mathematics

Article

On Opial's Type Integral Inequalities

Chang-Jian Zhao

Department of Mathematics, China Jiliang University, Hangzhou 310018, China; chjzhao@163.com

Received: 17 March 2019; Accepted: 18 April 2019; Published: 25 April 2019

Abstract: In the article we establish some new Opial's type inequalities involving higher order partial derivatives. These new inequalities, in special cases, yield Agarwal-Pang's, Pachpatte's and Das's type inequalities.

Keywords: opial integral inequality; Cauchy-Schwarz's inequality; Hölder's inequality

1. Introduction and Statement of Results

In 1960, Opial [1] established the following integral inequality:

Theorem 1. *Suppose $x \in C^1[0,a]$ satisfies $x(0) = x(a) = 0$ and $x(t) > 0$ for all $t \in (0,a)$. Then the integral inequality holds*

$$\int_0^a |x(t)x'(t)|\, dt \le \frac{a}{4}\int_0^a (x'(t))^2 dt, \tag{1}$$

where this constant $\frac{a}{4}$ is best possible.

The first natural extension of Opial's inequality (1) involving the higher order derivatives $x^{(n)}(s)(n \ge 1)$ instead of $x'(s)$ is embodied in the following:

Theorem 2 ([2])**.** *Let $x(t) \in C^{(n)}[0,a]$ be such that $x^{(i)}(0) = 0, 0 \le i \le n-1 (n \ge 1)$. Then the following inequality holds*

$$\int_0^a \left|x(t)x^{(n)}(t)\right| dt \le \frac{1}{2}a^n \int_0^a \left|x^{(n)}(t)\right|^2 dt. \tag{2}$$

A sharp version of (2) is the following:

Theorem 3 ([3])**.** *Let $x(t) \in C^{(n-1)}[0,a]$ be such that $x^{(i)}(0) = 0, 0 \le i \le n-1 (n \ge 1)$. Further, let $x^{(n-1)}(t)$ be absolutely continuous, and $\int_0^a \left|(x^{(n)}(t)\right|^2 dt < \infty$. Then the following inequality holds*

$$\int_0^a \left|x(t)x^{(n)}(t)\right| dt \le \frac{1}{2n!}\left(\frac{n}{2n-1}\right)^{1/2} a^n \int_0^a \left|x^{(n)}(t)\right|^2 dt. \tag{3}$$

A more general version of (3) was established in [4] as follows:

Theorem 4. *Let $r_k, 0 \le k \le n-1$ and ℓ be non-negative real numbers such that $\ell\sigma \ge 1$, where $\sigma = \sum_{k=0}^{n-1} r_k$, and let $p(t)$ be a non-negative continuous function on $[0,h]$. Further, let $x(t) \in C^{(n-1)}[0,h]$ be such that $x^{(i)}(0) = x^{(i)}(h) = 0, 0 \le i \le n-1$, and let $x^{(n-1)}(t)$ be absolutely continuous. Then the following inequality holds*

$$\int_0^h p(t)\left(\left|\prod_{\kappa=0}^{n} x^{(\kappa)}(t)\right|^{r_k}\right)^{\ell} dt \le \frac{1}{2\sigma}\left(\int_0^h [t(h-t)]^{(\ell\sigma-1)/2} p(t)dt\right)\sum_{k=0}^{n-1} r_k \left|x^{(k+1)}(t)\right|^{\ell\sigma} dt. \quad (4)$$

Opial's inequality and its generalizations, extensions and discretizations play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations [2,5–8]. The inequality (1) has received considerable attention and a large number of papers dealing with new proofs, extensions, generalizations, variants and discrete analogues of Opial's inequality have appeared in the literature [1,3,4,7,9–31]. For an extensive survey on these inequalities, see [2,8].

The first aim of the present paper is to establish a new Opial's type inequality involving higher order partial derivatives which is a generalization of inequality (4).

Theorem 5. *Let $r_{k_1,\ldots,k_n}, 0 \le k_i \le n_i - 1, 1 \le i \le n$ and ℓ be non-negative real numbers such that $\ell\sigma \ge 1$, where $\sigma = \sum_{k_1=0}^{n_1-1}\cdots\sum_{k_n=0}^{n_n-1} r_{k_1,\ldots,k_n}$, and let $p(s_1,\ldots,s_n)$ be a non-negative continuous function on $[0,a_1]\times\cdots\times[0,a_n]$. Let $x(s_1,\ldots,s_n)$ be a continuous function on $[0,a_1]\times\cdots\times[0,a_n]$ such that $\frac{\partial^{k_i}}{\partial s_i^{k_i}} x(s_1,\ldots,s_i,\ldots,s_n)|_{s_i=0 \text{ and } s_i=a_i} = 0, 1 \le i \le n$ and $0 \le k_i \le n_i - 1$. Further, let $\frac{\partial^{n_i}}{\partial s_i^{n_i}}\left(\frac{\partial^{n_i-1}}{\partial s_i^{n_i-1}} x(s_1,\ldots,s_i,\ldots,s_n)\right)$ and $\frac{\partial^{n_i-1}}{\partial s_i^{n_i-1}}\left(\frac{\partial^{n_i}}{\partial s_i^{n_i}} x(s_1,\ldots,s_i,\ldots,s_n)\right)$ be absolutely continuous on $[0,a_1]\times\cdots\times[0,a_n]$. Then*

