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Positive Solutions for a Hadamard Fractional p-Laplacian Three-Point Boundary Value Problem

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Abstract: This article is to study a three-point boundary value problem of Hadamard fractional p-Laplacian differential equation. When our nonlinearity grows $(p-1)$-superlinearly and $(p-1)$-sublinearly, the existence of positive solutions is obtained via fixed point index. Moreover, using an increasing operator fixed-point theorem, the uniqueness of positive solutions and uniform convergence sequences are also established.

Keywords: hadamard fractional differential equations; p-Laplacian boundary value problems; positive solutions; fixed point index

1. Introduction

In this paper, we study the existence and uniqueness of positive solutions for the Hadamard fractional p-Laplacian three-point boundary value problem

$$\begin{cases} D^{\alpha}(\varphi_p(D^{\beta}u(t))) = f(t,u(t)),\ t \in (1,e), \\ D^{\beta}u(1) = D^{\beta}u(e) = 0, u(1) = u'(1) = 0, u(e) = au(\xi), \end{cases} \tag{1}$$

where $\alpha \in (1,2]$, $\beta \in (2,3]$, and D^{α}, D^{β} are respectively the Hadamard fractional derivatives of orders α, β; $\xi \in (1,e)$, and $a \geq 0$ with $a(\log\xi)^{\beta-1} \in [0,1)$; note $\varphi_p(s) = |s|^{p-2}s$ is the p-Laplacian for $p > 1, s \in \mathbb{R}$.

Arafa et al. [1] introduced a fractional-order HIV-1 infection of CD4+T cells dynamics model and then used the generalised Euler method to find a numerical solution of the HIV-1 infection fractional order model: the model is

$$\begin{cases} D^{\alpha_1}(T) = s - KVT - dT + bI, \\ D^{\alpha_2}(I) = KVT - (b+\delta)I, \\ D^{\alpha_3}(V) = N\delta I - cV, \end{cases}$$

where $D^{\alpha_i}(i = 1,2,3)$ are fractional-order derivatives. Nonlinear analysis methods (such as fixed-point theorems, Leray–Schauder alternative, subsolution and supersolution methods and iterative techniques) are used to study various kinds of fractional-order equations (most of these results involve the Riemann–Liouville and Caputo-type fractional derivatives); see [2–52] and the references therein. In [2], the authors used a double iterative technique to study the unique solution of the p-Laplacian fractional boundary value problem

$$\begin{cases} -D_x^\alpha(\varphi_p(-D_x^\gamma z))(x) = f(x, z(x)), 0 < x < 1, \\ z(0) = 0, D_x^\gamma z(0) = D_x^\gamma z(1) = 0, z(1) = \int_0^1 z(x) d\chi(x), \end{cases} \tag{2}$$

where D_x^α, D_x^γ are the standard Riemann–Liouville derivatives. For the unique solution, they constructed uniform converged sequences, and provided estimates on the error and the convergence rate. In [3], the authors adopted some fixed-point theorems on cones to study the unique solution for the fractional p-Laplacian boundary value problem

$$\begin{cases} D_{0+}^\beta \varphi_p(D_{0+}^\alpha u(t)) + f(t, u(t)) = 0, t \in (0,1), \\ u(0) = u'(1) = u''(0) = D_{0+}^\alpha u(0) = 0, \end{cases} \tag{3}$$

where $D_{0+}^\alpha, D_{0+}^\beta$ are the Caputo fractional derivatives and they presented iterative schemes for the unique solution when f doesn't satisfy a Lipschitz condition. When nonlinearities satisfy a Lipschitz condition, we refer the reader to [4–9]. For example, the authors in [4] used Banach's contraction mapping principle to study the unique solution for the fractional Dirichlet boundary value problem

$$\begin{cases} D_{0+}^\alpha u(t) + f(t, u(t), u'(t)) = 0, t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{4}$$

where D_{0+}^α denotes the Riemann–Liouville fractional derivative. Positive solutions [16–35] and nontrivial solutions [36–52] were also studied for fractional-order equations. For example, the authors in [16] used the Guo–Krasnoselskii's fixed-point theorem and the Leggett–Williams fixed-point theorem to study the existence and multiplicity of positive solutions for the fractional boundary-value problem

$$\begin{cases} D^\alpha(\varphi_p(D^\alpha u(t))) = f(t, u(t)), t \in [0,1]_T, \\ u(0) = u(\sigma(1)) = D^\alpha u(0) = D^\alpha u(\sigma(1)) = 0, \end{cases} \tag{5}$$

where D^α is the conformable fractional derivative on time scales. In [17], the authors studied positive solutions for the fractional system

$$\begin{cases} D_{0+}^{\beta_1}(\varphi_p(D_{0+}^{\alpha_1} u(t))) + \lambda_1 f_1(t, u(t), v(t)) = 0, t \in (0,1), \\ D_{0+}^{\beta_2}(\varphi_p(D_{0+}^{\alpha_2} u(t))) + \lambda_2 f_2(t, u(t), v(t)) = 0, t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \varphi_p(D_{0+}^{\alpha_1} u(0)) = (\varphi_p(D_{0+}^{\alpha_1} u(1)))' = 0, \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \varphi_p(D_{0+}^{\alpha_2} v(0)) = (\varphi_p(D_{0+}^{\alpha_2} v(1)))' = 0, \\ u(1) = \mu_1 \int_0^1 g_1(s) v(s) dA_1(s), v(1) = \mu_2 \int_0^1 g_2(s) u(s) dA_2(s) \end{cases} \tag{6}$$

and obtained existence and nonexistence of positive solutions, and considered the impact of parameters on solutions. In [36], the authors used the Kuratowski noncompactness measure and the Sadovskii fixed-point theorem to study the impulsive fractional differential equations with the p-Laplacian operator

$$\begin{cases} D_{0+}^\beta(\varphi_p(D_{0+}^\alpha x))(t) = f(t, x(t), x'(t)), t \in (0,1), \\ \Delta x(t)|_{t=t_k} = I_k(x(t_k)), \Delta x'(t)|_{t=t_k} = J_k(x(t_k)), \\ D_{0+}^\alpha x(0) = 0, x(0) = x'(0) = \int_0^1 a_1(x(s)) ds, x(1) = x'(1) = \int_0^1 a_2(x(s)) ds. \end{cases} \tag{7}$$

