), the left Riemann–Liouville fractional derivative of order $\alpha(t)$ for function $x(t)$ are defined as the following two types*

$$D_{a+}^{\alpha(t)} x(t) = \left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-s)^{n-\alpha(t)-1}}{\Gamma(n-\alpha(t))} x(s) ds, \ \ t > a, \tag{5}$$

$$\boldsymbol{D}_{a+}^{\alpha(t)} x(t) = \left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-s)^{n-\alpha(s)-1}}{\Gamma(n-\alpha(s))} x(s) ds, \ \ t > a. \tag{6}$$

Definition 3. *Let $\alpha : [a, b] \to (n-1, n]$($n$ is a natural number), the left Caputo fractional derivative of order $\alpha(t)$ for function $x(t)$ are defined as the following two types*

$$^C D_{a+}^{\alpha(t)} x(t) = \int_a^t \frac{(t-s)^{n-\alpha(t)-1}}{\Gamma(n-\alpha(t))} x^{(n)}(s) ds, \ \ t > a, \tag{7}$$

$$^C \boldsymbol{D}_{a+}^{\alpha(t)} x(t) = \int_a^t \frac{(t-s)^{n-\alpha(s)-1}}{\Gamma(n-\alpha(s))} x^{(n)}(s) ds, \ \ t > a. \tag{8}$$

The problems denoted by the operator of variable order are apparently more complicated than the ones denoted by the operator of constant order. Recently, some authors have considered the applications of derivatives of variable order in various sciences such as anomalous diffusion modeling, mechanical applications, multi-fractional Gaussian noises. Among these, there have been many works dealing with numerical methods for some class of variable order fractional differential equations, for instance, [1–20].

We notice that, if the order $p(t)$ is a constant function q, then the Riemann–Liouville variable order fractional derivatives and integrals are the Riemann–Liouville fractional derivative and integral, respectively [21]. We know there are some important properties as following. Let $-\infty < b < \infty$.

Lemma 1. *[21] The Riemann–Liouville fractional integral defined for function $x(t) \in L(0, b)$ exists almost everywhere.*

Lemma 2. *[21] The equality $I_{0+}^{\gamma} I_{0+}^{\delta} x(t) = I_{0+}^{\delta} I_{0+}^{\gamma} x(t) = I_{0+}^{\gamma+\delta} x(t)$, $0 < \gamma < 1, 0 < \delta < 1$ holds for $x \in L(0, b)$.*

Lemma 2 is semigroup property for the Riemann–Liouville fractional integral, which is very crucial in obtaining the following Lemmas 3–5. In other words, without Lemma 2, one could not have Lemmas 3–5, for details, please refer to [21].

Lemma 3. *[21] The equality $D_{0+}^{\gamma} I_{0+}^{\gamma} x(t) = x(t)$, $0 < \gamma < 1$ holds for $x \in L(0, b)$.*

Lemma 4. *[21] Let $0 < \alpha < 1$, then the differential equation*

$$D_{0+}^{\alpha} x = 0, t > 0$$

has solution

$$x(t) = ct^{\alpha-1}, c \in R.$$

Lemma 5. *[21] Let $0 < \alpha < 1$, $x \in L(0,b)$, $D_{0+}^{\alpha} x \in L(0,b)$. Then the following equality holds*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = x(t) + ct^{\alpha-1}, c \in R.$$

These properties play a very important role in considering the existence of the solutions of differential equations for the Riemann–Liouville fractional derivative, for details, please refer to [22–26]. However, from [15–18], for general functions $h(t), g(t)$, we notice that the semigroup property does not hold, i.e., $I_{a+}^{h(t)} I_{a+}^{g(t)} \neq I_{a+}^{h(t)+g(t)}$. Thus, it brings us extreme difficulties, that we cannot get these properties like Lemmas 3–5 for the variable order fractional operators (integral and derivative). Without these properties for variable order fractional derivative and integral, we can hardly consider the existence of solutions of differential equations for variable order derivative by means of nonlinear functional analysis (for instance, some fixed point theorems).

In [18], by means of Banach contraction principle, we considered the uniqueness result of solutions to initial value problems of differential equations of variable order

$$\begin{cases} D_{0+}^{q(t)} x(t) = f(t,x), 0 < t \leq T, \\ x(0) = 0, \end{cases} \tag{9}$$

where $0 < T < +\infty$, $D_{0+}^{q(t)}$ denotes derivative of variable order defined by (2), and $q : [0,T] \to (0,1]$ is a piecewise constant function with partition $P = \{[0,T_1], (T_1,T_2], (T_2,T_3], \cdots, (T_{N^*-1},T]\}$ (N^* is a given natural number) of the finite interval $[0,T]$, i.e.,

$$q(t) = \sum_{k=1}^{N^*} q_k I_k(t), t \in [0,T],$$

where $0 < q_k \leq 1, k = 1,2,\cdots,N^*$ are constants, and I_k is the indicator of the interval $[T_{k-1},T_k]$, $k = 1,2,\cdots,N^*$(here $T_0 = 0, T_{N^*} = T$), that is $I_k = 1$ for $t \in [T_{k-1},T_k]$, $I_k = 0$ for elsewhere.

In this paper, we will consider the existence of solutions to the problem (1) for variable orders $p(t), q(t)$ are not piecewise constants. Based on some analysis, we will introduce the concept of approximate solution to the problem (1). Then, according to our discussion and analysis, we explore the unique existence of the approximate solution of the problem (1).

This paper is organized as follows. In Section 2, we provide some facts to the variable order integral and derivative through several examples. Also, we state some results which will play a very important role in obtaining our main results. In Section 3, we set forth our main result. Finally, two examples are given.

2. Some Preliminaries on Approximate Solution

In this section, we give some preliminaries on approximate solutions to the initial value problem (1). First of all, we use an example to illustrate the claim: for general function $p(t), q(t)$, the Riemann–Liouville variable order fractional integral does not have the semigroup property.

Example 1. *Let $p(t) = \frac{t}{6} + \frac{1}{3}$, $q(t) = \frac{t}{4} + \frac{1}{4}$, $f(t) = 1, 0 \leq t \leq 3$. Now, we calculate $I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=1}$ and $I_{0+}^{p(t)+q(t)} f(t)|_{t=1}$ which are defined in (3).*

For $1 \le t \le 3$, *we have*

$$\begin{aligned} I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) = & \int_0^t \frac{(t-s)^{\frac{t}{6}+\frac{1}{3}-1}}{\Gamma(\frac{t}{6}+\frac{1}{3})} \int_0^s \frac{(s-\tau)^{\frac{s}{4}+\frac{1}{4}-1}}{\Gamma(\frac{s}{4}+\frac{1}{4})} d\tau ds \\ = & \int_0^t \frac{(t-s)^{\frac{t}{6}-\frac{2}{3}} s^{\frac{1}{4}+\frac{s}{4}}}{\Gamma(\frac{t}{6}+\frac{1}{3})\Gamma(\frac{5}{4}+\frac{s}{4})} ds \\ = & \int_0^1 \frac{(t-s)^{\frac{t}{6}-\frac{2}{3}} s^{\frac{1}{4}+\frac{s}{4}}}{\Gamma(\frac{t}{6}+\frac{1}{3})\Gamma(\frac{5}{4}+\frac{s}{4})} ds + \int_1^t \frac{(t-s)^{\frac{t}{6}-\frac{2}{3}} s^{\frac{1}{4}+\frac{s}{4}}}{\Gamma(\frac{t}{6}+\frac{1}{3})\Gamma(\frac{5}{4}+\frac{s}{4})} ds. \end{aligned}$$

We set $M_1 = \max_{1 \le t \le 3} |\frac{1}{\Gamma(p(t))}|$ *and* $M_2 = \max_{1 \le s \le 3} |\frac{1}{\Gamma(\frac{5}{4}+\frac{s}{4})}|$. *For* $1 \le t \le 3$, *it holds*

$$\begin{aligned} \left|\int_1^t \frac{(t-s)^{\frac{t}{6}-\frac{2}{3}} s^{\frac{1}{4}+\frac{s}{4}}}{\Gamma(\frac{t}{6}+\frac{1}{3})\Gamma(\frac{5}{4}+\frac{s}{4})} ds\right| = & \left|\int_1^t 3^{\frac{t}{6}-\frac{2}{3}} (\frac{t-s}{3})^{\frac{t}{6}-\frac{2}{3}} \frac{s^{\frac{1}{4}+\frac{s}{4}}}{\Gamma(\frac{t}{6}+\frac{1}{3})\Gamma(\frac{5}{4}+\frac{s}{4})} ds\right| \\ \le & M_1 M_2 \int_1^t 3^{\frac{1}{2}-\frac{2}{3}} (\frac{t-s}{3})^{\frac{1}{6}-\frac{2}{3}} s ds \\ \le & M_1 M_2 \int_1^t 3^{\frac{1}{3}} (t-s)^{-\frac{1}{2}} 3 ds \\ = & 2 \times 3^{\frac{4}{3}} M_1 M_2 (t-1)^{\frac{1}{2}}, \end{aligned}$$

hence, we have

$$\left[\int_1^t \frac{(t-s)^{\frac{t}{6}-\frac{2}{3}} s^{\frac{1}{4}+\frac{s}{4}}}{\Gamma(\frac{t}{6}+\frac{1}{3})\Gamma(\frac{5}{4}+\frac{s}{4})} ds\right]_{t=1} = 0.$$

So, we get

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=1} = \int_0^1 \frac{(1-s)^{-\frac{1}{2}} s^{\frac{1}{4}+\frac{s}{4}}}{\Gamma(\frac{1}{2})\Gamma(\frac{5}{4}+\frac{s}{4})} ds \approx 1.063$$

and

$$I_{0+}^{p(t)+q(t)} f(t)|_{t=1} = \int_0^1 \frac{(1-s)^{p(1)+q(1)-1}}{\Gamma(p(1)+q(1))} ds = \int_0^1 ds = 1.$$

Therefore,

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=1} \ne I_{0+}^{p(t)+q(t)} f(t)|_{t=1}.$$

Without the semigroup property of the Riemann–Liouville variable order fractional integral, we can assure that the variable order fractional integration operator of non-constant continuous functions $p(t)$ for $x(t)$ does not have the properties like Lemmas 3–5. Consequently, we cannot transform differential equations of variable order into an integral equation.