$$\begin{aligned}
&\int_0^{a_1}\cdots\int_0^{a_n} p(s_1,\ldots,s_n)\left(\prod_{k_1=0}^{n_1-1}\cdots\prod_{k_n=0}^{n_n-1}\left|\frac{\partial^{k_1+\cdots+k_n}}{\partial s_1^{k_1}\cdots\partial s_n^{k_n}} x(s_1,\ldots,s_n)\right|^{r_{k_1,\ldots,k_n}}\right)^{\ell} ds_1\cdots ds_n \\
&\le \frac{1}{2\sigma}\left(\int_0^{a_1}\cdots\int_0^{a_n}\left[\prod_{i=1}^{n} s_i(a_i-s_i)\right]^{(\ell\sigma-1)/2} p(s_1,\ldots,s_n)ds_1\cdots ds_n\right)\times \quad (5)\\
&\times \sum_{k_1=0}^{n_1-1}\cdots\sum_{k_n=0}^{n_n-1} r_{k_1,\ldots,k_n}\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{(k_1+1)+\cdots+(k_n+1)}}{\partial s_1^{k_1+1}\cdots\partial s_n^{k_n+1}} x(s_1,\ldots,s_n)\right|^{\ell\sigma} ds_1\cdots ds_n.
\end{aligned}$$

Remark 1. *Let $x(s_1,\ldots,s_n)$ and $p(s_1,\ldots,s_n)$ reduce to $x(t)$ and $p(t)$, respectively, and with suitable modifications, then (5) changes to (4). A special case of Theorem 5 was proved in [31].*

A result involving two functions and their higher order derivatives is embodied in [18]:

Theorem 6. *For $j = 1,2$ let $x_j(t) \in C^{n-1}[0,a]$ be such that $x_j^{(i)}(0) = 0, 0 \le i \le n-1$. Further, let $x_j^{(n-1)}$ be absolutely continuous, and $\int_0^a |x_j^{(n)}(t)|^2 dt < \infty$. Then*

$$\int_0^a \left\{\left|x_2(t)x_1^{(n)}(t)\right| + \left|x_1(t)x_2^{(n)}(t)\right|\right\} dt \le \frac{1}{2n!}\left(\frac{n}{2n-1}\right)^{1/2} a^n \int_0^a \left(\left|x_1^{(n)}(t)\right|^2 + \left|x_2^{(n)}(t)\right|^2\right) dt, \quad (6)$$

The second aim of the present paper is to establish a new Opial's type inequality involving higher order partial derivatives which is a generalization of inequality (6).

Theorem 7. *For $j = 1,2$, let $x_j(s_1,\ldots,s_n)$ be a continuous function on $[0,a_1]\times\cdots\times[0,a_n]$ such that $\frac{\partial^{\kappa_1+\cdots+\kappa_{i-1}+\kappa_{i+1}+\cdots+\kappa_n}}{\partial s_1^{\kappa_1}\cdots\partial s_{i-1}^{\kappa_{i-1}}\partial s_{i+1}^{\kappa_{i+1}}\cdots\partial s_1^{\kappa_n}} x_j(s_1,\ldots,s_{i-1},s_i,s_{i+1},\ldots,s_n)|_{s_i=0 \text{ and } s_i=a_i} = 0,\ 0 \le \kappa_i \le n_i - 1$, and $i = 1,\ldots,n$.*

Further, let $\frac{\partial^{n_i}}{\partial s_i^{n_i}}\left(\frac{\partial^{n_i-1}}{\partial s_i^{n_i-1}}x_j(s_1,\ldots,s_i,\ldots,s_n)\right)$ *and* $\frac{\partial^{n_i-1}}{\partial s_i^{n_i-1}}\left(\frac{\partial^{n_i}}{\partial s_i^{n_i}}x_j(s_1,\ldots,s_i,\ldots,s_n)\right)$ *be absolutely continuous on* $[0,a_1]\times\cdots\times[0,a_n]$*, and* $\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_j(s_1,\ldots,s_n)\right|^2 ds_1\cdots ds_n$*, exist. Then*

$$\begin{aligned}&\int_0^{a_1}\cdots\int_0^{a_n}\left\{\left|\frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial s_1^{\kappa_1}\cdots\partial s_n^{\kappa_n}}x_1(s_1,\ldots,s_n)\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_2(s_1,\ldots,s_n)\right|\right.\\&\left.+\left|\frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial s_1^{\kappa_1}\cdots\partial s_n^{\kappa_n}}x_2(s_1,\ldots,s_n)\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_1(s_1,\ldots,s_n)\right|\right\}ds_1\cdots ds_n \qquad (7)\\&\le\sqrt{2}D(n_1,\ldots,n_n,\kappa_1,\ldots,\kappa_n)\prod_{i=1}^n a_i^{n_i-\kappa_i}\int_0^{a_1}\cdots\int_0^{a_n}\left\{\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_1(s_1,\ldots,s_n)\right|^2\right.\\&\left.+\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_2(s_1,\ldots,s_n)\right|^2\right\}ds_1\cdots ds_n,\end{aligned}$$

where

$$D(n_1,\ldots,n_n,\kappa_1,\ldots,\kappa_n)=\frac{1}{4\prod_{i=1}^n(n_i-\kappa_i)!}\left(\frac{\prod_{i=1}^n(n_i-\kappa_i)}{\prod_{i=1}^n(2n_i-2\kappa_i-1)}\right)^{1/2}.$$