Hadamard fractional-order problems were briefly discussed in the literature; see [53–72] and the references therein. Yang in [53] used the comparison principle and the monotone iterative technique combined with the subsolution and supersolution method to study the existence of extremal solutions for Hadamard fractional differential equations with Cauchy initial value conditions

$$\begin{cases}(D^{\alpha}_{a+}x)(t) = f(t,x(t),y(t)), (J^{1-\alpha}_{a+}x)(a^+) = x_0^*, \alpha \in (0,1], t \in (a,b],\\ (D^{\alpha}_{a+}y)(t) = g(t,x(t),y(t)), (J^{1-\alpha}_{a+}y)(a^+) = y_0^*, \alpha \in (0,1], t \in (a,b],\end{cases} \tag{8}$$

where D^{α}_{a+}, J^{α}_{a+} are the left-sided Hadamard fractional derivative and integral of order α, respectively. In [54], the authors used fixed point methods to study the existence of positive solutions for Hadamard fractional integral boundary value problems

$$\begin{cases}D^{\beta}(\varphi_p(D^{\alpha}u(t))) = f(t,u(t)), t \in (1,e),\\ u(1) = D^{\alpha}u(1) = u'(1) = u'(e) = 0, \varphi_p(D^{\alpha}u(e)) = \mu \int_1^e \varphi_p(D^{\alpha}u(t))\frac{dt}{t}.\end{cases} \tag{9}$$

In this paper, we study the existence of positive solutions for the Hadamard fractional p-Laplacian three-point boundary value problem (1). Note: (i) we establish some relations from the corresponding problem without the p-Laplacian operator, and use some $(p-1)$-superlinearly and $(p-1)$-sublinearly conditions for the nonlinearity to obtain positive solutions for (1); (ii) using an increasing operator fixed-point theorem, we obtain the unique solution for (1), and establish uniformconverged sequences for this solution.

2. Preliminaries

In this paper, we only provide the definition for the Hadamard fractional derivative; for more details about Hadamard fractional calculus, see the book [73].

Definition 1. *The Hadamard derivative of fractional order q for a function $g : [1,\infty) \to \mathbb{R}$ is defined as*

$$D^q g(t) = \frac{1}{\Gamma(n-q)}\left(t\frac{d}{dt}\right)^n \int_1^t (\log t - \log s)^{n-q-1} g(s)\frac{ds}{s}, n-1 < q < n,$$

where $n = [q]+1$, $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

In what follows, we calculate the Green's functions associated with (1). We let $\varphi_p(D^{\beta}u(t)) = -v(t)$ for $t \in [1,e]$. Then, from (1) we obtain

$$\begin{cases}-D^{\alpha}v(t) = f(t,u(t)),\ t \in (1,e),\\ v(1) = v(e) = 0.\end{cases} \tag{10}$$

Lemma 1. *The boundary value problem (10) takes the form*

$$v(t) = \int_1^e G_{\alpha}(t,s) f(s,u(s))\frac{ds}{s}, \tag{11}$$

where

$$G_{\alpha}(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases}(\log t)^{\alpha-1}(1-\log s)^{\alpha-1} - (\log t - \log s)^{\alpha-1}, & 1 \le s \le t \le e,\\ (\log t)^{\alpha-1}(1-\log s)^{\alpha-1}, & 1 \le t \le s \le e.\end{cases}$$

Proof. We use ideas in Lemma 2 of [59]. For some $c_i \in \mathbb{R} (i = 1,2)$, we have

$$v(t) = c_1(\log t)^{\alpha-1} + c_2(\log t)^{\alpha-2} - \frac{1}{\Gamma(\alpha)}\int_1^t (\log t - \log s)^{\alpha-1} f(s,u(s))\frac{ds}{s}.$$

From the condition $v(1) = 0$, we have $c_2 = 0$. Hence,

$$v(t) = c_1(\log t)^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_1^t (\log t - \log s)^{\alpha-1} f(s, u(s)) \frac{ds}{s}.$$

Substituting e into the above equation, and using $u(e) = 0$, we obtain

$$v(e) = c_1 - \frac{1}{\Gamma(\alpha)} \int_1^e (1 - \log s)^{\alpha-1} f(s, u(s)) \frac{ds}{s} = 0.$$

Then,

$$c_1 = \frac{1}{\Gamma(\alpha)} \int_1^e (1 - \log s)^{\alpha-1} f(s, u(s)) \frac{ds}{s} = 0.$$

Consequently, we have

$$\begin{aligned} v(t) &= \frac{1}{\Gamma(\alpha)} \int_1^e (\log t)^{\alpha-1} (1 - \log s)^{\alpha-1} f(s, u(s)) \frac{ds}{s} - \frac{1}{\Gamma(\alpha)} \int_1^t (\log t - \log s)^{\alpha-1} f(s, u(s)) \frac{ds}{s} \\ &= \int_1^e G_\alpha(t, s) f(s, u(s)) \frac{ds}{s}. \end{aligned}$$

This completes the proof. □

Note that $\varphi_p(D^\beta u(t)) = -v(t)$. Then, $\varphi_p(-D^\beta u(t)) = v(t)$ and $-D^\beta u(t) = \varphi_q(v(t))$, where q is a constant with $q^{-1} + p^{-1} = 1$. Then, from (1), we have

$$\begin{cases} -D^\beta u(t) = \varphi_q(v(t)), t \in (1, e), \\ u(1) = u'(1) = 0, u(e) = au(\xi). \end{cases} \tag{12}$$