Let $L[x(t);s]$, $L[I_{0+}^{p(t)}x(t);s]$, $L[D_{0+}^{p(t)}x(t);s]$ denote the Laplace transforms of functions $x(t)$, $I_{0+}^{p(t)}x(t)$ and $D_{0+}^{p(t)}x(t)$. We have not found out the explicit connection between $L[x(t);s]$ and $L[I_{0+}^{p(t)}x(t);s]$, as a result, we have not found out the explicit connection between $L[x(t);s]$ and $L[D_{0+}^{p(t)}x(t);s]$.

Example 2. *Let* $p(t) = \frac{1}{\sqrt{t+1}}, t \ge 0$. *We consider the Laplace transforms of functions* $t(t \ge 0)$ *and* $I_{0+}^{p(t)}t(t \ge 0)$ *defined in (3). We can know that*

$$L[t;s] = \int_0^\infty e^{-st} t dt = \frac{1}{s^2}, \tag{10}$$

$$
\begin{aligned}
L[I_{0+}^{(t+1)^{-\frac{1}{2}}}t;s] = & \int_0^\infty e^{-st}\int_0^t \frac{(t-\tau)^{(t+1)^{-\frac{1}{2}}-1}}{\Gamma((t+1)^{-\frac{1}{2}})}\tau d\tau dt \\
= & \int_0^\infty e^{-st}\int_\tau^\infty \frac{(t-\tau)^{(t+1)^{-\frac{1}{2}}-1}}{\Gamma((t+1)^{-\frac{1}{2}})}\tau dt d\tau \\
= & \int_0^\infty e^{-s(\tau+r)}\int_0^\infty \frac{r^{(\tau+r+1)^{-\frac{1}{2}}-1}}{\Gamma((\tau+r+1)^{-\frac{1}{2}})}\tau dr d\tau \\
= & \int_0^\infty e^{-s\tau}\tau\int_0^\infty e^{-sr}\frac{r^{(\tau+r+1)^{-\frac{1}{2}}-1}}{\Gamma((\tau+r+1)^{-\frac{1}{2}})}dr d\tau. \quad (11)
\end{aligned}
$$

By (10) or (11), we do not get the explicit connection between $L[t;s]$ and $L[I_{0+}^{(t+1)^{-\frac{1}{2}}}t;s]$.

In view of this example, the definition of variable order fractional derivative and the connection between the Laplace transforms of function $x(t)$ and its derivative $x'(t)$, we cannot obtain the Laplace transform formula for variable order fractional derivatives (2). Based on these facts, we cannot get the explicit expression of the solutions for the problem (1).

Throughout this paper, we assume that

(A_1) Let $p:[0,+\infty)\to(0,1)$ and $q:[0,+\infty)\to(0,1)$ be continuous functions, $q(t)<p(t)$ for all $t\in[0,+\infty)$, and that $p(t),q(t)$ satisfy

$$\lim_{t\to+\infty}p(t)=\rho_1,\ \lim_{t\to+\infty}q(t)=\rho_2, 0<\rho_1,\rho_2<1. \quad (12)$$

The following result is necessary in our next analysis of main result.

Lemma 6. *Let condition (A_1) hold. Then there exist positive constant T, natural number n^* and intervals $[0,T_1],(T_1,T_2],\cdots,(T_{n^*-1},T]$ $(T,+\infty)$($n^*\in N$) and functions $\alpha:[0,+\infty)\to(0,1)$ and $\beta:[0,+\infty)\to(0,1)$ defined by*

$$\alpha(t)=\sum_{k=1}^{n^*}p_kI_k(t)+\rho_1I_T(t),\ \ t\in[0,+\infty), \quad (13)$$

$$\beta(t)=\sum_{k=1}^{n^*}q_kI_k(t)+\rho_2I_T(t),\ \ t\in[0,+\infty), \quad (14)$$

where $p_k,q_k\in(0,1)$, $I_k(t)$ is the indicator of the interval $[T_{k-1},T_k]$ ($k=1,2,\cdots,n^*$, here $T_0=0,T_{n^*}=T$), i.e., $I_k(t)=1$ for $t\in[T_{k-1},T_k]$, $I_k(t)=0$ for t lying in elsewhere; $I_T(t)$ is the indicator of interval $(T,+\infty)$, i.e., $I_T(t)=1$ for $t\in(T,+\infty)$, $I_T(t)=0$ for t lying in elsewhere, such that for arbitrary small $\varepsilon>0$,

$$|p(t)-\alpha(t)|<\varepsilon,|q(t)-\beta(t)|<\varepsilon,\ \ 0\le t<+\infty. \quad (15)$$

Proof. By (12), for $\forall\,\varepsilon>0$, there exist $\overline{T}_1,\overline{T}_2>0$, such that

$$|p(t)-\rho_1|<\varepsilon,t>\overline{T}_1;|p(t)-\rho_2|<\varepsilon,t>\overline{T}_2.$$

Let $T=\max\{\overline{T}_1,\overline{T}_2\}$, then, for $\forall\,\varepsilon>0$, we have that

$$|p(t)-\rho_1|<\varepsilon,|p(t)-\rho_2|<\varepsilon,t>T. \quad (16)$$

We know that $p:[0,T]\to(0,1)$, $q:[0,T]\to(0,1)$ are continuous functions. Since $p(t)$ is right continuous at point 0, then, for arbitrary small $\varepsilon>0$, there is $\delta_{01}>0$ such that

$$|p(t)-p(0)|<\varepsilon,\ \ \text{for}\ \ 0\le t\le\delta_{01}.$$

Since $q(t)$ is right continuous at point 0, then, for arbitrary small $\varepsilon > 0$, there is $\delta_{02} > 0$ such that

$$|q(t) - q(0)| < \varepsilon, \text{ for } 0 \le t \le \delta_{02}.$$

Then for arbitrary small $\varepsilon > 0$, takeing $\delta_0 = \min\{\delta_{01}, \delta_{02}\}$, it holds

$$|p(t) - p(0)| < \varepsilon, \ |q(t) - q(0)| < \varepsilon, \text{ for } 0 \le t \le \delta_0. \tag{17}$$

We take point $\delta_0 \doteq T_1$ (if $T_1 < T$, we consider continuities of $p(t), q(t)$ at point T_1, otherwise, we end this procedure). Since $p(t)$ is right continuous at point T_1, so, for arbitrary small $\varepsilon > 0$, there is $\delta_{11} > 0$ such that

$$|p(t) - p(T_1)| < \varepsilon, \text{ for } T_1 \le t \le T_1 + \delta_{11},$$

Since $q(t)$ is right continuous at point T_1, then, for arbitrary small $\varepsilon > 0$, there is $\delta_{12} > 0$ such that

$$|q(t) - q(T_1)| < \varepsilon, \text{ for } T_1 \le t \le T_1 + \delta_{12}.$$

Hence, for arbitrary small $\varepsilon > 0$, taking $\delta_1 = \min\{\delta_{11}, \delta_{12}\}$, it holds

$$|p(t) - p(T_1)| < \varepsilon, \ |q(t) - q(T_1)| < \varepsilon, \text{ for } T_1 \le t \le T_1 + \delta_1. \tag{18}$$

We take point $T_1 + \delta_1 \doteq T_2$ (if $T_2 < T$, we consider continuities of $p(t), q(t)$ at point T_2, otherwise, we end this procedure). Since $p(t)$ is right continuous at point T_2, so, for arbitrary small $\varepsilon > 0$, there is $\delta_{21} > 0$ such that

$$|p(t) - p(T_2)| < \varepsilon, \text{ for } T_2 \le t \le T_2 + \delta_{21}.$$

Since $q(t)$ is right continuous at point T_2, so, for arbitrary small $\varepsilon > 0$, there is $\delta_{22} > 0$ such that

$$|q(t) - q(T_2)| < \varepsilon, \text{ for } T_2 \le t \le T_2 + \delta_{22}.$$

Thus, for arbitrary small $\varepsilon > 0$, taking $\delta_2 = \min\{\delta_{21}, \delta_{22}\}$, it holds

$$|p(t) - p(T_2)| < \varepsilon, \ |q(t) - q(T_2)| < \varepsilon, \text{ for } T_2 \le t \le T_2 + \delta_2. \tag{19}$$

We take point $T_2 + \delta_2 \doteq T_3$ (if $T_3 < T$, we consider continuities of $p(t), q(t)$ at point T_3, otherwise, we end this procedure). Since $p(t)$ is right continuous at point T_3, so, for arbitrary small $\varepsilon > 0$, there is $\delta_{31} > 0$ such that

$$|p(t) - p(T_3)| < \varepsilon, \text{ for } T_3 \le t \le T_3 + \delta_{31},$$

Since $q(t)$ is right continuous at point T_3, so, for arbitrary small $\varepsilon > 0$, there is $\delta_{32} > 0$ such that

$$|q(t) - q(T_3)| < \varepsilon, \text{ for } T_3 \le t \le T_3 + \delta_{32}.$$

Therefore, for arbitrary small $\varepsilon > 0$, taking $\delta_3 = \min\{\delta_{31}, \delta_{32}\}$, it holds

$$|p(t) - p(T_3)| < \varepsilon, \ |q(t) - q(T_3)| < \varepsilon, \text{ for } T_3 \le t \le T_3 + \delta_3. \tag{20}$$

Since $[0, T]$ is a finite interval, then, continuing this analysis procedure, we could obtain that there exist $\delta_{n^*-2} > 0$, $\delta_{n^*-1} > 0$ ($n^* \in N$) such that $T_{n^*-2} + \delta_{n^*-2} \doteq T_{n^*-1} < T$, $T_{n^*-1} + \delta_{n^*-1} \ge T$, such that for arbitrary small $\varepsilon > 0$, it holds

$$|p(t) - p(T_{n^*-1})| < \varepsilon, |q(t) - q(T_{n^*-1})| < \varepsilon \text{ for } T_{n^*-1} \le t \le T, \tag{21}$$

From (16)–(21), we could let

$$p(0) \doteq p_1, p(T_1) \doteq p_2, p(T_2) \doteq p_3, p(T_3) \doteq p_4, \cdots, p(T_{n^*-1}) \doteq p_{n^*},$$

$$q(0) \doteq q_1, q(T_1) \doteq q_2, q(T_2) \doteq q_3, q(T_3) \doteq q_4, \cdots, q(T_{n^*-1}) \doteq q_{n^*}.$$