Remark 2. *Taking for* $\kappa_1=\cdots=\kappa_n=0$ *and* $n_1=\cdots=n_n=1$ *in (7) and for* $j=1,2$*, let* $x_j(s_1,\ldots,s_n)$ *reduce to* $x_j(t)$*, respectively and with suitable modifications, then (7) changes to (6). A special case of Theorem 7 was proved in [31].*

2. Proofs of Main Results

In order to prove Theorem 5, we need the following lemma.

Lemma 1. *Let* $\lambda\ge 1$ *be a real number, and let* $p(s_1,\ldots,s_n)$ *be a nonnegative and continuous functions on* $[0,a_1]\times\cdots\times[0,a_n]$*. Further, let* $x(s_1,\ldots,s_n)$ *be an absolutely continuous function on* $[0,a_1]\times\cdots\times[0,a_n]$*, with* $x(s_1,\ldots,s_{i-1},0,s_{i+1},\ldots,s_n)=0$*,* $x(s_1,\ldots,s_{i-1},a_i,s_{i+1},\ldots,s_n)=0$ *and* $i=1,\ldots,n$*. Then*

$$\begin{aligned}&\int_0^{a_1}\cdots\int_0^{a_n}p(s_1,\ldots,s_n)|x(s_1,\ldots,s_n)|^\lambda ds_1\cdots ds_n\\&\le\frac{1}{2}\left(\int_0^{a_1}\cdots\int_0^{a_n}\left[\prod_{i=1}^n s_i(a_i-s_i)\right]^{(\lambda-1)/2}p(s_1,\ldots,s_n)ds_1\cdots ds_n\right) \qquad (8)\\&\times\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^n}{\partial s_1\ldots\partial s_n}x(s_1,\ldots,s_n)\right|^\lambda dsdt.\end{aligned}$$

Proof. From the hypotheses, we have

$$x(s_1,\ldots,s_n)=\int_0^{s_1}\cdots\int_0^{s_n}\frac{\partial^n}{\partial s_1\cdots\partial s_n}x(s_1,\ldots,s_n)ds_1\cdots ds_n.$$

By Hölder's inequality with indices λ and $\lambda/(\lambda-1)$, it follows that

$$|x(s_1,\ldots,s_n)|^{\lambda/2}\le\left[\left(\int_0^{s_1}\cdots\int_0^{s_n}\left|\frac{\partial^n}{\partial s_1\cdots\partial s_n}x(s_1,\ldots,s_n)\right|ds_1\cdots ds_n\right)^\lambda\right]^{1/2}$$

$$\leq \left(\prod_{i=1}^{n} s_i\right)^{(\lambda-1)/2} \left(\int_0^{s_1} \cdots \int_0^{s_n} \left|\frac{\partial^n}{\partial s_1 \cdots \partial s_n} x(s_1, \ldots, s_n)\right|^{\lambda} ds_1 \cdots ds_n\right)^{1/2}. \tag{9}$$

Similarly

$$|x(s_1, \ldots, s_n)|^{\lambda/2} \leq \left(\prod_{i=1}^{n} (a_i - s_i)\right)^{(\lambda-1)/2} \left(\int_{s_1}^{a_1} \cdots \int_{s_n}^{a_n} \left|\frac{\partial^n}{\partial s_1 \cdots \partial s_n} x(s_1, \ldots, s_n)\right|^{\lambda} ds_1 \cdots ds_n\right)^{1/2}. \tag{10}$$

Now a multiplication of (9) and (10), and by the elementary inequality $2\sqrt{\alpha\beta} \leq \alpha + \beta, \alpha \geq 0, \beta \geq 0$ gives