Lemma 2. *The boundary value problem* (12) *is equivalent to the integral equation*

$$u(t) = \int_1^e G_\beta(t, s) \varphi_q\left(v(s)\right) \frac{ds}{s}, \tag{13}$$

where

$$\begin{aligned} G_{1\beta}(t, s) &= \frac{1}{\Gamma(\beta)} \begin{cases} (\log t)^{\beta-1}(1 - \log s)^{\beta-1} - (\log t - \log s)^{\beta-1}, & 1 \le s \le t \le e, \\ (\log t)^{\beta-1}(1 - \log s)^{\beta-1}, & 1 \le t \le s \le e, \end{cases} \\ G_\beta(t, s) &= G_{1\beta}(t, s) + \frac{a(\log t)^{\beta-1}}{(1 - a(\log \xi)^{\beta-1})\Gamma(\beta)} G_{1\beta}(\xi, s). \end{aligned} \tag{14}$$

Proof. We follow the ideas in Lemma 1. For some $c_i \in \mathbb{R} (i = 1, 2, 3)$, we have

$$u(t) = c_1(\log t)^{\beta-1} + c_2(\log t)^{\beta-2} + c_3(\log t)^{\beta-3} - \frac{1}{\Gamma(\beta)} \int_1^t (\log t - \log s)^{\beta-1} \varphi_q\left(v(s)\right) \frac{ds}{s}.$$

Then, $u(1) = u'(1) = 0$ implies $c_2 = c_3 = 0$. Consequently, we have

$$u(t) = c_1(\log t)^{\beta-1} - \frac{1}{\Gamma(\beta)} \int_1^t (\log t - \log s)^{\beta-1} \varphi_q\left(v(s)\right) \frac{ds}{s}.$$

Substituting e, ξ into the above equation, and using $u(e) = au(\xi)$, we obtain

$$c_1 - \frac{1}{\Gamma(\beta)} \int_1^e (1 - \log s)^{\beta-1} \varphi_q\left(v(s)\right) \frac{ds}{s} = ac_1(\log \xi)^{\beta-1} - \frac{a}{\Gamma(\beta)} \int_1^\xi (\log \xi - \log s)^{\beta-1} \varphi_q\left(v(s)\right) \frac{ds}{s}.$$

Solving this equation, we have

$$c_1 = \frac{1}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}\int_1^e (1-\log s)^{\beta-1}\varphi_q(v(s))\frac{ds}{s} - \frac{a}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}\int_1^\xi (\log\xi-\log s)^{\beta-1}\varphi_q(v(s))\frac{ds}{s}.$$

As a result, we obtain

$$\begin{aligned}
u(t) &= \frac{(\log t)^{\beta-1}}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}\int_1^e (1-\log s)^{\beta-1}\varphi_q(v(s))\frac{ds}{s} - \frac{a(\log t)^{\beta-1}}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}\int_1^\xi \\
&\quad (\log\xi-\log s)^{\beta-1}\varphi_q(v(s))\frac{ds}{s} - \frac{1}{\Gamma(\beta)}\int_1^t (\log t-\log s)^{\beta-1}\varphi_q(v(s))\frac{ds}{s} \\
&= \frac{(\log t)^{\beta-1}}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}\int_1^e (1-\log s)^{\beta-1}\varphi_q(v(s))\frac{ds}{s} - \frac{a(\log t)^{\beta-1}}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}\int_1^\xi \\
&\quad (\log\xi-\log s)^{\beta-1}\varphi_q(v(s))\frac{ds}{s} - \frac{1}{\Gamma(\beta)}\int_1^t (\log t-\log s)^{\beta-1}\varphi_q(v(s))\frac{ds}{s} + \frac{1}{\Gamma(\beta)}\int_1^e (\log t)^{\beta-1} \\
&\quad (1-\log s)^{\beta-1}\varphi_q(v(s))\frac{ds}{s} - \frac{1}{\Gamma(\beta)}\int_1^e (\log t)^{\beta-1}(1-\log s)^{\beta-1}\varphi_q(v(s))\frac{ds}{s} \\
&= \int_1^e G_{1\beta}(t,s)\varphi_q(v(s))\frac{ds}{s} + \frac{a(\log t)^{\beta-1}}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}\int_1^e (\log\xi)^{\beta-1}(1-\log s)^{\beta-1}\varphi_q(v(s))\frac{ds}{s} \\
&\quad - \frac{a(\log t)^{\beta-1}}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}\int_1^\xi (\log\xi-\log s)^{\beta-1}\varphi_q(v(s))\frac{ds}{s} \\
&= \int_1^e G_{1\beta}(t,s)\varphi_q(v(s))\frac{ds}{s} + \frac{a(\log t)^{\beta-1}}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}\int_1^e G_{1\beta}(\xi,s)\varphi_q(v(s))\frac{ds}{s} \\
&= \int_1^e G_\beta(t,s)\varphi_q(v(s))\frac{ds}{s}.
\end{aligned}$$

This completes the proof. □

Note $v(t) = \int_1^e G_\alpha(t,s)f(s,u(s))\frac{ds}{s}, t \in [1,e]$, and we have that (1) is equivalent to the Hammerstein type integral equation

$$u(t) = \int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}. \tag{15}$$

Let $E := C[1,e]$, $\|u\| := \max_{t\in[1,e]}|u(t)|$, $P := \{u \in E : u(t) \geq 0, \forall t \in [1,e]\}$. Then, $(E, \|\cdot\|)$ is a real Banach space and P a cone on E. From (15), we define an operator $A : E \to E$ as follows:

$$(Au)(t) = \int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s},\ u \in E. \tag{16}$$

Note that our functions G_α, G_β, f are continuous, so the operator A is a completely continuous operator. Moreover, if there is a $u \in E$ is a fixed point of A, then from Lemmas 1–2, we have that u is a solution for (1). Therefore, in what follows, we turn to study the existence of fixed points of the operator A.