Thus, we define functions $\alpha, \beta : [0, +\infty) \to (0, 1)$ as following

$$\alpha(t) = \begin{cases} p_1, & t \in [0, T_1], \\ p_2, & t \in (T_1, T_2], \\ \vdots \\ p_{n^*}, & t \in (T_{n^*-1}, T], \\ \rho_1, & t \in (T, +\infty), \end{cases} \qquad \beta(t) = \begin{cases} q_1, & t \in [0, T_1], \\ q_2, & t \in (T_1, T_2], \\ \vdots \\ q_{n^*}, & t \in (T_{n^*-1}, T], \\ \rho_2, & t \in (T, +\infty). \end{cases}$$

Hence, from the previous arguments, for arbitrary small $\varepsilon > 0$, we have

$$\begin{cases} |p(t) - p_1| < \varepsilon, |q(t) - q_1| < \varepsilon, & \text{for } t \in [0, T_1], \\ |p(t) - p_2| < \varepsilon, |q(t) - q_2| < \varepsilon, & \text{for } t \in (T_1, T_2], \\ \vdots \\ |p(t) - p_{n^*}| < \varepsilon, |q(t) - q_{n^*}| < \varepsilon, & \text{for } t \in (T_{n^*-1}, T], \\ |p(t) - \rho_1| < \varepsilon, |q(t) - \rho_2| < \varepsilon, & \text{for } t \in (T, +\infty). \end{cases} \tag{22}$$

Thus, we complete this proof. □

The following example illustrates that the semigroup property of the variable order fractional integral does not holds for the piecewise constant functions $p(t)$ and $q(t)$ defined in the same partition of finite interval $[a, b]$.

Example 3. *Let* $p(t) = \begin{cases} 4, & 0 \le t \le 1, \\ 3, & 1 < t \le 4, \end{cases}$ $q(t) = \begin{cases} 3, & 0 \le t \le 1, \\ 2, & 1 < t \le 4, \end{cases}$ *and* $f(t) = 1, 0 \le t \le 4$. *We'll verify* $I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=3} \neq I_{0+}^{p(t)+q(t)} f(t)|_{t=3}$, *here, the variable order fractional integral is defined in (3). For* $1 \le t \le 4$, *we have*

$$\begin{aligned} & I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) \\ = \; & \int_0^1 \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{3-1}}{\Gamma(3)} d\tau ds + \int_1^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{2-1}}{\Gamma(2)} d\tau ds \\ = \; & \int_0^1 \frac{(t-s)^{p(t)-1} s^3}{6\Gamma(p(t))} ds + \int_1^t \frac{(t-s)^{p(t)-1} s^2}{2\Gamma(2)\Gamma(p(t))} ds, \end{aligned}$$

thus, we have

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=3} = \int_0^1 \frac{(3-s)^2 s^3}{6\Gamma(3)} ds + \int_1^3 \frac{(3-s)^2 s^2}{2\Gamma(2)\Gamma(3)} ds = \frac{245}{144}.$$

$$I_{0+}^{p(t)+q(t)} f(t)|_{t=3} = \int_0^3 \frac{(3-s)^{p(3)+q(3)-1}}{\Gamma(p(3)+q(3))} ds = \frac{3^{3+2}}{\Gamma(1+3+2)} = \frac{81}{40}.$$

Therefore, we obtain

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=3} \neq I_{0+}^{p(t)+q(t)} f(t)|_{t=3},$$

which implies that the semigroup property of the variable order fractional integral does not hold for the piecewise constant functions $p(t)$ *and* $q(t)$ *defined in the same partition* $[0,1], (1,4]$ *of finite interval* $[0,4]$.

Lemma 7. *[10] Suppose* $\beta > 0$, $a(t)$ *is a nonnegative nondecreasing function locally integrable on* $0 \leq t < L$ *(some* $L \leq +\infty$*) and* $g(t)$ *is a nonnegative nondecreasing continuous function defined on* $0 \leq t < L$, $g(t) \leq M$ *(constant), and suppose u(t) is nonnegative and locally integrable on* $0 \leq t < L$ *with*

$$u(t) \leq a(t) + g(t) \int_0^t (t-s)^{\beta-1} u(s) ds$$

on this interval. Then

$$u(t) \leq a(t) E_\beta (g(t)\Gamma(\beta) t^\beta), 0 \leq t < L,$$

where E_β *is the Mittag–Leffler function defined by* $E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta+1)}$.

3. Existence of Approximate Solution

According to the previous arguments, we do not transform the problem (1) into an integral equation. Here, we consider the unique existence of approximate solution of the problem (1). In this section, we present our main results.

Now we make the following assumptions:

(A_2) $f : [0,+\infty) \times R^2 \to R$ be a continuous function, and there exist positive constants $\lambda > \{\rho_1, \rho_2\}$, $c_1, c_2 > 0$ satisfying

$$\frac{c_1}{\Gamma(1+\rho_1)} + \frac{c_2}{\Gamma(1+\rho_1-\rho_2)} < 1,$$

such that

$$|f(t,(1+t^\lambda)x_1,(1+t^\lambda)y_1) - f(t,(1+t^\lambda)x_2,(1+t^\lambda)y_2)| \leq c_1|x_1-x_2| + c_2|y_1-y_2|, \quad (23)$$

where ρ_1, ρ_2 are the constants in $(A1)$.

(A_3) $f(t,0,0) (t \in (0,+\infty))$ satisfies

$$\lim_{t\to+\infty} \frac{1}{1+t^\lambda} \int_0^t (t-s)^{\rho_1-\rho_2-1} |f(s,0,0)| ds = 0.$$

Let B_i denote the Banach spaces defined as

$$B_i = \{x | x \in C[0,T_i]\}$$

with the norm

$$\|x\|_{B_i} = \max_{t\in[0,T_i]} |x(t)|, \quad (24)$$

where T_i is the constant obtained in Lemma 6, $i = 1, \cdots, n^* (T_{n^*} = T)$. Let

$$E = \left\{ x \middle| x \in C[0,+\infty), \sup_{t\geq 0} \frac{|x(t)|}{1+t^\lambda} < \infty \right\}$$

with the norm

$$\|x\|_E = \sup_{t\geq 0} \frac{|x(t)|}{1+t^{\lambda}}, \tag{25}$$

where $\lambda > \{\rho_1, \rho_2\}$. Then, by the same arguments as in Lemma 2.2 of [22], we know that $(E, \|\cdot\|_E)$ is a Banach space, here we omit this proof.

Now, we consider the following initial value problem

$$\begin{cases} D_{0+}^{\alpha(t)} x(t) = f(t, x, D_{0+}^{\beta(t)} x), 0 < t < +\infty, \\ x(0) = 0, \end{cases} \tag{26}$$

where $\alpha(t), \beta(t)$ are defined in (13) and (14).

In order to obtain our main results, we start off by carrying on essential analysis to the equation of (26).

By (13) and (14), we get

$$\int_0^t \frac{(t-s)^{-\alpha(t)}}{\Gamma(1-\alpha(t))} x(s) ds = \sum_{k=1}^{n^*} I_k(t) \int_0^t \frac{(t-s)^{-p_k}}{\Gamma(1-p_k)} x(s) ds + I_T(t) \int_0^t \frac{(t-s)^{-\rho_1}}{\Gamma(1-\rho_1)} x(s) ds,$$

$$\int_0^t \frac{(t-s)^{-\beta(t)}}{\Gamma(1-\beta(t))} x(s) ds = \sum_{k=1}^{n^*} I_k(t) \int_0^t \frac{(t-s)^{-q_k}}{\Gamma(1-q_k)} x(s) ds + I_T(t) \int_0^t \frac{(t-s)^{-\rho_2}}{\Gamma(1-\rho_2)} x(s) ds \doteq h_{\beta,x}(t),$$

So, the equation of (26) can be written by

$$\frac{d}{dt}\left(\sum_{k=1}^{n^*} I_k(t) \int_0^t \frac{(t-s)^{-p_k}}{\Gamma(1-p_k)} x(s) ds + I_T(t) \int_0^t \frac{(t-s)^{-\rho_1}}{\Gamma(1-\rho_1)} x(s) ds\right) = f(t, x, \frac{d}{dt} h_{\beta,x}(t)), \ \ 0 < t < +\infty. \tag{27}$$

Then, Equation (27) in the interval $(0, T_1]$ can be written by

$$\frac{d}{dt} \int_0^t \frac{(t-s)^{-p_1}}{\Gamma(1-p_1)} x(s) ds = D_{0+}^{p_1} x(t) = f(t, x, D_{0+}^{q_1} x), 0 < t \leq T_1. \tag{28}$$

The Equation (27) in the interval $(T_1, T_2]$ can be written by

$$\frac{d}{dt} \int_0^t \frac{(t-s)^{-p_2}}{\Gamma(1-p_2)} x(s) ds = f(t, x, \frac{d}{dt} \int_0^t \frac{(t-s)^{-q_2}}{\Gamma(1-q_2)} x(s) ds), \ \ T_1 < t \leq T_2. \tag{29}$$

The Equation (27) in the interval $(T_2, T_3]$ can be written by

$$\frac{d}{dt} \int_0^t \frac{(t-s)^{-p_3}}{\Gamma(1-p_3)} x(s) ds = f(t, x, \frac{d}{dt} \int_0^t \frac{(t-s)^{-q_3}}{\Gamma(1-q_3)} x(s) ds), \ \ T_2 < t \leq T_3. \tag{30}$$

The Equation (27) in the interval $(T_{i-1}, T_i], i = 4, 5, \cdots, n^*$ $(T_{n^*} = T)$ can be written by

$$\frac{d}{dt} \int_0^t \frac{(t-s)^{-p_i}}{\Gamma(1-p_i)} x(s) ds = f(t, x, \frac{d}{dt} \int_0^t \frac{(t-s)^{-q_i}}{\Gamma(1-q_i)} x(s) ds), \ \ T_{i-1} < t \leq T_i. \tag{31}$$

The Equation (27) in the interval $(T, +\infty)$ can be written by

$$\frac{d}{dt} \int_0^t \frac{(t-s)^{-\rho_1}}{\Gamma(1-\rho_1)} x(s) ds = f(t, x, \frac{d}{dt} \int_0^t \frac{(t-s)^{-\rho_2}}{\Gamma(1-\rho_2)} x(s) ds), \ \ T < t < +\infty. \tag{32}$$

Now, we present the definition of a solution to the problem (26), which is crucial in our work.