$$\begin{aligned}
|x(s_1, \ldots, s_n)|^{\lambda} &\leq \left[\prod_{i=1}^{n} s_i(a_i - s_i)\right]^{(\lambda-1)/2} \left(\int_0^{s_1} \cdots \int_0^{s_n} \left|\frac{\partial^n}{\partial s_1 \cdots \partial s_n} x(s_1, \ldots, s_n)\right|^{\lambda} ds_1 \cdots ds_n\right)^{1/2} \\
&\quad \times \left(\int_{s_1}^{a_1} \cdots \int_{s_n}^{a_n} \left|\frac{\partial^n}{\partial s_1 \cdots \partial s_n} x(s_1, \ldots, s_n)\right|^{\lambda} ds_1 \cdots ds_n\right)^{1/2} \\
&\leq \frac{1}{2}\left[\prod_{i=1}^{n} s_i(a_i - s_i)\right]^{(\lambda-1)/2} \left\{\int_0^{s_1} \cdots \int_0^{s_n} \left|\frac{\partial^n}{\partial s_1 \cdots \partial s_n} x(s_1, \ldots, s_n)\right|^{\lambda} ds_1 \cdots ds_n \right. \qquad (11) \\
&\quad \left. + \int_{s_1}^{a_1} \cdots \int_{s_n}^{a_n} \left|\frac{\partial^n}{\partial s_1 \cdots \partial s_n} x(s_1, \ldots, s_n)\right|^{\lambda} ds_1 \cdots ds_n\right\} \\
&= \frac{1}{2}\left[\prod_{i=1}^{n} s_i(a_i - s_i)\right]^{(\lambda-1)/2} \int_0^{a_1} \cdots \int_0^{a_n} \left|\frac{\partial^n}{\partial s_1 \cdots \partial s_n} x(s_1, \ldots, s_n)\right|^{\lambda} ds_1 \cdots ds_n.
\end{aligned}$$

Multiplying the both sides of (11) by $p(s_1, \ldots, s_n)$ and integrating both sides over s_i from 0 to $a_i, i = 1, \ldots, n$ respectively, we obtain

$$\begin{aligned}
&\int_0^{a_1} \cdots \int_0^{a_n} p(s_1, \ldots, s_n)|x(s_1, \ldots, s_n)|^{\lambda} ds_1 \cdots ds_n \\
&\leq \frac{1}{2}\int_0^{a_1} \cdots \int_0^{a_n} \left(\left[\prod_{i=1}^{n} s_i(a_i - s_i)\right]^{(\lambda-1)/2} p(s_1, \ldots, s_n) \times \right. \\
&\quad \left. \times \int_0^{a_1} \cdots \int_0^{a_n} \left|\frac{\partial^n}{\partial s_1 \cdots \partial s_n} x(s_1, \ldots, s_n)\right|^{\lambda} ds_1 \cdots ds_n\right) ds_1 \cdots ds_n \\
&= \frac{1}{2}\left(\int_0^{a_1} \cdots \int_0^{a_n} \left[\prod_{i=1}^{n} s_i(a_i - s_i)\right]^{(\lambda-1)/2} p(s_1, \ldots, s_n) ds_1 \cdots ds_n\right) \\
&\quad \times \int_0^{a_1} \cdots \int_0^{a_n} \left|\frac{\partial^n}{\partial s_1 \cdots \partial s_n} x(s_1, \ldots, s_n)\right|^{\lambda} ds_1 \cdots ds_n.
\end{aligned}$$

□

Proof of Theorem 5. We recall that for the real numbers $\alpha_{k_1,\ldots,k_n}, r_{k_1,\ldots,k_n} \geq 0, 0 \leq k_i \leq n_i - 1, 1 \leq i \leq n$, and any $\ell \geq 1$, the following elementary inequality holds

$$\prod_{k_1=0}^{n_1-1} \cdots \prod_{k_n=0}^{n_n-1} \alpha_{k_1,\ldots,k_n}^{r_{k_1,\ldots,k_n}} \leq \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_n=0}^{n_n-1} r_{k_1,\ldots,k_n} \alpha_{k_1,\ldots,k_n} \leq \left(\sum_{k_1=0}^{n_1-1} \cdots \sum_{k_n=0}^{n_n-1} r_{k_1,\ldots,k_n} \alpha_{k_1,\ldots,k_n}^{\ell}\right)^{1/\ell}. \tag{12}$$

From the inequality (12), we have

$$
\begin{aligned}
&\left(\prod_{k_1=0}^{n_1-1}\cdots\prod_{k_n=0}^{n_n-1}\left|\frac{\partial^{k_1+\cdots+k_n}}{\partial s_1^{k_1}\cdots\partial s_n^{k_n}}x(s_1,\ldots,s_n)\right|^{r_{k_1,\ldots,k_n}}\right)^{\ell}\\
&=\left(\prod_{k_1=0}^{n_1-1}\cdots\prod_{k_n=0}^{n_n-1}\left|\frac{\partial^{k_1+\cdots+k_n}}{\partial s_1^{k_1}\cdots\partial s_n^{k_n}}x(s_1,\ldots,s_n)\right|^{r_{k_1,\ldots,k_n}/\sigma}\right)^{\ell\sigma} \qquad (13)\\
&\le\left(\sum_{k_1=0}^{n_1-1}\cdots\sum_{k_n=0}^{n_n-1}\frac{r_{k_1,\ldots,k_n}}{\sigma}\left|\frac{\partial^{k_1+\cdots+k_n}}{\partial s_1^{k_1}\cdots\partial s_n^{k_n}}x(s_1,\ldots,s_n)\right|\right)^{\ell\sigma}\\
&\le\sum_{k_1=0}^{n_1-1}\cdots\sum_{k_n=0}^{n_n-1}\frac{r_{k_1,\ldots,k_n}}{\sigma}\left|\frac{\partial^{k_1+\cdots+k_n}}{\partial s_1^{k_1}\cdots\partial s_n^{k_n}}x(s_1,\ldots,s_n)\right|^{\ell\sigma}.
\end{aligned}
$$