Lemma 3 (see [21] (Lemma 3.2). *Let $\beta \in (n-1,n]$, and $n \geq 3$. Then, the function G has the properties:*
(R1) $G(t,s) = G(1-s,1-t)$, for $t,s \in [0,1]$,
(R2) $t^{\beta-1}(1-t)s(1-s)^{\beta-1} \leq \Gamma(\beta)G(t,s) \leq (\beta-1)s(1-s)^{\beta-1}$, for $t,s \in [0,1]$,
(R3) $t^{\beta-1}(1-t)s(1-s)^{\beta-1} \leq \Gamma(\beta)G(t,s) \leq (\beta-1)t^{\beta-1}(1-t)$, for $t,s \in [0,1]$, where

$$G(t,s) = \frac{1}{\Gamma(\beta)}\begin{cases} t^{\beta-1}(1-s)^{\beta-1} - (t-s)^{\beta-1}, & 0 \leq s \leq t \leq 1, \\ t^{\beta-1}(1-s)^{\beta-1}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Lemma 4. *Let* $\phi(t) = \frac{(\log t)^{\beta-1}(1-\log t)}{\Gamma(\beta)}$, $\eta(s) = \frac{\log s(1-\log s)^{\beta-1}}{\Gamma(\beta)}$, *for* $t, s \in [1, e]$. *Then, the functions* G_α, G_β *have the properties:*

(I1) $G_\alpha \in C([1,e] \times [1,e], \mathbb{R}^+)$ *and* $\Gamma(\alpha)G_\alpha(t,s) \leq 1$, *for* $t, s \in [1, e]$,

(I2) $\phi(t)\eta(s)\frac{(1-a(\log\xi)^{\beta-1}+a\phi(\xi))\Gamma(\beta)}{1-a(\log\xi)^{\beta-1}} \leq G_\beta(t,s) \leq \frac{(a+(1-a(\log\xi)^{\beta-1})\Gamma(\beta))(\beta-1)}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}\eta(s)$, *for* $t, s \in [1, e]$,

(I3) $G_\beta(t,s) \leq \frac{(1-a(\log\xi)^{\beta-1}+a\phi(\xi))(\beta-1)(\log t)^{\beta-1}}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}$, *for* $t, s \in [1, e]$.

Proof. From the definition of G_α, we easily have (I1). From Lemma 3, in $G(t,s)$, using $\log t, \log s$ to replace t, s, we have

$$\Gamma(\beta)\phi(t)\eta(s) \leq G_{1\beta}(t,s) \leq (\beta-1)\eta(s), \text{ for } t, s \in [1, e], \tag{17}$$

and

$$G_{1\beta}(t,s) \leq (\beta-1)\phi(t), \text{ for } t, s \in [1, e]. \tag{18}$$

Consequently, from (17), we have

$$\begin{aligned} G_\beta(t,s) &= G_{1\beta}(t,s) + \frac{a(\log t)^{\beta-1}}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}G_{1\beta}(\xi,s) \\ &\leq (\beta-1)\eta(s) + \frac{a(\log t)^{\beta-1}}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}(\beta-1)\eta(s) \\ &= \frac{(a+(1-a(\log\xi)^{\beta-1})\Gamma(\beta))(\beta-1)}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}\eta(s), \text{ for } t, s \in [1, e], \end{aligned}$$

and

$$\begin{aligned} G_\beta(t,s) &= G_{1\beta}(t,s) + \frac{a(\log t)^{\beta-1}}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}G_{1\beta}(\xi,s) \\ &\geq \Gamma(\beta)\phi(t)\eta(s) + \frac{a(\log t)^{\beta-1}}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}\Gamma(\beta)\phi(\xi)\eta(s) \\ &\geq \Gamma(\beta)\phi(t)\eta(s) + \frac{a(\log t)^{\beta-1}(1-\log t)}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}\Gamma(\beta)\phi(\xi)\eta(s) \\ &= \Gamma(\beta)\phi(t)\eta(s) + \frac{a}{1-a(\log\xi)^{\beta-1}}\Gamma(\beta)\phi(t)\phi(\xi)\eta(s) \\ &= \Gamma(\beta)\phi(t)\eta(s)\frac{1-a(\log\xi)^{\beta-1}+a\phi(\xi)}{1-a(\log\xi)^{\beta-1}}, \text{ for } t, s \in [1, e]. \end{aligned}$$

This implies that (I2) holds. Finally, from (18), we obtain

$$\begin{aligned} G_\beta(t,s) &= G_{1\beta}(t,s) + \frac{a(\log t)^{\beta-1}}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}G_{1\beta}(\xi,s) \\ &\leq (\beta-1)\frac{(\log t)^{\beta-1}(1-\log t)}{\Gamma(\beta)} + \frac{a(\log t)^{\beta-1}}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}(\beta-1)\phi(\xi) \\ &\leq \frac{(\beta-1)(\log t)^{\beta-1}}{\Gamma(\beta)}\left(1 + \frac{a\phi(\xi)}{1-a(\log\xi)^{\beta-1}}\right) \\ &= \frac{(1-a(\log\xi)^{\beta-1}+a\phi(\xi))(\beta-1)(\log t)^{\beta-1}}{(1-a(\log\xi)^{\beta-1})\Gamma(\beta)}, \text{ for } t, s \in [1, e]. \end{aligned}$$

Thus, (I3) holds. This completes the proof. □

For convenience, we define three positive constants

$$\kappa_1 = \frac{(1 - a(\log\xi)^{\beta-1} + a\phi(\xi))\Gamma(\beta)}{1 - a(\log\xi)^{\beta-1}}, \kappa_2 = \frac{(a + (1 - a(\log\xi)^{\beta-1})\Gamma(\beta))(\beta-1)}{(1 - a(\log\xi)^{\beta-1})\Gamma(\beta)},$$

$$\kappa_3 = \frac{(1 - a(\log\xi)^{\beta-1} + a\phi(\xi))(\beta-1)}{(1 - a(\log\xi)^{\beta-1})\Gamma(\beta)}.$$