Definition 4. *We say the problem (26) exists one unique solution, if there are unique functions $u_i(t)$, $i = 1, 2, \cdots, n^*$, such that $u_1 \in C[0, T_1]$ satisfying Equation (28) and $u_1(0) = 0$; $u_2 \in C[0, T_2]$ satisfying Equation (29) and $u_2(0) = 0$; $u_3 \in C[0, T_3]$ satisfying Equation (30) and $u_3(0) = 0$; $u_i \in C[0, T_i]$ satisfying Equation (31) and $u_i(0) = 0$ $(i = 4, 5, \cdots, n^*)(T_{n^*} = T)$; $u_T \in C[0, +\infty)$ satisfying Equation (32) and $u_T(0) = 0$.*

The following is the definition of approximate solution of the problem (1).

Definition 5. *If there exist $T > 0$, natural number $n^* \in N$ and intervals $[0, T_1], (T_1, T_2], \cdots, (T_{n^*-1}, T]$, $(T, +\infty)$ and functions defined in Equations (13) and (14), such that the problem (26) exists one unique solution, then, we say this solution of the problem (26) is one unique approximate solution of the problem (1).*

Our main result is as follows.

Theorem 1. *Let conditions $(A_1), (A_2), (A_3)$ hold, then the problem (1) exists one unique approximate solution.*

Proof of Theorem 1. From Definitions 4 and 5 and Lemma 6, we only need to consider the unique existence of solution of the problem (26). According to the above analysis, equation of problem (26) can be written as the Equation (27). So Equation (26) in the interval $(0, T_1]$ can be written as (28). Applying operator $I_{0+}^{p_1}$ to both sides of (28), by Lemma 5, we have

$$x(t) = ct^{p_1-1} + \frac{1}{\Gamma(p_1)} \int_0^t (t-s)^{p_1-1} f(s, x(s), D_{0+}^{q_1} x(s)) ds, \;\; 0 < t \leq T_1.$$

By $x(0) = 0$ and the assumption of function f, we get $c = 0$, that is

$$x(t) = \frac{1}{\Gamma(p_1)} \int_0^t (t-s)^{p_1-1} f(s, x(s), D_{0+}^{q_1} x(s)) ds, \;\; 0 \leq t \leq T_1. \tag{33}$$

Let $D_{0+}^{q_1} x(t) = y(t)$, then, according to $x(0) = 0$ and Lemma 5, we get that

$$x(t) = I_{0+}^{q_1} y(t),$$

hence we will consider existence of solution to integral equation as following

$$y(t) = \frac{1}{\Gamma(p_1 - q_1)} \int_0^t (t-s)^{p_1-q_1-1} f(s, I_{0+}^{q_1} y(s), y(s)) ds, \;\; 0 \leq t \leq T_1. \tag{34}$$

Obviously, if $y^* \in B_1 = C[0, T_1]$ is a solution of (34), then, applying operator $I_{0+}^{q_1}$ on both sides of (34), from Lemma 2, it holds

$$I_{0+}^{q_1} y^*(t) = I_{0+}^{q_1} I_{0+}^{p_1-q_1} f(t, I_{0+}^{q_1} y^*(t), y^*(t)) = I_{0+}^{p_1} f(t, I_{0+}^{q_1} y^*(t), y^*(t)), \;\; 0 \leq t \leq T_1,$$

let

$$I_{0+}^{q_1} y^*(t) = x^*(t), 0 \leq t \leq T,$$

as a result, we have that

$$x^*(t) = I_{0+}^{p_1} f(t, x^*(t), D_{0+}^{q_1} x^*(t)), \;\; 0 \leq t \leq T_1,$$

that is, $x^* \in B_1 = C[0, T_1]$ is a solution of (33), thus, we know that $x^* \in B_1 = C[0, T_1]$ is a solution of Equation (28) with zero initial value condition.

Define operator $F : B_1 \to B_1$ by

$$Fy(t) = \frac{1}{\Gamma(p_1 - q_1)} \int_0^t (t-s)^{p_1-q_1-1} f(s, I_{0+}^{q_1} y(s), y(s)) ds, \ \ 0 \le t \le T_1. \tag{35}$$

From the continuity of function f and the standard arguments, we know that the operator $F : B_1 \to B_1$ is well defined. Let $M = \max_{0 \le t \le T} |f(t, 0, 0)|$. Let Ω_1 be a bounded, convex and closed subset of B_1 defined by

$$\Omega_1 = \{y | y \in B_1; |y(t)| \le K_1 e^{R_1^2 t^{p_1-q_1}}, 0 \le t \le T_1\},$$

where

$$K_1 = \frac{2MT_1^{p_1-q_1}}{\Gamma(1+p_1-q_1)},$$

$R_1 \in N$ satisfying

$$R_1 > \{1, (\frac{2d_1(1+T_1^{p_1-q_1})}{p_1-q_1})^{\frac{1}{p_1-q_1}}\},$$

here $d_1 = \frac{1}{\Gamma(p_1-q_1)} \left[\frac{c_1 T_1^{q_1}}{\Gamma(1+q_1)} + c_2 \right]$ (c_1, c_2 are the constants appearing in condition (A_2)).

By the analogy way as in [23], we could verify that $F : \Omega_1 \to \Omega_1$ is well defined. In fact, for $y \in \Omega_1$, since

$$\begin{aligned} |I_{0+}^{q_1} y(s)| \le & \quad \frac{1}{\Gamma(q_1)} \int_0^s (s-\tau)^{q_1-1} |y(\tau)| d\tau \\ \le & \quad \frac{K_1}{\Gamma(q_1)} \int_0^s (s-\tau)^{q_1-1} e^{R_1^2 \tau^{p_1-q_1}} d\tau \\ \le & \quad \frac{K_1}{\Gamma(q_1)} \int_0^s (s-\tau)^{q_1-1} e^{R_1^2 s^{p_1-q_1}} d\tau \\ = & \quad \frac{K_1}{\Gamma(1+q_1)} s^{q_1} e^{R_1^2 s^{p_1-q_1}} \\ \le & \quad \frac{K_1 T_1^{q_1}}{\Gamma(1+q_1)} e^{R_1^2 s^{p_1-q_1}}. \end{aligned}$$

Now, $y \in \Omega_1$, by estimations above and (A_2), we get

$$\begin{aligned} & |Fy(t)| \\ \le & \quad \frac{1}{\Gamma(p_1-q_1)} \int_0^t (t-s)^{p_1-q_1-1} |f(s, I_{0+}^{q_1} y(s), y(s))| ds \\ = & \quad \frac{1}{\Gamma(p_1-q_1)} \int_0^t (t-s)^{p_1-q_1-1} |f(s, I_{0+}^{q_1} y(s), y(s)) - f(s,0,0) + f(s,0,0)| ds \\ \le & \quad \frac{MT_1^{p_1-q_1}}{\Gamma(1+p_1-q_1)} + \frac{1}{\Gamma(p_1-q_1)} \int_0^t (t-s)^{p_1-q_1-1} (c_1 \frac{|I_{0+}^{q_1} y(s)|}{1+s^\lambda} + c_2 \frac{|y(s)|}{1+s^\lambda}) ds \\ \le & \quad \frac{K_1}{2} + \frac{1}{\Gamma(p_1-q_1)} \int_0^t (t-s)^{p_1-q_1-1} (c_1 |I_{0+}^{q_1} y(s)| + c_2 |y(s)|) ds \end{aligned}$$

$$
\begin{aligned}
&\le \frac{K_1}{2} + \frac{1}{\Gamma(p_1-q_1)}\int_0^t (t-s)^{p_1-q_1-1}\left(\frac{K_1 c_1 T_1^{q_1}}{\Gamma(1+q_1)} e^{R_1^2 s^{p_1-q_1}} + c_2 K_1 e^{R_1^2 s^{p_1-q_1}}\right) ds \\
&\le \frac{K_1}{2} + K_1 d_1 \Bigg[\sum_{i=1}^{R_1-1}\int_{\frac{(i-1)t}{R_1}}^{\frac{it}{R_1}} (t-s)^{p_1-q_1-1} e^{R_1^2 s^{p_1-q_1}} ds \\
&\quad + \int_{\frac{(R_1-1)t}{R_1}}^{t} (t-s)^{p_1-q_1-1} e^{R_1^2 s^{p_1-q_1}} ds\Bigg] \\
&\le \frac{K_1}{2} + K_1 d_1 \Bigg[\sum_{i=1}^{R_1-1}\int_{\frac{(i-1)t}{R_1}}^{\frac{it}{R_1}} R_1^{1-p_1+q_1}(R_1-i)^{p_1-q_1-1} t^{p_1-q_1-1} e^{R_1^2 s^{p_1-q_1}} ds \\
&\quad + \int_{\frac{(R_1-1)t}{R_1}}^{t} (t-s)^{p_1-q_1-1} e^{R_1^2 t^{p_1-q_1}} ds\Bigg] \\
&\le \frac{K_1}{2} + K_1 d_1 \Bigg[\sum_{i=1}^{R_1-1}\int_{\frac{(i-1)t}{R_1}}^{\frac{it}{R_1}} R_1^{1-p_1+q_1} t^{p_1-q_1-1} e^{R_1^2 s^{p_1-q_1}} ds \\
&\quad + \int_{\frac{(R_1-1)t}{R_1}}^{t} (t-s)^{p_1-q_1-1} e^{R_1^2 t^{p_1-q_1}} ds\Bigg] \\
&= \frac{K_1}{2} + K_1 d_1 R_1^{1-p_1+q_1}\int_0^{\frac{(R_1-1)t}{R_1}} t^{p_1-q_1-1} e^{R_1^2 s^{p_1-q_1}} ds + \frac{K_1 d_1 R_1^{q_1-p_1} T_1^{p_1-q_1}}{p_1-q_1} e^{R_1^2 t^{p_1-q_1}} \\
&\le \frac{K_1}{2} + K_1 d_1 R_1^{1-p_1+q_1}\int_0^{\frac{(R_1-1)t}{R_1}} s^{p_1-q_1-1} e^{R_1^2 s^{p_1-q_1}} ds + \frac{K_1 d_1 R_1^{q_1-p_1} T_1^{p_1-q_1}}{p_1-q_1} e^{R_1^2 t^{p_1-q_1}} \\
&\le \frac{K_1}{2} + \frac{K_1 d_1 R_1^{1-p_1+q_1}}{R_1^2(p_1-q_1)} e^{R_1^2 (\frac{(R_1-1)t}{R_1})^{p_1-q_1}} + \frac{K_1 d_1 R_1^{q_1-p_1} T_1^{p_1-q_1}}{p_1-q_1} e^{R_1^2 t^{p_1-q_1}} \\
&\le \frac{K_1}{2} + \frac{K_1 d_1 R_1^{-1-p_1+q_1}}{p_1-q_1} e^{R_1^2 t^{p_1-q_1}} + \frac{K_1 d_1 R_1^{q_1-p_1} T_1^{p_1-q_1}}{p_1-q_1} e^{R_1^2 t^{p_1-q_1}} \\
&\le \frac{K_1}{2} e^{R_1^2 t^{p_1-q_1}} + \frac{K_1 d_1 (1+T_1^{p_1-q_1})}{p_1-q_1} R_1^{q_1-p_1} e^{R_1^2 t^{p_1-q_1}} \\
&\le \frac{K_1}{2} e^{R_1^2 t^{p_1-q_1}} + \frac{K_1}{2} e^{R_1^2 t^{p_1-q_1}} = K_1 e^{R_1^2 t^{p_1-q_1}},
\end{aligned}
$$