Multiplying (13) by $p(s_1,\ldots,s_n)$, integrating the two sides of (13) over s_i from 0 to $a_i, i,\ldots,n$, respectively, and then applying the Lemma 1 to the right side again, we observe

$$
\begin{aligned}
&\int_0^{a_1}\cdots\int_0^{a_n}p(s_1,\ldots,s_n)\left(\prod_{k_1=0}^{n_1-1}\cdots\prod_{k_n=0}^{n_n-1}\left|\frac{\partial^{k_1+\cdots+k_n}}{\partial s_1^{k_1}\cdots\partial s_n^{k_n}}x(s_1,\ldots,s_n)\right|^{r_{k_1,\ldots,k_n}}\right)^{\ell}ds_1\cdots ds_n\\
&\le\frac{1}{\sigma}\sum_{k_1=0}^{n_1-1}\cdots\sum_{k_n=0}^{n_n-1}r_{k_1,\ldots,k_n}\int_0^{a_1}\cdots\int_0^{a_n}p(s_1,\ldots,s_n)\left|\frac{\partial^{k_1+\cdots+k_n}}{\partial s_1^{k_1}\cdots\partial s_n^{k_n}}x(s_1,\ldots,s_n)\right|^{\ell\sigma}ds_1\cdots ds_n\\
&\le\frac{1}{\sigma}\sum_{k_1=0}^{n_1-1}\cdots\sum_{k_n=0}^{n_n-1}\frac{r_{k_1,\ldots,k_n}}{2}\left(\int_0^{a_1}\cdots\int_0^{a_n}\left[\prod_{i=1}^{n}s_i(a_i-s_i)\right]^{(\ell\sigma-1)/2}p(s_1,\ldots,s_n)ds_1\cdots ds_n\right)\times\\
&\times\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{(k_1+1)+\cdots+(k_n+1)}}{\partial s_1^{k_1+1}\cdots\partial s_n^{k_n+1}}x(s_1,\ldots,s_n)\right|^{\ell\sigma}ds_1\cdots ds_n\\
&=\frac{1}{2\sigma}\left(\int_0^{a_1}\cdots\int_0^{a_n}\left[\prod_{i=1}^{n}s_i(a_i-s_i)\right]^{(\ell\sigma-1)/2}p(s_1,\ldots,s_n)ds_1\cdots ds_n\right)\times\\
&\times\sum_{k_1=0}^{n_1-1}\cdots\sum_{k_n=0}^{n_n-1}r_{k_1,\ldots,k_n}\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{(k_1+1)+\cdots+(k_n+1)}}{\partial s_1^{k_1+1}\cdots\partial s_n^{k_n+1}}x(s_1,\ldots,s_n)\right|^{\ell\sigma}ds_1\cdots ds_n.
\end{aligned}
$$

This completes the proof of Theorem 5.

Proof of Theorem 7. From the hypotheses of the Theorem 7, we have for $0\le\kappa_i\le n_i-1, i=1,\ldots,n$

$$
\begin{aligned}
&\left|\frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial s_1^{\kappa_1}\cdots\partial s_n^{\kappa_n}}x_1(s_1,\ldots,s_n)\right|\le\frac{1}{\prod_{i=1}^{n}(n_i-\kappa_i-1)!}\\
&\times\int_0^{s_1}\cdots\int_0^{s_n}\prod_{i=1}^{n}(s_i-\sigma_i)^{n_i-\kappa_i-1}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial\sigma_1^{n_1}\cdots\partial\sigma_n^{n_n}}x_1(\sigma_1,\ldots,\sigma_n)\right|d\sigma_1\cdots d\sigma_n. \qquad (14)
\end{aligned}
$$

Multiplying (14) by $\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_2(s_1,\ldots,s_n)\right|$ and using Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
&\left|\frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial s_1^{\kappa_1}\cdots\partial s_n^{\kappa_n}}x_1(s_1,\ldots,s_n)\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_2(s_1,\ldots,s_n)\right| \\
\le{}& \frac{1}{\prod_{i=1}^n(n_i-\kappa_i-1)!}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_2(s_1,\ldots,s_n)\right|\left(\int_0^{s_1}\cdots\int_0^{s_n}\prod_{i=1}^n(s_i-\sigma_i)^{2(n_i-\kappa_i-1)}d\sigma_1\cdots d\sigma_n\right)^{1/2} \\
&\times\left(\int_0^{s_1}\cdots\int_0^{s_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial \sigma_1^{n_1}\cdots\partial \sigma_n^{n_n}}x_1(\sigma_1,\ldots,\sigma_n)\right|^2 d\sigma_1\cdots d\sigma_n\right)^{1/2} \qquad (15)\\
={}& \frac{\prod_{i=1}^n s_i^{n_i-\kappa_i-1/2}}{\prod_{i=1}^n(n_i-\kappa_i-1)!\Big[(2n_i-2\kappa_i-1)\Big]^{1/2}}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_2(s_1,\ldots,s_n)\right| \\
&\times\left(\int_0^{s_1}\cdots\int_0^{s_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial \sigma_1^{n_1}\cdots\partial \sigma_n^{n_n}}x_1(\sigma_1,\ldots,\sigma_n)\right|^2 d\sigma_1\cdots d\sigma_n\right)^{1/2}.
\end{aligned}
$$