Lemma 5. *Let $z \in P$ and $\mu(\tau) = \int_1^e \eta(s)G_\alpha(s,\tau)\frac{ds}{s}$, for $\tau \in [1,e]$. Then, we have the following two integral inequalities*

$$\int_1^e G_\beta(t,s)\int_1^e G_\alpha(s,\tau)z(\tau)\frac{d\tau}{\tau}\frac{ds}{s} \geq \kappa_1\phi(t)\int_1^e z(\tau)\mu(\tau)\frac{d\tau}{\tau}, \text{ for } t \in [1,e], \tag{19}$$

and

$$\int_1^e G_\beta(t,s)\int_1^e G_\alpha(s,\tau)z(\tau)\frac{d\tau}{\tau}\frac{ds}{s} \leq \kappa_2\int_1^e z(\tau)\mu(\tau)\frac{d\tau}{\tau}, \text{ for } t \in [1,e]. \tag{20}$$

This is a direct result from Lemma 4(I2), so we omit the details.

Lemma 6 (see [74] (Lemma 2.6))**.** *Let $\theta > 0$ and $\varphi \in P$. Then,*

$$\left(\int_0^1 \varphi(t)dt\right)^\theta \leq \int_0^1 \varphi^\theta(t)dt, \text{ if } \theta \geq 1, \text{ and } \left(\int_0^1 \varphi(t)dt\right)^\theta \geq \int_0^1 \varphi^\theta(t)dt, \text{ if } 0 < \theta \leq 1.$$

Lemma 7 (see [75])**.** *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set and that $A : \overline{\Omega} \cap P \to P$ is a continuous compact operator. If there exists a $\omega_0 \in P\backslash\{0\}$ such that*

$$\omega - A\omega \neq \lambda\omega_0, \forall\lambda \geq 0, \omega \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 0$, where i denotes the fixed point index on P.

Lemma 8 (see [75])**.** *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $A : \overline{\Omega} \cap P \to P$ is a continuous compact operator. If*

$$\omega - \lambda A\omega \neq 0, \forall\lambda \in [0,1], \omega \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 1$.

Lemma 9 (see [75])**.** *Let E be a partially ordered Banach space, and $x_0, y_0 \in E$ with $x_0 \leq y_0$, $D = [x_0, y_0]$. Suppose that $A : D \to E$ satisfies the following conditions:*

(i) A is an increasing operator;

(ii) $x_0 \leq Ax_0, y_0 \geq Ay_0$, i.e., x_0 and y_0 is a subsolution and a supersolution of A;

(iii) A is a completely continuous operator.

Then, A has the smallest fixed point x^* and the largest fixed point y^* in $[x_0, y_0]$, respectively. Moreover, $x^* = \lim_{n\to\infty} A^n x_0$ and $y^* = \lim_{n\to\infty} A^n y_0$.

3. Positive Solutions for (1)

For convenience, let

$$\kappa_3 = \kappa_1\int_1^e \mu(t)\phi(t)\frac{dt}{t}, \ \kappa_4 = \kappa_2\int_1^e \mu(t)\frac{dt}{t}.$$

First, we list assumptions for our nonlinearity f:

(H1) $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$, (H2) there exist $c_1 > 0, a_1 > \begin{cases} \kappa_3^{1-p}, & 1 < p \leq 2, \\ \kappa_3^{1-p}\Gamma^{2-p}(\alpha), & p \geq 2, \end{cases}$ such that

$$f(t,z) \geq a_1 z^{p-1} - c_1, \ \forall (t,z) \in [1,e] \times \mathbb{R}^+,$$

(H3) there exist $r_1 > 0, a_2 \in \begin{cases} (0, \kappa_4^{1-p}\Gamma^{2-p}(\alpha)), & 1 < p \leq 2, \\ (0, \kappa_4^{-1}\kappa_3^{2-p}), & p \geq 2, \end{cases}$ such that

$$f(t,z) \leq a_2 z^{p-1}, \ \forall (t,z) \in [1,e] \times [0,r_1],$$

(H4) there exist $r_2 > 0, a_3 > \begin{cases} \kappa_3^{1-p}, & 1 < p \leq 2, \\ \kappa_3^{1-p}\Gamma^{2-p}(\alpha), & p \geq 2, \end{cases}$ such that

$$f(t,z) \geq a_3 z^{p-1}, \ \forall (t,z) \in [1,e] \times [0,r_2],$$

(H5) there exist $c_2 > 0, a_4 \in \begin{cases} (0, \kappa_4^{1-p}2^{p-2}\Gamma^{2-p}(\alpha)), & 1 < p \leq 2, \\ (0, \kappa_4^{-1}\kappa_3^{2-p}), & p \geq 2, \end{cases}$ such that

$$f(t,z) \leq a_4 z^{p-1} + c_2, \ \forall (t,z) \in [1,e] \times \mathbb{R}^+,$$

(H6) there exists $k \in (0,1)$ such that $f(t,\lambda u) \geq \lambda^{k(p-1)} f(t,u), \forall \lambda \in [0,1], t \in [1,e]$.

(H7) $f(t,u)$ is increasing with respect to u, i.e., $f(t,u_1) \leq f(t,u_2)$ if $u_1 \leq u_2$, and $f(t,0) \not\equiv 0$, $\forall t \in [1,e]$.