which implies that $F:\Omega_1 \to \Omega_1$ is well defined. By the standard arguments, we could know that $F:\Omega_1 \to \Omega_1$ is a completely operator. Hence, the Schauder fixed point theorem assures that operator F has at least one fixed point $y_1(t) \in \Omega_1$. Obviously, $y_1(0) = 0$. Now, we will verify the uniqueness of solution to the integral Equation (34). We notice that: for $0 \le s \le t \le T_1$, if $0 \le t-s \le 1$, then $(t-s)^{p_1-1} \le (t-s)^{p_1-q_1-1}$; if $t-s \ge 1$, then $(t-s)^{p_1-q_1-1} \le (t-s)^{p_1-1}$. As a result, we take

$$\max\{(t-s)^{p_1-1}, (t-s)^{p_1-q_1-1}\} \doteq (t-s)^{\alpha-1},$$

where α denotes p_1 or p_1-q_1. Now, let $u_1(t), u_2(t)$ zre two solutions of the integral Equation (34), by expression above and (A_2), we get

$$
\begin{aligned}
&|u_1(t)-u_2(t)| \\
&\le \frac{1}{\Gamma(p_1-q_1)}\int_0^t (t-s)^{p_1-q_1-1}(c_1 \frac{|I_{0+}^{q_1}(u_1(s)-u_2(s))|}{1+s^\lambda} + c_2 \frac{|u_1(s)-u_2(s)|}{1+s^\lambda}) ds
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(p_1-q_1)}\int_0^t (t-s)^{p_1-q_1-1}(c_1|I_{0+}^{q_1}(u_1(s)-u_2(s))|+c_2|u_1(s)-u_2(s)|)ds \\
&\le \frac{c_1}{\Gamma(p_1-q_1)\Gamma(q_1)}\int_0^t (t-s)^{p_1-q_1-1}\int_0^s (s-\tau)^{q_1-1}|u_1(\tau)-u_2(\tau)|d\tau ds \\
&\quad + \frac{c_2}{\Gamma(p_1-q_1)}\int_0^t (t-s)^{p_1-q_1-1}|u_1(s)-u_2(s)|ds \\
&= \frac{c_1}{\Gamma(p_1-q_1)\Gamma(q_1)}\int_0^t \int_\tau^t (t-s)^{p_1-q_1-1}(s-\tau)^{q_1-1}|u_1(\tau)-u_2(\tau)|ds d\tau \\
&\quad + \frac{c_2}{\Gamma(p_1-q_1)}\int_0^t (t-s)^{p_1-q_1-1}|u_1(s)-u_2(s)|ds \\
&= \frac{c_1}{\Gamma(p_1)}\int_0^t (t-\tau)^{p_1-1}|u_1(\tau)-u_2(\tau)|d\tau + \frac{c_2}{\Gamma(p_1-q_1)}\int_0^t (t-s)^{p_1-q_1-1}|u_1(s)-u_2(s)|ds \\
&\le \frac{c_1}{\Gamma(p_1)}\int_0^t (t-\tau)^{\alpha-1}|u_1(\tau)-u_2(\tau)|d\tau + \frac{c_2}{\Gamma(p_1-q_1)}\int_0^t (t-s)^{\alpha-1}|u_1(s)-u_2(s)|ds \\
&= [\frac{c_1}{\Gamma(p_1)} + \frac{c_2}{\Gamma(p_1-q_1)}]\int_0^t (t-\tau)^{\alpha-1}|u_1(\tau)-u_2(\tau)|d\tau,
\end{aligned}
$$

by Lemma 7, we obtain that $u_1(t)=u_2(t), 0\le t\le T_1$, this assures the uniqueness of solution of (34). As a result, by some arguments above, $x_1(t)=I_{0+}^{q_1}y_1(t)$ is one unique solution of the Equation (28) with zero initial value condition.

Also, we have obtained that the Equation (27) in the interval $(T_1,T_2]$ can be written by (29). In order to consider the existence result of solutions to (29), we may discuss the following equation defined on interval $(0,T_2]$

$$
\frac{d}{dt}\int_0^t \frac{(t-s)^{-p_2}x(s)}{\Gamma(1-p_2)}ds = D_{0+}^{p_2}x(t) = f(t,x,\frac{d}{dt}\int_0^t \frac{(t-s)^{-q_2}x(s)}{\Gamma(1-q_2)}ds) = f(t,x,D_{0+}^{q_2}x). \tag{36}
$$

It is clear that if function $x\in C[0,T_2]$ satisfies the Equation (36), then $x(t)$ must satisfy the Equation (29). In fact, if $x^*\in C[0,T_2]$ with $x^*(0)=0$ is a solution of the Equation (36) with initial value condition $x(0)=0$, that is

$$
\begin{aligned}
&D_{0+}^{p_2}x^*(t) \\
=\ & \frac{d}{dt}\int_0^t \frac{(t-s)^{-p_2}x^*(s)}{\Gamma(1-p_2)}ds \\
=\ & f(t,x^*(t),D_{0+}^{q_2}x^*(t)) = f(t,x^*(t),\frac{d}{dt}\int_0^t \frac{(t-s)^{-q_2}x^*(s)}{\Gamma(1-q_2)}ds),\ \ 0<t\le T_2;\ \ x^*(0)=0.
\end{aligned}
$$

Hence, from the equality above, we have that $x^*\in C[0,T_2]$ with $x^*(0)=0$ satisfies the equation

$$
\frac{d}{dt}\int_0^t \frac{(t-s)^{-p_2}x^*(s)}{\Gamma(1-p_2)}ds = f(t,x^*(t),\frac{d}{dt}\int_0^t \frac{(t-s)^{-q_2}x^*(s)}{\Gamma(1-q_2)}ds),\ \ T_1\le t\le T_2,
$$

which means the function $x^*\in C[0,T_2]$ with $x^*(0)=0$ is a solution of the Equation (29).

Based on this fact, we consider the existence of solutions to the Equation (36) with initial value condition $x(0)=0$.

Now, applying operator $I_{0+}^{p_2}$ on both sides of (36), by Lemma 5, we have

$$x(t) = ct^{p_2-1} + \frac{1}{\Gamma(p_2)} \int_0^t (t-s)^{p_2-1} f(s, x(s), D_{0+}^{q_2} x(s)) ds, \;\; 0 < t \leq T_2.$$

By initial value condition $x(0) = 0$, we have $c = 0$, that is

$$x(t) = \frac{1}{\Gamma(p_2)} \int_0^t (t-s)^{p_2-1} f(s, x(s), D_{0+}^{q_2} x(s)) ds, \;\; 0 \leq t \leq T_2. \tag{37}$$

Let $D_{0+}^{q_2} x(t) = y(t)$, then, according to $x(0) = 0$ and Lemma 5, we get that

$$x(t) = I_{0+}^{q_2} y(t),$$

hence we will consider existence of solution to integral equation as following

$$y(t) = \frac{1}{\Gamma(p_2 - q_2)} \int_0^t (t-s)^{p_2-q_2-1} f(s, I_{0+}^{q_2} y(s), y(s)) ds, \;\; 0 \leq t \leq T_2. \tag{38}$$

Obviously, if $y^* \in B_2 = C[0, T_2]$ is a solution of (38), then, by (38) and Lemma 2, it holds

$$I_{0+}^{q_2} y^*(t) = I_{0+}^{q_2} I_{0+}^{p_2-q_2} f(t, I_{0+}^{q_2} y^*(t), y^*(t)) = I_{0+}^{p_2} f(t, I_{0+}^{q_2} y^*(t), y^*(t)), \;\; 0 \leq t \leq T_2,$$

let

$$I_{0+}^{q_2} y^*(t) = x^*(t), 0 \leq t \leq T_2,$$

as a result, we have that

$$x^*(t) = I_{0+}^{p_2} f(t, x^*(t), D_{0+}^{q_2} x^*(t)), \;\; 0 \leq t \leq T_2,$$

that is, $x^* \in B_2 = C[0, T_2]$ is a solution of (37), hence, $x^* \in B_2 = C[0, T_2]$ is a solution of Equation (29) with zero initial value condition.

Define operator $F : B_2 \to B_2$ by

$$Fy(t) = \frac{1}{\Gamma(p_2 - q_2)} \int_0^t (t-s)^{p_2-q_2-1} f(s, I_{0+}^{q_2} y(s), y(s)) ds, \;\; 0 \leq t \leq T_2.$$

From the continuity of function f and the standard arguments, we know that the operator $F : B_1 \to B_2$ is well defined. Let Ω_2 be a bounded, convex and closed subset of B_2 defined by

$$\Omega_2 = \{y | y \in B_2; |y(t)| \leq K_2 e^{R_2^2 t^{p_2-q_2}}, 0 \leq t \leq T_2\},$$

where

$$K_2 = \frac{2MT_2^{p_2-q_2}}{\Gamma(1 + p_2 - q_2)},$$

$R_2 \in N$ satisfying

$$R_2 > \{1, (\frac{2d_2(1 + T_2^{p_2-q_2})}{p_2 - q_2})^{\frac{1}{p_2-q_2}}\},$$

here $d_2 = \frac{1}{\Gamma(p_2-q_2)} \left[\frac{c_1 T_2^{q_2}}{\Gamma(1+q_2)} + c_2 \right]$ (c_1, c_2 are the constants appearing in condition (A_2)). By the same arguments above, there exists $y_2 \in \Omega_2$ such that $x_2(t) = I_{0+}^{q_2} y_2(t)$ is one unique solution of the Equation (29) with zero initial value condition.