Integrating the two sides of (15) over s_i from 0 to $a_i, i=1,\ldots,n$, respectively and then applying the Cauchy-Schwarz inequality to the right side again, we observe

$$
\begin{aligned}
&\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial s_1^{\kappa_1}\cdots\partial s_n^{\kappa_n}}x_1(s_1,\ldots,s_n)\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_2(s_1,\ldots,s_n)\right|ds_1\cdots ds_n \\
\le{}& \frac{1}{\prod_{i=1}^n(n_i-\kappa_i-1)!\Big[(2n_i-2\kappa_i-1)\Big]^{1/2}}\left(\int_0^{a_1}\cdots\int_0^{a_n}\prod_{i=1}^n s_i^{2n_i-2\kappa_i-1}ds_1\cdots ds_n\right)^{1/2} \qquad (16)\\
&\times\Bigg\{\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_2(s_1,\ldots,s_n)\right|^2 \\
&\times\left(\int_0^{s_1}\cdots\int_0^{s_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial \sigma_1^{n_1}\cdots\partial \sigma_n^{n_n}}x_1(\sigma_1,\ldots,\sigma_n)\right|^2 d\sigma_1\cdots d\sigma_n\right)ds_1\cdots ds_n\Bigg\}^{1/2}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial s_1^{\kappa_1}\cdots\partial s_n^{\kappa_n}}x_2(s_1,\ldots,s_n)\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_1(s_1,\ldots,s_n)\right|ds_1\cdots ds_n \\
\le{}& \frac{1}{\prod_{i=1}^n(n_i-\kappa_i-1)!\Big[(2n_i-2\kappa_i-1)\Big]^{1/2}}\left(\int_0^{a_1}\cdots\int_0^{a_n}\prod_{i=1}^n s_i^{2n_i-2\kappa_i-1}ds_1\cdots ds_n\right)^{1/2} \qquad (17)\\
&\times\Bigg\{\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_1(s_1,\ldots,s_n)\right|^2 \\
&\times\left(\int_0^{s_1}\cdots\int_0^{s_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial \sigma_1^{n_1}\cdots\partial \sigma_n^{n_n}}x_2(\sigma_1,\ldots,\sigma_n)\right|^2 d\sigma_1\cdots d\sigma_n\right)ds_1\cdots ds_n\Bigg\}^{1/2}.
\end{aligned}
$$

Taking the sum of the two sides of (16) and (17), and in view of the elementary inequality $\alpha^{1/2}+\beta^{1/2}\leq[2(\alpha+\beta)]^{1/2}, \alpha\geq 0, \beta\geq 0$, we have

$$\begin{aligned}
&\int_0^{a_1}\cdots\int_0^{a_n}\left\{\left|\frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial s_1^{\kappa_1}\cdots\partial s_n^{\kappa_n}}x_1(s_1,\ldots,s_n)\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_2(s_1,\ldots,s_n)\right|\right.\\
&\left.+\left|\frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial s_1^{\kappa_1}\cdots\partial s_n^{\kappa_n}}x_2(s_1,\ldots,s_n)\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_1(s_1,\ldots,s_n)\right|\right\}ds_1\cdots ds_n\\
\leq\;&\frac{1}{\prod_{i=1}^n(n_i-\kappa_i-1)!\left[(2n_i-2\kappa_i-1)\right]^{1/2}}\left(\int_0^{a_1}\cdots\int_0^{a_n}\prod_{i=1}^n s_i^{2n_i-2\kappa_i-1}ds_1\cdots ds_n\right)^{1/2}\\
&\times\left\{2\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_2(s_1,\ldots,s_n)\right|^2\right.\\
&\times\left(\int_0^{s_1}\cdots\int_0^{s_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial\sigma_1^{n_1}\cdots\partial\sigma_n^{n_n}}x_1(\sigma_1,\ldots,\sigma_n)\right|^2 d\sigma_1\cdots d\sigma_n\right)ds_1\cdots ds_n\\
&+2\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_1(s_1,\ldots,s_n)\right|^2\\
&\left.\times\left(\int_0^{s_1}\cdots\int_0^{s_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial\sigma_1^{n_1}\cdots\partial\sigma_n^{n_n}}x_2(\sigma_1,\ldots,\sigma_n)\right|^2 d\sigma_1\cdots d\sigma_n\right)ds_1\cdots ds_n\right\}^{1/2}.
\end{aligned}\tag{18}$$