Let

$$\begin{aligned} P_0 &= \left\{ z \in P : z(t) \geq \frac{(1 - a(\log\xi)^{\beta-1} + a\phi(\xi))\Gamma^2(\beta)}{(a + (1 - a(\log\xi)^{\beta-1})\Gamma(\beta))(\beta-1)} \phi(t)\|z\|, \ \forall t \in [1,e] \right\} \\ &= \left\{ z \in P : z(t) \geq \frac{\kappa_1}{\kappa_2}\phi(t)\|z\|, \ \forall t \in [1,e] \right\}. \end{aligned}$$

Then, we have the following lemma.

Lemma 10. *Suppose that (H1) holds. Then, $A(P) \subset P_0$.*

Proof. From Lemma 4(I2), for $u \in P$, we have

$$\begin{aligned} (Au)(t) &= \int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\leq \int_1^e \frac{(a + (1 - a(\log\xi)^{\beta-1})\Gamma(\beta))(\beta-1)}{(1 - a(\log\xi)^{\beta-1})\Gamma(\beta)}\eta(s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}, \end{aligned}$$

and

$$\begin{aligned} (Au)(t) &\geq \int_1^e \Gamma(\beta)\phi(t)\eta(s)\frac{1 - a(\log\xi)^{\beta-1} + a\phi(\xi)}{1 - a(\log\xi)^{\beta-1}}\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &= \frac{(1 - a(\log\xi)^{\beta-1} + a\phi(\xi))\Gamma^2(\beta)}{(a + (1 - a(\log\xi)^{\beta-1})\Gamma(\beta))(\beta-1)}\phi(t)\int_1^e \frac{(a + (1 - a(\log\xi)^{\beta-1})\Gamma(\beta))(\beta-1)}{(1 - a(\log\xi)^{\beta-1})\Gamma(\beta)}\eta(s) \\ &\quad \cdot \varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\geq \frac{(1 - a(\log\xi)^{\beta-1} + a\phi(\xi))\Gamma^2(\beta)}{(a + (1 - a(\log\xi)^{\beta-1})\Gamma(\beta))(\beta-1)}\phi(t)\|Au\|, \text{ for } t \in [1,e]. \end{aligned}$$

Therefore, $(Au)(t) \geq \frac{(1-a(\log\xi)^{\beta-1}+a\phi(\xi))\Gamma^2(\beta)}{(a+(1-a(\log\xi)^{\beta-1})\Gamma(\beta))(\beta-1)}\phi(t)\|Au\|$, for $t \in [1,e]$. This completes the proof. □

Let $B_\rho = \{u \in P : \|u\| < \rho\}$, for $\rho > 0$.

Theorem 1. *Suppose that (H1)–(H3) hold. Then, (1) has at least one positive solution.*

Proof. Let $S_1 = \{u \in P : u = Au + \lambda\psi, \forall\lambda \geq 0\}$, where $\psi \in P_0$ is a fixed element. We prove that S_1 is bounded in P. If $u \in S_1$, then, from Lemma 10, we have $u \in P_0$, and $u(t) \geq (Au)(t)$ for $t \in [1,e]$. Now, we consider two cases. □

Case 1. Let $p \geq 2$. Then, we have $\frac{1}{p-1} \in (0,1]$. From (H2), we have

$$f^{\frac{1}{p-1}}(t,z) + c_1^{\frac{1}{p-1}} \geq (f(t,z)+c_1)^{\frac{1}{p-1}} \geq (a_1 z^{p-1})^{\frac{1}{p-1}} = a_1^{\frac{1}{p-1}} z, \text{ for } (t,z) \in [1,e]\times\mathbb{R}^+.$$

Consequently, from (19) and Lemma 6, we obtain

$$\begin{aligned}
u(t) &\geq \int_1^e G_\beta(t,s)\left(\int_1^e G_\alpha(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s} \\
&= \Gamma^{-\frac{1}{p-1}}(\alpha)\int_1^e G_\beta(t,s)\left(\int_1^e \Gamma(\alpha)G_\alpha(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s} \\
&= \Gamma^{-\frac{1}{p-1}}(\alpha)\int_1^e G_\beta(t,s)\left(\int_1^e \Gamma(\alpha)G_\alpha(s,\tau)f(\tau,u(\tau))d\log\tau\right)^{\frac{1}{p-1}}\frac{ds}{s} \\
&= \Gamma^{-\frac{1}{p-1}}(\alpha)\int_1^e G_\beta(t,s)\left(\int_0^1 \Gamma(\alpha)G_\alpha(s,e^x)f(e^x,u(e^x))dx\right)^{\frac{1}{p-1}}\frac{ds}{s} \\
&\geq \Gamma^{-\frac{1}{p-1}}(\alpha)\int_1^e G_\beta(t,s)\int_0^1 (\Gamma(\alpha)G_\alpha(s,e^x))^{\frac{1}{p-1}}f^{\frac{1}{p-1}}(e^x,u(e^x))dx\frac{ds}{s} \\
&\geq \Gamma^{-\frac{1}{p-1}}(\alpha)\int_1^e G_\beta(t,s)\int_0^1 (\Gamma(\alpha)G_\alpha(s,e^x))f^{\frac{1}{p-1}}(e^x,u(e^x))dx\frac{ds}{s} \\
&= \Gamma^{\frac{p-2}{p-1}}(\alpha)\int_1^e G_\beta(t,s)\int_1^e G_\alpha(s,\tau)f^{\frac{1}{p-1}}(\tau,u(\tau))\frac{d\tau}{\tau}\frac{ds}{s} \\
&\geq \kappa_1\phi(t)\Gamma^{\frac{p-2}{p-1}}(\alpha)\int_1^e \mu(\tau)f^{\frac{1}{p-1}}(\tau,u(\tau))\frac{d\tau}{\tau} \\
&\geq \kappa_1\phi(t)\Gamma^{\frac{p-2}{p-1}}(\alpha)\int_1^e \mu(\tau)\left(a_1^{\frac{1}{p-1}}u(\tau) - c_1^{\frac{1}{p-1}}\right)\frac{d\tau}{\tau}.
\end{aligned} \tag{21}$$