In a similar way, for $i = 3, \cdots, n^*$, we get that the Equation (31) defined on $(T_{i-1}, T_i]$ $(T_{n^*} = T)$ has one solution $x_i(t) \in \Omega_i \subset B_i$ with $x_i(0) = 0$, where

$$\Omega_i = \{y | y \in B_i; |y(t)| \leq K_i e^{R_i^2 t^{p_i - q_i}}, 0 \leq t \leq T_i\},$$

$$K_i = \frac{2MT_i^{p_i - q_i}}{\Gamma(1 + p_i - q_i)},$$

$R_i \in N$ satisfying

$$R_i > \left\{1, \left(\frac{2d_i(1 + T_i^{p_i - q_i})}{p_i - q_i}\right)^{\frac{1}{p_i - q_i}}\right\},$$

here $d_i = \frac{1}{\Gamma(p_i - q_i)} \left[\frac{c_1 T_i^{q_i}}{\Gamma(1+q_i)} + c_2\right]$ (c_1, c_2 are the constants appearing in condition (A_2)), $i = 3, 4, \cdots, n^*$, $T_{n^*} = T$.

Finally, we get that the Equation (27) in the interval $(T, +\infty)$ can be written by (32). In order to consider the existence result of solutions to (32), we may discuss the following equation defined on interval $(0, +\infty)$

$$\frac{d}{dt} \int_0^t \frac{(t - s)^{-\rho_1}}{\Gamma(1 - \rho_1)} x(s) ds = D_{0+}^{\rho_1} x(t) = f(t, x, D_{0+}^{\rho_2} x), \quad 0 < t < +\infty. \tag{39}$$

We see that, if function $x \in C[0, +\infty)$ satisfies the Equation (39), then $x(t)$ must satisfy the Equation (32). In fact, if $x^* \in C[0, +\infty)$ with $x^*(0) = 0$ is a solution of the Equation (39) with initial value condition $x(0) = 0$, that is

$$\begin{aligned} D_{0+}^{\rho_1} x^*(t) = & \quad \frac{d}{dt} \int_0^t \frac{(t - s)^{-\rho_1} x^*(s)}{\Gamma(1 - \rho_1)} ds = f(t, x^*(t), D_{0+}^{\rho_2} x^*) \\ = & \quad f(t, x^*(t), \frac{d}{dt} \int_0^t \frac{(t - s)^{-\rho_2} x^*(s)}{\Gamma(1 - \rho_2)} ds), \quad 0 < t < +\infty; \quad x^*(0) = 0. \end{aligned}$$

Hence, from the equality above, we have $x^* \in C[0, +\infty)$ with $x^*(0) = 0$ satisfying the equation

$$\frac{d}{dt} \int_0^t \frac{(t - s)^{-\rho_1} x(s)}{\Gamma(1 - \rho_1)} ds = f(t, x(t), \frac{d}{dt} \int_0^t \frac{(t - s)^{-\rho_2} x(s)}{\Gamma(1 - \rho_2)} ds), \quad T < t < +\infty,$$

which means the function $x^* \in C[0, +\infty)$ with $x^*(0) = 0$ is a solution of the Equation (32).

Based on this fact, we will consider the existence of solutions to the Equation (39) with initial value condition $x(0) = 0$.

Now, applying operator $I_{0+}^{\rho_1}$ on both sides of (39), by Lemma 5, we have that

$$x(t) = ct^{\rho_1 - 1} + \frac{1}{\Gamma(\rho_1)} \int_0^t (t - s)^{\rho_1 - 1} f(s, x(s), D_{0+}^{\rho_2} x(s)) ds, \quad 0 < t < +\infty.$$

By initial value condition $x(0) = 0$, we have $c = 0$, that is

$$x(t) = \frac{1}{\Gamma(\rho_1)} \int_0^t (t - s)^{\rho_1 - 1} f(s, x(s), D_{0+}^{\rho_2} x(s)) ds, \quad 0 \leq t < +\infty. \tag{40}$$

Similar to arguments above, we let $D_{0+}^{\rho_2} x(t) = y(t)$, then, according to $x(0) = 0$ and Lemma 5, we get that

$$x(t) = I_{0+}^{\rho_2} y(t),$$

hence we will consider existence of solution to integral equation as following

$$y(t) = \frac{1}{\Gamma(\rho_1-\rho_2)} \int_0^t (t-s)^{\rho_1-\rho_2-1} f(s, I_{0+}^{\rho_2} y(s), y(s)) ds, \ \ 0 \le t < +\infty. \tag{41}$$

Obviously, if $y^* \in E$ is a solution of (41), then, by (41) and Lemma 2, it holds

$$I_{0+}^{\rho_2} y^*(t) = I_{0+}^{\rho_2} I_{0+}^{\rho_1-\rho_2} f(t, I_{0+}^{\rho_2} y^*(t), y^*(t)) = I_{0+}^{\rho_1} f(t, I_{0+}^{\rho_2} y^*(t), y^*(t)), \ \ 0 \le t < +\infty.$$

Let

$$I_{0+}^{\rho_2} y^*(t) = x^*(t), 0 \le t < +\infty.$$

As a result, we have that

$$x^*(t) = I_{0+}^{\rho_1} f(t, x^*(t), D_{0+}^{\rho_2} x^*(t)), \ \ 0 \le t < +\infty,$$

that is, $x^* \in E$ is a solution of (40), hence, $x^* \in E$ is a solution of Equation (32) with zero initial value condition.

Defining operator $F : E \to E$ as follows

$$Fy(t) = \frac{1}{\Gamma(\rho_1-\rho_2)} \int_0^t (t-s)^{\rho_1-\rho_2-1} f(s, I_{0+}^{\rho_2} y(s), y(s)) ds, 0 \le t < +\infty.$$

To get the operator $F : E \to E$ is well defined. First, we verify that $Fy \in C[0, +\infty)$ for $x \in E$. In fact, for the case of $t_0 \in (0, +\infty)$, take $t > t_0$, $t - t_0 < 1$, then

$$(t_0 - s)^{\rho_1-1} > (t-s)^{\rho_1-1}, 0 \le s < t_0.$$

Now, for $y \in E$, it holds

$$\begin{aligned} \frac{|I_{0+}^{\rho_2} y(s)|}{1+s^\lambda} &\le \frac{\int_0^s (s-\tau)^{\rho_2-1} |y(\tau)| d\tau}{\Gamma(\rho_2)(1+s^\lambda)} \\ &\le \frac{\int_0^s (s-\tau)^{\rho_2-1} (1+\tau^\lambda) \|y\|_E d\tau}{\Gamma(\rho_2)(1+s^\lambda)} \\ &\le \frac{\int_0^s (s-\tau)^{\rho_2-1} (1+s^\lambda) \|y\|_E d\tau}{\Gamma(\rho_2)(1+s^\lambda)} \\ &= \frac{\|y\|_E s^{\rho_2}}{\Gamma(1+\rho_2)}, \end{aligned}$$

thus, for $y \in E$, we have

$$\begin{aligned} |Fy(t)| \le & \frac{1}{\Gamma(\rho_1-\rho_2)} \int_0^{t_0} ((t_0-s)^{\rho_1-\rho_2-1} - (t-s)^{\rho_1-\rho_2-1})(c_1 \frac{|I_{0+}^{\rho_2} y(s)|}{1+s^\lambda} + c_2 \frac{|y(s)|}{1+s^\lambda}) ds \\ & + \frac{1}{\Gamma(\rho_1-\rho_2)} \int_{t_0}^t (t-s)^{\rho_1-\rho_2-1} (c_1 \frac{|I_{0+}^{\rho_2} y(s)|}{1+s^\lambda} + c_2 \frac{|y(s)|}{1+s^\lambda}) ds \\ & + \frac{1}{\Gamma(\rho_1-\rho_2)} \int_0^{t_0} ((t_0-s)^{\rho_1-\rho_2-1} - (t-s)^{\rho_1-\rho_2-1}) |f(s,0,0)| ds \\ & + \frac{1}{\Gamma(\rho_1-\rho_2)} \int_{t_0}^t (t-s)^{\rho_1-\rho_2-1} |f(s,0,0)| ds \end{aligned}$$

$$
\begin{aligned}
\leq\quad & \frac{\|y\|_E}{\Gamma(\rho_1-\rho_2)}\int_0^{t_0}((t_0-s)^{\rho_1-\rho_2-1}-(t-s)^{\rho_1-\rho_2-1})\left(c_1\frac{s^{\rho_2}}{\Gamma(1+\rho_2)}+c_2\right)ds\\
&+\frac{\|y\|_E}{\Gamma(\rho_1-\rho_2)}\int_{t_0}^{t}(t-s)^{\rho_1-\rho_2-1}\left(c_1\frac{s^{\rho_2}}{\Gamma(1+\rho_2)}+c_2\right)ds\\
&+\frac{\max_{0\leq t\leq t_0+1}|f(t,0,0)|}{\Gamma(\rho_1-\rho_2)}\int_0^{t_0}((t_0-s)^{\rho_1-\rho_2-1}-(t-s)^{\rho_1-\rho_2-1})ds\\
&+\frac{\max_{0\leq t\leq t_0+1}|f(t,0,0)|}{\Gamma(\rho_1-\rho_2)}\int_{t_0}^{t}(t-s)^{\rho_1-\rho_2-1}ds.
\end{aligned}
$$

We will consider the four terms above, respectively. For $0<\eta<\rho_1-\rho_2$, it is easy to show that

$$\int_0^t (t-s)^{\rho_1-\rho_2-1}s^{\eta}ds=\frac{\Gamma(1+\eta)\Gamma(\rho_1-\rho_2)t^{\rho_1-\rho_2+\eta}}{\Gamma(1+\rho_1-\rho_2+\eta)}.$$