On the other hand, note the derivation rule of integral upper bound function and the derivation rule of product function, we have

$$\begin{aligned}
&\frac{\partial^n}{\partial s_1\cdots\partial s_n}\left[\left(\int_0^{s_1}\cdots\int_0^{s_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial\sigma_1^{n_1}\cdots\partial\sigma_n^{n_n}}x_1(\sigma_1,\ldots,\sigma_n)\right|^2 d\sigma_1\cdots d\sigma_n\right)\right.\\
&\left.\times\left(\int_0^{s_1}\cdots\int_0^{s_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial\sigma_1^{n_1}\cdots\partial\sigma_n^{n_n}}x_2(\sigma_1,\ldots,\sigma_n)\right|^2 d\sigma_1\cdots d\sigma_n\right)\right]\\
=\;&\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_1(s_1,\ldots,s_n)\right|^2\left(\int_0^{s_1}\cdots\int_0^{s_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial\sigma_1^{n_1}\cdots\partial\sigma_n^{n_n}}x_2(\sigma_1,\ldots,\sigma_n)\right|^2 d\sigma_1\cdots d\sigma_n\right)\\
&+\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1}\cdots\partial s_n^{n_n}}x_2(s_1,\ldots,s_n)\right|^2\left(\int_0^{s_1}\cdots\int_0^{s_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial\sigma_1^{n_1}\cdots\partial\sigma_n^{n_n}}x_1(\sigma_1,\ldots,\sigma_n)\right|^2 d\sigma_1\cdots d\sigma_n\right)
\end{aligned}\tag{19}$$

and

$$\begin{aligned}
&\left(\int_0^{a_1}\cdots\int_0^{a_n}\prod_{i=1}^n s_i^{2n_i-2\kappa_i-1}ds_1\cdots ds_n\right)^{1/2}\\
&=\left(\int_0^{a_1}s_1^{2n_1-2k_1-1}ds_1\cdot\int_0^{a_2}s_2^{2n_2-2k_2-1}ds_2\cdot\cdots\cdot\int_0^{a_n}s_n^{2n_n-2k_n-1}ds_n\right)^{1/2}\\
&=\frac{\prod_{i=1}^n a_i^{n_i-\kappa_i}}{2\left[\prod_{i=1}^n(n_i-\kappa_i)\right]^{1/2}}.
\end{aligned}\tag{20}$$

From (18)–(20) and in view the elementary inequality $(\alpha\beta)^{1/2} \leq \frac{1}{2}(\alpha+\beta), \alpha \geq 0, \beta \geq 0$, we have

$$\int_0^{a_1} \cdots \int_0^{a_n} \left\{ \left| \frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial s_1^{\kappa_1} \cdots \partial s_n^{\kappa_n}} x_1(s_1,\ldots,s_n) \frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1} \cdots \partial s_n^{n_n}} x_2(s_1,\ldots,s_n) \right| \right.$$

$$\left. + \left| \frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial s_1^{\kappa_1} \cdots \partial s_n^{\kappa_n}} x_2(s_1,\ldots,s_n) \frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1} \cdots \partial s_n^{n_n}} x_1(s_1,\ldots,s_n) \right| \right\} ds_1 \cdots ds_n$$

$$\leq 2D(n_1,\ldots,n_n,\kappa_1,\ldots,\kappa_n) \prod_{i=1}^{n} a_i^{n_i-\kappa_i} \left[2 \left(\int_0^{a_1} \cdots \int_0^{a_n} \left| \frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1} \cdots \partial s_n^{n_n}} x_1(s_1,\ldots,s_n) \right|^2 ds_1 \cdots ds_n \right) \right.$$

$$\left. \times \left(\int_0^{a_1} \cdots \int_0^{a_n} \left| \frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1} \cdots \partial s_n^{n_n}} x_2(s_1,\ldots,s_n) \right|^2 ds_1 \cdots ds_n \right) \right]^{1/2}$$

$$\leq \sqrt{2} D(n_1,\ldots,n_n,\kappa_1,\ldots,\kappa_n) \prod_{i=1}^{n} a_i^{n_i-\kappa_i} \int_0^{a_1} \cdots \int_0^{a_n} \left\{ \left| \frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1} \cdots \partial s_n^{n_n}} x_1(s_1,\ldots,s_n) \right|^2 \right.$$

$$\left. + \left| \frac{\partial^{n_1+\cdots+n_n}}{\partial s_1^{n_1} \cdots \partial s_n^{n_n}} x_2(s_1,\ldots,s_n) \right|^2 \right\} ds_1 \cdots ds_n,$$

where

$$D(n_1,\ldots,n_n,\kappa_1,\ldots,\kappa_n) = \frac{1}{4\prod_{i=1}^{n}(n_i-\kappa_i)!} \left(\frac{\prod_{i=1}^{n}(n_i-\kappa_i)}{\prod_{i=1}^{n}(2n_i-2\kappa_i-1)} \right)^{1/2}, \; i-1,\ldots,n.$$

Funding: Research is supported by National Natural Science Foundation of China (11371334, 10971205).

Conflicts of Interest: The author declares no conflict of interest.