Multiplying by $\mu(t)$ on both sides of (21) and integrating over $[1,e]$, we obtain

$$\begin{aligned}
\int_1^e u(t)\mu(t)\frac{dt}{t} &\geq \kappa_1\Gamma^{\frac{p-2}{p-1}}(\alpha)\int_1^e \mu(t)\phi(t)\frac{dt}{t}\int_1^e \mu(t)\left(a_1^{\frac{1}{p-1}}u(t) - c_1^{\frac{1}{p-1}}\right)\frac{dt}{t} \\
&= \kappa_3\Gamma^{\frac{p-2}{p-1}}(\alpha)\int_1^e \mu(t)\left(a_1^{\frac{1}{p-1}}u(t) - c_1^{\frac{1}{p-1}}\right)\frac{dt}{t}.
\end{aligned}$$

Solving this inequality, we have

$$\int_1^e u(t)\mu(t)\frac{dt}{t} \leq \frac{\kappa_3 c_1^{\frac{1}{p-1}}\Gamma^{\frac{p-2}{p-1}}(\alpha)\int_1^e \mu(t)\frac{dt}{t}}{\kappa_3 a_1^{\frac{1}{p-1}}\Gamma^{\frac{p-2}{p-1}}(\alpha) - 1}.$$

Note that, for $u \in P_0$, we get

$$\int_1^e \frac{\kappa_1}{\kappa_2}\phi(t)\|u\|\mu(t)\frac{dt}{t} \le \frac{\kappa_3 c_1^{\frac{1}{p-1}}\Gamma^{\frac{p-2}{p-1}}(\alpha)\int_1^e \mu(t)\frac{dt}{t}}{\kappa_3 a_1^{\frac{1}{p-1}}\Gamma^{\frac{p-2}{p-1}}(\alpha)-1}, \text{ and } \|u\| \le \frac{\kappa_4 c_1^{\frac{1}{p-1}}\Gamma^{\frac{p-2}{p-1}}(\alpha)}{\kappa_3 a_1^{\frac{1}{p-1}}\Gamma^{\frac{p-2}{p-1}}(\alpha)-1}.$$

Case 2. Let $p \in (1,2]$. Then, we have $p-1 \in (0,1]$. Note that $\frac{G_\beta(t,s)}{\kappa_3} \le 1$, for $t,s \in [1,e]$, by (H2), (19) and Lemma 6 we have

$$\begin{aligned}
u^{p-1}(t) &\ge \left(\int_1^e G_\beta(t,s)\left(\int_1^e G_\alpha(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}}\frac{ds}{s}\right)^{p-1} \\
&= \kappa_3^{p-1}\left(\int_1^e \frac{G_\beta(t,s)}{\kappa_3}\left(\int_1^e G_\alpha(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} d\log s\right)^{p-1} \\
&= \kappa_3^{p-1}\left(\int_0^1 \frac{G_\beta(t,e^x)}{\kappa_3}\left(\int_1^e G_\alpha(e^x,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)^{\frac{1}{p-1}} dx\right)^{p-1} \\
&\ge \kappa_3^{p-1}\int_0^1 \left(\frac{G_\beta(t,e^x)}{\kappa_3}\right)^{p-1}\int_1^e G_\alpha(e^x,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}dx \\
&\ge \kappa_3^{p-2}\int_1^e G_\beta(t,s)\int_1^e G_\alpha(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\frac{ds}{s} \\
&\ge \kappa_1\kappa_3^{p-2}\phi(t)\int_1^e \mu(\tau)f(\tau,u(\tau))\frac{d\tau}{\tau} \\
&\ge \kappa_1\kappa_3^{p-2}\phi(t)\int_1^e \mu(\tau)(a_1u^{p-1}(\tau)-c_1)\frac{d\tau}{\tau}.
\end{aligned} \tag{22}$$

Multiplying by $\mu(t)$ on both sides of (22) and integrating over $[1,e]$, we conclude that

$$\int_1^e \mu(t)u^{p-1}(t)\frac{dt}{t} \ge \kappa_1\kappa_3^{p-2}\int_1^e \mu(t)\phi(t)\frac{dt}{t}\int_1^e \mu(t)(a_1u^{p-1}(t)-c_1)\frac{dt}{t}.$$

Solving this inequality, we have

$$\int_1^e \mu(t)u^{p-1}(t)\frac{dt}{t} \le \frac{\kappa_3^{p-1}c_1\int_1^e \mu(t)\frac{dt}{t}}{\kappa_3^{p-1}a_1-1}.$$

Noting that $u \in P_0$, we have

$$\|u\|^{p-1} \le \frac{\kappa_3^{p-1}\kappa_2^{p-2}\kappa_1^{1-p}c_1\kappa_4}{\kappa_3^{p-1}a_1-1}\left(\int_1^e \mu(t)\phi^{p-1}(t)\frac{dt}{t}\right)^{-1}.$$

The above two cases imply that S_1 is bounded in P. Then, we can choose

$$R_1 > \begin{cases} \sqrt[p-1]{\frac{\kappa_3^{p-1}\kappa_2^{p-2}\kappa_1^{1-p}c_1\kappa_4}{\kappa_3^{p-1}a_1-1}\left(\int_1^e \mu(t)\phi^{p-1}(t)\frac{dt}{t}\right)^{-1}}, & 1 < p \le 2, \\ \frac{\kappa_4 c_1^{\frac{1}{p-1}}\Gamma^{\frac{p-2}{p-1}}(\alpha)}{\kappa_3 a_1^{\frac{1}{p-1}}\Gamma^{\frac{p-2}{p-1}}(\alpha)-1}, & p \ge 2, \end{cases}$$

such that

$$u \ne Au + \lambda\psi, \text{ for } u \in \partial B_{R_1} \cap P, \ \forall \lambda \ge 0.$$

As a result, Lemma 7 implies that

$$i(A, B_{R_1} \cap P, P) = 0. \tag{23}$$

For r_1 in (H3), we now prove that

$$u \neq \lambda Au, \text{ for } u \in \partial B_{r_1} \cap P, \ \forall \lambda \in [0,1]. \tag{24}$$

If this claim isn't true, then there exist $u \in \partial B_{r_1} \cap P$ and $\lambda \in [0,1]$ such that $u = \lambda Au$, and $u(t) \leq (Au)(t)$, for $t \in [1,e]$. Now, we consider two cases.