Hence, for any given $\varepsilon>0$, there exists a $\delta_1>0$, such that, when $0\leq t_0\leq\delta_1$, it holds that

$$\frac{c_1\|y\|_E}{\Gamma(\rho_1-\rho_2)\Gamma(1+\rho_2)}\int_0^{t_0}(t_0-s)^{\rho_1-\rho_2-1}s^{\rho_2}ds<\frac{\varepsilon}{4},\frac{c_2\|y\|_E}{\Gamma(\rho_2-\rho_2)}\int_0^{t_0}(t_0-s)^{\rho_1-\rho_2-1}ds<\frac{\varepsilon}{4}. \tag{42}$$

Moreover, we get

$$
\begin{aligned}
&\int_{\delta_1}^{t_0}((t_0-s)^{\rho_1-\rho_2-1}-(t-s)^{\rho_1-\rho_2-1})s^{\rho_2}ds\\
\leq\quad & t_0^{\rho_2}\int_{\delta_1}^{t_0}((t_0-s)^{\rho_1-\rho_2-1}-(t-s)^{\rho_1-\rho_2-1})ds\\
=\quad & \frac{t_0^{\rho_2}}{\rho_1-\rho_2}((t_0-\delta_1)^{\rho_1-\rho_2}-(t-\delta_1)^{\rho_1-\rho_2}+(t-t_0)^{\rho_1-\rho_2})\\
\leq\quad & \frac{t_0^{\rho_2}}{\rho_1-\rho_2}(t-t_0)^{\rho_1-\rho_2},
\end{aligned}
$$

$$\int_{\delta_1}^{t_0}((t_0-s)^{\rho_1-\rho_2-1}-(t-s)^{\rho_1-\rho_2-1})ds\leq\frac{1}{\rho_1-\rho_2}(t-t_0)^{\rho_1-\rho_2},$$

hence, we know that there exists $\delta_2>0$ such that for $0<t-t_0<\delta_2$, we have

$$
\begin{aligned}
\frac{c_1\|y\|_E}{\Gamma(\rho_1-\rho_2)\Gamma(1+\rho_2)}\int_{\delta_1}^{t_0}((t_0-s)^{\rho_1-\rho_2-1}-(t-s)^{\rho_1-\rho_2-1})s^{\rho_2}ds<\frac{\varepsilon}{4},\\
\frac{c_2\|y\|_E}{\Gamma(\rho_2-\rho_2)}\int_{\delta_1}^{t_0}((t_0-s)^{\rho_1-\rho_2-1}-(t-s)^{\rho_1-\rho_2-1})ds<\frac{\varepsilon}{4},
\end{aligned}
$$

together with (42), it leads to

$$\int_0^{t_0}((t_0-s)^{\rho_1-\rho_2-1}-(t-s)^{\rho_1-\rho_2-1})(\frac{c_1\|y\|_E s^{\rho_2}}{\Gamma(\rho_1-\rho_2)\Gamma(1+\rho_2)}+\frac{c_2\|y\|_E}{\Gamma(\rho_2-\rho_2)})ds<\varepsilon.$$

By the direct calculation, we have

$$\int_{t_0}^{t}(t-s)^{\rho_1-\rho_2-1}s^{\rho_2}ds\leq(t_0+1)^{\rho_2}\frac{(t-t_0)^{\rho_1-\rho_2}}{\rho_1-\rho_2},$$

$$\int_{t_0}^{t}(t-s)^{\rho_1-\rho_2-1}ds \le \frac{(t-t_0)^{\rho_1-\rho_2}}{\rho_1-\rho_2},$$

which implies that there exists $\delta_3 > 0$ such that for $0 < t - t_0 < \delta_3$, we get

$$\int_{t_0}^{t}(t-s)^{\rho_1-\rho_2-1}\Big(\frac{c_1\|y\|_E s^{\rho_2}}{\Gamma(\rho_1-\rho_2)\Gamma(1+\rho_2)}+\frac{c_2\|y\|_E}{\Gamma(\rho_2-\rho_2)}\Big)ds < \varepsilon.$$

By the same arguments, we get that these estimations still hold for the last two terms above. Hence, we obtain $Fx(t)$ is continuous on point t_0. In view of the arbitrariness of t_0, we have $Fx \in C(0, +\infty)$.

For the case of $t_0 = 0$, by (A_2), for $y \in E$, take $t < 1$, then

$$\begin{aligned}
|Fy(t)| = \quad & \Big|\frac{1}{\Gamma(\rho_1-\rho_2)}\int_0^t (t-s)^{\rho_1-\rho_2-1} f(s, I_{0+}^{\rho_2}y(s), y(s))ds\Big| \\
\le \quad & \frac{\|y\|_E}{\Gamma(\rho_1-\rho_2)}\int_0^t (t-s)^{\rho_1-\rho_2-1})(c_1\frac{s^{\rho_2}}{\Gamma(1+\rho_2)}+c_2)ds \\
& +\frac{\max_{0\le t\le 1}|f(t,0,0)|}{\Gamma(\rho_1-\rho_2)}\int_0^t (t-s)^{\rho_1-\rho_2-1}ds,
\end{aligned}$$

From the previous arguments, we could know that $Fy(t)$ is continuous on point 0. As a result, we have $Fy \in C[0, +\infty)$ for $x \in E$.

By the similar arguments, for $y \in E$, by (A_2), we have

$$\begin{aligned}
\Big|\frac{Fy(t)}{1+t^\lambda}\Big| \le \quad & \frac{1}{\Gamma(\rho_1-\rho_2)(1+t^\lambda)}\int_0^t (t-s)^{\rho_1-\rho_2-1}\Big(c_1\frac{|I_{0+}^{\rho_2}y(s)|}{1+s^\lambda}+c_2\frac{|y(s)|}{1+s^\lambda}\Big)ds \\
& +\frac{1}{\Gamma(\rho_1-\rho_2)(1+t^\lambda)}\int_0^t (t-s)^{\rho_1-\rho_2-1}|f(s,0,0)|ds \\
\le \quad & \frac{\|y\|_E}{\Gamma(\rho_1-\rho_2)(1+t^\lambda)}\int_0^t (t-s)^{\rho_1-\rho_2-1}\Big(c_1\frac{s^{\rho_2}}{\Gamma(1+\rho_2)}+c_2\Big)ds \\
& +\frac{1}{\Gamma(\rho_1-\rho_2)(1+t^\lambda)}\int_0^t (t-s)^{\rho_1-\rho_2-1}|f(s,0,0)|ds \\
= \quad & \frac{\|y\|_E}{1+t^\lambda}\Big[\frac{c_1 t^{\rho_1}}{\Gamma(1+\rho_1)}+\frac{c_2 t^{\rho_1-\rho_2}}{\Gamma(1+\rho_1-\rho_2)}\Big] \\
& +\frac{1}{\Gamma(\rho_1-\rho_2)(1+t^\lambda)}\int_0^t (t-s)^{\rho_1-\rho_2-1}|f(s,0,0)|ds,
\end{aligned}$$

according to these estimations and (A_2), we ge that $\lim_{t\to+\infty}\frac{Fy(t)}{1+t^\lambda} = 0$. Hence, $F : E \to E$ is well defined.

Now, for $x, y \in E$, by a similar way, we get

$$\begin{aligned}
& \frac{|Fx(t)-Fy(t)|}{1+t^\lambda} \\
\le \quad & \frac{1}{\Gamma(\rho_1-\rho_2)}\int_0^t (t-s)^{\rho_1-\rho_2-1}\Big(c_1\frac{I_{0+}^{\rho_2}|x(s)-y(s)|}{1+s^\lambda}+c_2\frac{|x(s)-y(s)|}{1+s^\lambda}\Big)ds \\
\le \quad & \frac{\|x-y\|_E}{1+t^\lambda}\Big[\frac{c_1 t^{\rho_1}}{\Gamma(1+\rho_1)}+\frac{c_2 t^{\rho_1-\rho_2}}{\Gamma(1+\rho_1-\rho_2)}\Big] \\
\le \quad & \Big[\frac{c_1}{\Gamma(1+\rho_1)}+\frac{c_2}{\Gamma(1+\rho_1-\rho_2)}\Big]\|x-y\|_E,
\end{aligned}$$

which implies that the operator $F : E \to E$ is a contraction operator, so the Banach contraction principle assures that the operator F has a unique fixed point $y_T(t) \in E$. According to some arguments above, we obtain that $x_T(t) = I_{0+}^{\rho_2} y_T(t)$ is one unique solution of the Equation (32) with zero initial value condition. Thus, according to Definition 5, we obtain that the problem (1) has one unique approximate solution. $\square$

Example 4. *Now, we consider the initial value problem as following*

$$\begin{cases} D_{0+}^{\frac{1}{2}+\frac{t}{200(1+t^2)}} x(t) = \frac{\Gamma(\frac{3}{2})x^4}{12(1+t^2)^4(1+x^4)} + \frac{\Gamma(\frac{7}{6})(D_{0+}^{\frac{1}{3}+\frac{t}{600(1+t^2+t^3)}} x)^2}{12(1+t^2)^2(1+(D_{0+}^{\frac{1}{3}+\frac{t}{600(1+t^2+t^3)}} x)^2)}, 0 < t < +\infty, \\ x(0) = 0. \end{cases} \tag{43}$$

We let

$$p(t) = \frac{1}{2} + \frac{t}{200(1+t^2)}, q(t) = \frac{1}{3} + \frac{t}{600(1+t^2+t^3)}, 0 \le t < +\infty,$$

$$f(t, x(t), y(t)) = \frac{\Gamma(\frac{3}{2})x^4(t)}{12(1+t^2)^4(1+x^4(t))} + \frac{\Gamma(\frac{7}{6})y^2(t)}{12(1+t^2)^2(1+y^2(t))}, 0 < t < +\infty, x(t), y(t) \in R.$$