References

1. Opial Z. Sur une inégalité. *Ann. Polon. Math.* **1960**, *8*, 29–32. [CrossRef]
2. Agarwal, R.P.; Pang, P.Y.H. *Opial Inequalities with Applications in Differential and Difference Equations*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1995.
3. Das, K.M. An inequality similar to Opial's inequality. *Proc. Am. Math. Soc.* **1969**, *22*, 258–261.
4. Pachpatte, B.G. On some new generalizations of Opial inequalit. *Demonstr. Math.* **1986**, *19*, 281–291.
5. Agarwal, R.P.; Lakshmikantham, V. *Uniqueness and Nonuniqueness Criteria for Ordinary Differential Equations*; World Scientific: Singapore, 1993.
6. Bainov, D.; Simeonov, P. *Integral Inequalities and Applications*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1992.
7. Li, J.D. Opial-type integral inequalities involving several higher order derivatives. *J. Math. Anal. Appl.* **1992**, *167*, 98–100.
8. Mitrinovič, D.S.; Pečarić, J.E.; Fink, A.M. *Inequalities involving Functions and Their Integrals ang Derivatives*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1991.
9. Agarwal, R.P.; Thandapani, E. On some new integrodifferential inequalities. *Anal. sti. Univ. "Al. I. Cuza" din Iasi* **1982**, 28, 123–126.
10. Başcı, Y.; Baleanu, D. New aspects of Opial-type integral inequalities. *Adv. Differ. Equ.* **2018**, *2018*, 452. [CrossRef]
11. Beesack, P.R. On an integral inequality of Z. Opial. *Trans. Am. Math. Soc.* **1962**, *104*, 470–475. [CrossRef]
12. Costa, T.M.; Román-Flores, H.; Chalco-Cano, Y. Opial-type inequalities for interval-valued functions. *Fuzzy Sets Syst.* **2019**, *358*, 48–63. [CrossRef]
13. Godunova, E.; Levin, V.I. On an inequality of Maron. *Mat. Zametki* **1967**, *2*, 221–224.

14. Hua, L.K. On an inequality of Opial. *Sci. Sin.* **1965**, *14*, 789–790.
15. Mirković, T.Z. Dynamic Opial diamond-α integral inequalities involving the power of a function. *J. Inequal. Appl.* **2017**, *2017*, 139. [CrossRef]
16. Mirković, T.Z.; Tričković, S.B.; Stanković, M.S. Opial inequality in q-calculus. *J. Inequal. Appl.* **2018**, *2018*, 347. [CrossRef]
17. Mitrinovič, D.S. *Analytic Inequalities*; Springer-Verlag: Berlin, Germany; New York, NY, USA, 1970.
18. Pachpatte, B.G. On Opial-type integral inequalities. *J. Math. Anal. Appl.* **1986**, *120*, 547–556. [CrossRef]
19. Pečarić, J.E. *An Integral Inequality, in Analysis, Geometry, and Groups: A Riemann Legacy Volume, Part II*; Srivastava, H.M., Rassias, T.M., Eds.; Hadronic Press: Palm Harbor, FL, USA, 1993; pp. 472–478.
20. Pečarić, J.E.; Brnetić, I. Note on generalization of Godunova-Levin-Opial inequality. *Demonstr. Math.* **1997**, *30*, 545–549. [CrossRef]
21. Pečarić, J.E.; Brnetić, I. Note on the Generalization of Godunova-Levin-Opial inequality in Several Independent Variables. *J. Math. Anal. Appl.* **1997**, *215*, 274–282. [CrossRef]
22. Rauf, K.; Anthonio, Y.O. Results on an integral inequality of the Opial-type. *Glob. J. Pure Appl. Sci.* **2017**, *23*, 151–156. [CrossRef]
23. Rauf, K.; Anthonio, Y.O. Time scales on Opial-type inequalities. *J. Inequal. Spec. Funct.* **2017**, *8*, 86–98.
24. Rozanova, G.I. Integral inequalities with derivatives and with arbitrary convex functions. *Moskov. Gos. Ped. Inst. Vcen. Zap.* **1972**, *460*, 58–65.
25. Saker, S.H.; Abdou, D.M.; Kubiaczyk, I. Opial and Polya type inequalities via convexity. *Fasc. Math.* **2018**, *60*, 145–159. [CrossRef]
26. Tomovski, Z.; Pečarić, J.; Farid, G. Weighted Opial-type inequalities for fractional integral and differential operators involving generalized Mittag-Leffler functions. *Eur. J. Pure Appl. Math.* **2017**, *10*, 419–439.
27. Yang, G.S. Inequality of Opial-type in two variables. *Tamkung J. Math.* **1982**, *13*, 255–259.
28. Zhao, C.-J.; Bencze, M. On Agarwal-Pang type integral inequalities. *Ukrainan Math. J.* **2012**, *64*, 200–209. [CrossRef]
29. Zhao, C.-J.; Cheung, W.S. On Opial inequalities involving higher order derivatives. *Bull. Korean Math. Soc.* **2012**, *49*, 1263–1274. [CrossRef]
30. Zhao, C.-J.; Cheung, W.S. On Opial-Dan's type inequalities. *Bull. Malaysian Math. Soc.* 2014, **37**, 1169–1175.
31. Zhao, C.-J.; Cheung W.S. On Opial-type integral inequalities and applications. *Math. Inequ. Appl.* **2014**, *17* , 223–232. [CrossRef]