Obviously, we get $\lim_{t\to+\infty} p(t) = \frac{1}{2}$ *and* $\lim_{t\to+\infty} q(t) = \frac{1}{3}$*, thus, p satisfies* (A_1) *with* $\rho_1 = \frac{1}{2}$*,* $\rho_2 = \frac{1}{3}$*. That* $f(t, 0, 0) = 0$*. In addition, for all* $0 \le t < +\infty, x(t), y(t) \in R$*, from the differentiation mean theorem, we get*

$$\begin{aligned} & |f(t,(1+t^2)x_1,(1+t^2)y_1) - f(t,(1+t^2)x_2,(1+t^2)y_2)| \\ \le\ & \frac{\Gamma(\frac{3}{2})}{12}|\frac{x_1^4}{1+(1+t^2)^4x_1^4} - \frac{x_2^4}{1+(1+t^2)^4x_2^4}| \\ & + \frac{\Gamma(\frac{7}{6})}{12}|\frac{y_1^2(t)}{1+(1+t^2)^2y_1^2} - \frac{y_2^2}{1+(1+t^2)^2y_2^2})| \\ \le\ & \frac{\Gamma(\frac{3}{2})}{3}|x_1 - x_2| + \frac{\Gamma(\frac{7}{6})}{3}|y_1 - y_1|, \end{aligned}$$

which implies that f satisfies (A_2) *with* $c_1 = \frac{\Gamma(\frac{3}{2})}{3}$*,* $c_2 = \frac{\Gamma(\frac{7}{6})}{3}$*, which satisfies*

$$\begin{aligned} & \frac{c_1}{\Gamma(1+\rho_1)} + \frac{c_2}{\Gamma(1+\rho_1-\rho_2)} \\ =\ & \frac{\Gamma(\frac{3}{2})}{3}\frac{1}{\Gamma(1+\frac{1}{2})} + \frac{\Gamma(\frac{7}{6})}{3}\frac{1}{\Gamma(1+\frac{1}{2}-\frac{1}{3})} \\ =\ & \frac{2}{3} < 1. \end{aligned}$$

For given arbitrary small $\varepsilon = \frac{1.1}{100}$*, there exists* $T = \frac{22}{\varepsilon} = 2000$*, such that*

$$|p(t) - \frac{1}{2}| = \frac{t}{200(1+t^2)} < \frac{1}{t} \le \frac{1}{T} = \frac{\varepsilon}{22} < \varepsilon, t \ge T,$$

$$|q(t) - \frac{1}{3}| = \frac{t}{600(1+t^2+t^3)} < \frac{1}{t} \le \frac{1}{T} = \frac{\varepsilon}{22} < \varepsilon, t \ge T.$$

Now, we consider function $p(t)$ restricted on interval $[0, T] = [0, 2000]$. By the right continuity of function $p(t)$ at point 0, for $\varepsilon = \frac{1.1}{100}$, taking $\delta_0 = 2$, when $0 \le t \le \delta_0 = 2$, we have

$$|p(t) - p(0)| = |\frac{t}{200(1+t^2)}| \le \frac{t}{200} < \frac{\delta_0}{200} = \frac{1}{100} < \frac{1.1}{100} = \varepsilon.$$

$$|q(t) - q(0)| = |\frac{t}{600(1+t^2+t^3)}| \le \frac{t}{600} < \frac{\delta_0}{200} = \frac{1}{100} < \frac{1.1}{100} = \varepsilon.$$

We get $t_1 = \delta_0 = 2$. By the right continuity of functions $p(t)$, $q(t)$ at the point t_1, for $\varepsilon = \frac{1.1}{100}$, taking $\delta_1 = 2$, when $0 \le t - t_1 \le \delta_1$, by differential mean value theorem, we have

$$\begin{aligned} |p(t) - p(t_1)| = \quad & |\frac{t}{200(1+t^2)} - \frac{t_1}{200(1+t_1^2)}| \\ \le \quad & |\frac{1-\xi^2}{200(1+\xi^2)^2}||t - t_1| \\ \le \quad & \frac{1+\xi^2}{200(1+\xi^2)^2}|t - t_1| \\ \le \quad & \frac{1}{200}|t - t_1| \\ < \quad & \frac{\delta_1}{200} = \frac{1}{100} < \frac{1.1}{100} = \varepsilon, \end{aligned}$$

$$\begin{aligned} |q(t) - q(t_1)| = \quad & |\frac{t}{600(1+t^2+t^3)} - \frac{t_1}{600(1+t_1^2+t_1^3)}| \\ \le \quad & |\frac{1-\eta^2-2\eta^3}{600(1+\eta^2+\eta^3)^2}||t - t_1| \\ \le \quad & |\frac{1+\eta^2+2\eta^3}{600(1+\eta^2+\eta^3)^2}||t - t_1| \\ \le \quad & \frac{3}{600}|t - t_1| \\ < \quad & \frac{\delta_1}{200} = \frac{1}{100} < \frac{1.1}{100} = \varepsilon, \end{aligned}$$

where $t_1 < \xi < t$, $t_1 < \eta < t$. We let $t_2 = t_1 + \delta_1 = 4$. By the right continuity of function $p(t)$ at point t_2, for $\varepsilon = \frac{1.1}{100}$, taking $\delta_2 = 2$, when $0 \le t - t_1 \le \delta_2$, by the same reasons above, we have

$$|p(t) - p(t_1)| = \quad |\frac{t}{200(1+t^2)} - \frac{t_2}{20(1+t_2^2)}| < \frac{\delta_2}{200} = \frac{1}{100} < \frac{1.1}{100} = \varepsilon,$$

$$|q(t) - q(t_1)| = \quad |\frac{t}{600(1+t^2+t^3)} - \frac{t_2}{600(1+t_2^2+t^3)}| < \frac{\delta_2}{200} = \frac{1}{100} < \frac{1.1}{100} = \varepsilon,$$

Continuing this procession, from $t_{n-1} = 2(n-1) < 2000$, $t_n = t_{n-1} + \delta_{n-1} = 2(n-1) + 2 = 2000$, we get $n = 1000$. Thus, let

$$p_1 \doteq p(0) = \frac{1}{2}, p_2 \doteq p(t_1) = p(2) = \frac{1}{2} + \frac{2}{200(1+4)},$$

$$p_3 \doteq p(t_2) = p(4) = \frac{1}{2} + \frac{4}{200 \times (1+16)}, \cdots, p_{1000} = p(t_{999}) = p(1998) = \frac{1}{2} + \frac{1998}{200 \times (1+1998^2)}.$$

$$q_1 \doteq q(0) = \frac{1}{3}, q_2 \doteq q(t_1) = q(2) = \frac{1}{3} + \frac{2}{600(1+4+8)},$$

$$q_3 \doteq q(t_2) = q(4) = \frac{1}{3} + \frac{4}{600 \times (1+16+64)}, \cdots,$$

$$q_{1000} = q(t_{999}) = q(1998) = \frac{1}{3} + \frac{1998}{600 \times (1+1998^2+1998^3)}.$$

As a result, we get intervals $[0,2], (2,4], \cdots, (1998, 2000], (2000, +\infty)$ *and function* $\alpha(t)$ *defined by*

$$\alpha(t) = \begin{cases} p_1 = \frac{1}{2}, \ \textit{for} \ \ t \in [0,2], \\ p_2 = \frac{1}{2} + \frac{2}{200\times(1+4)}, \ \textit{for} \ \ t \in (2,4], \\ p_3 = \frac{1}{2} + \frac{4}{200\times(1+16)}, \ \textit{for} \ \ t \in (4,6], \\ \cdots, \\ p_{1000} = \frac{1}{2} + \frac{1998}{200\times(1+1998^2)}, \ \textit{for} \ \ t \in (1998, 2000] \\ \rho_1 = \frac{1}{2}, \qquad \textit{for} \quad t \in (2000, +\infty). \end{cases} \tag{44}$$

$$\beta(t) = \begin{cases} q_1 = \frac{1}{3}, \ \textit{for} \ \ t \in [0,2], \\ q_2 = \frac{1}{3} + \frac{2}{600\times(1+4+8)}, \ \textit{for} \ \ t \in (2,4], \\ q_3 = \frac{1}{3} + \frac{4}{600\times(1+16+64)}, \ \textit{for} \ \ t \in (4,6], \\ \cdots, \\ q_{1000} = \frac{1}{3} + \frac{1998}{6000\times(1+1998^2+1998^3)}, \ \textit{for} \ \ t \in (1998, 2000] \\ \rho_2 = \frac{1}{3}, \qquad \textit{for} \quad t \in (2000, +\infty). \end{cases}$$

By Definitions 4 and 5 and the arguments of Theorem 1, the problem (43) has one unique approximate solution.

Remark 1. *From Lemma 6 and Definition 5, we may take arbitrary small* ε*, such that the problem (43) has one unique approximate solution. This means that the proximity is very high.*

Example 5. *Finally, we calculate the approximate solution of the following initial value problem for linear equation*

$$D_{0+}^{\frac{1}{2}+\frac{t}{200(1+t^2)}} x(t) = t^{\frac{1}{4}}, x(0) = 0, 0 < t < +\infty, \tag{45}$$

According to analysis in Example 4, we get intervals $[0,2], (2,4], \cdots, (1998, 2000], (2000, +\infty)$ *and function* $\alpha(t)$ *defined in (44). By Definitions 4 and 5, we calculate out the approximate solution of the problem (45) as following*

$$\begin{cases} x_1(t) = \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{4})} t^{\frac{3}{4}} \in C[0,2], \\ x_2(t) = \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{4} + \frac{2}{200\times(1+4)})} t^{\frac{3}{4} + \frac{2}{200\times(1+4)}} \in C[0,4], \\ x_3(t) = \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{4} + \frac{4}{200\times(1+16)})} t^{\frac{3}{4} + \frac{4}{200\times(1+16)}} \in C[0,6], \\ \cdots, \\ x_{1000} = \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{4} + \frac{1998}{200\times(1+1998^2)})} t^{\frac{3}{4} + \frac{1998}{200\times(1+1998^2)}} \in C[0,2000], \\ x_{2000}(t) = \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{4})} t^{\frac{3}{4}} \in C[0,+\infty). \end{cases}$$

Remark 2. *By the characters of variable order derivative, we cannot get accurate solution of the problem (45). Hence, the approximate solution given by us is significative.*

4. Conclusions

In this paper, we have obtained the unique existence result of approximate solution of initial value problem for fractional differential equation of variable order involving with the variable order derivative defined on the half-axis. Through discussing the characters of variable order calculus(integral and derivative), we introduce the concept of approximate solution to the problem. Based on our discussion and analysis, using the fixed point theorem, we have found the unique existence results. As applications, two examples are presented to illustrate the main results. The issue of the existence and qualitative analysis of approximate solution of initial value problems for fractional differential equation of variable order is interesting. In the future, we will consider the existence and qualitative analysis of approximate solution of initial value problem for singular fractional differential equation of variable order.

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