Case 1. Let $p \geq 2$. Then, we have $p - 1 \geq 1$. From (20), (H3) and Lemma 6, we get

$$\begin{aligned}
u^{p-1}(t) &\leq \left(\int_1^e G_\beta(t,s) \left(\int_1^e G_\alpha(s,\tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right)^{p-1} \\
&= \kappa_3^{p-1} \left(\int_0^1 \frac{G_\beta(t,e^x)}{\kappa_3} \left(\int_1^e G_\alpha(e^x,\tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right)^{\frac{1}{p-1}} dx \right)^{p-1} \\
&\leq \kappa_3^{p-2} \int_0^1 G_\beta(t,e^x) \int_1^e G_\alpha(e^x,\tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} dx \\
&= \kappa_3^{p-2} \int_1^e G_\beta(t,s) \int_1^e G_\alpha(s,\tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \frac{ds}{s} \\
&\leq \kappa_3^{p-2} \kappa_2 \int_1^e \mu(\tau) a_2 u^{p-1}(\tau) \frac{d\tau}{\tau}.
\end{aligned} \tag{25}$$

Multiplying by $\mu(t)$ on both sides of (25) and integrating over $[1,e]$, we find

$$\int_1^e \mu(t) u^{p-1}(t) \frac{dt}{t} \leq \kappa_3^{p-2} a_2 \kappa_4 \int_1^e \mu(t) u^{p-1}(t) \frac{dt}{t}.$$

This implies that

$$\int_1^e \mu(t) u^{p-1}(t) \frac{dt}{t} = 0, \text{ and } u(t) \equiv 0, \text{ for } t \in [1,e],$$

since $\mu(t) \not\equiv 0$, for $t \in [1,e]$. This contradicts $u \in \partial B_{r_1} \cap P$, $r_1 > 0$.

Case 2. Let $p \in (1,2]$. Then, we have $\frac{1}{p-1} \geq 1$. From (20), (H3) and Lemma 6, we obtain

$$\begin{aligned}
u(t) &\leq \int_1^e G_\beta(t,s) \left(\int_1^e G_\alpha(s,\tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right)^{\frac{1}{p-1}} \frac{ds}{s} \\
&= \Gamma^{-\frac{1}{p-1}}(\alpha) \int_1^e G_\beta(t,s) \left(\int_0^1 \Gamma(\alpha) G_\alpha(s,e^x) f(e^x, u(e^x)) dx \right)^{\frac{1}{p-1}} \frac{ds}{s} \\
&\leq \Gamma^{-\frac{1}{p-1}}(\alpha) \int_1^e G_\beta(t,s) \int_0^1 (\Gamma(\alpha) G_\alpha(s,e^x))^{\frac{1}{p-1}} f^{\frac{1}{p-1}}(e^x, u(e^x)) dx \frac{ds}{s} \\
&\leq \Gamma^{\frac{p-2}{p-1}}(\alpha) \int_1^e G_\beta(t,s) \int_1^e G_\alpha(s,\tau) f^{\frac{1}{p-1}}(\tau, u(\tau)) \frac{d\tau}{\tau} \frac{ds}{s} \\
&\leq \Gamma^{\frac{p-2}{p-1}}(\alpha) \kappa_2 \int_1^e \mu(\tau) a_2^{\frac{1}{p-1}} u(\tau) \frac{d\tau}{\tau}.
\end{aligned} \tag{26}$$

Multiplying by $\mu(t)$ on both sides of the preceding inequalities and integrating over $[1,e]$, we find

$$\int_1^e \mu(t) u(t) \frac{dt}{t} \leq \Gamma^{\frac{p-2}{p-1}}(\alpha) \kappa_4 a_2^{\frac{1}{p-1}} \int_1^e \mu(t) u(t) \frac{dt}{t}.$$

Note that $\mu(t) \not\equiv 0$, for $t \in [1, e]$, and this implies that

$$\int_1^e \mu(t)u^{p-1}(t)\frac{dt}{t} = 0, \text{ and } u(t) \equiv 0, \text{ for } t \in [1, e].$$

This contradicts $u \in \partial B_{r_1} \cap P, r_1 > 0$.

Combining the above two cases, we have that (24) holds. Then, from Lemma 8, we obtain

$$i(A, B_{r_1} \cap P, P) = 1. \tag{27}$$

Note that we can also take $R_1 > r_1$ such that (23) is still true. Thus, from (23) and (27), we have

$$i(A, (B_{R_1} \backslash \overline{B}_{r_1}) \cap P, P) = i(A, B_{R_1} \cap P, P) - i(A, B_{r_1} \cap P, P) = -1,$$

and hence A has at least one fixed point in $(B_{R_1} \backslash \overline{B}_{r_1}) \cap P$, i.e., (1) has at least one positive solution. This completes the proof.

Theorem 2. *Suppose that (H1), and (H4)–(H5) hold. Then, (1) has at least one positive solution.*

Proof. We can use similar methods as in Theorem 1 to provide the proof. We first prove that

$$u \neq Au + \lambda\widetilde{\psi}, \text{ for } u \in \partial B_{r_2} \cap P, \forall \lambda \geq 0, \tag{28}$$

where $\widetilde{\psi} \in P$ is a given element, and r_2 is defined in (H4). Otherwise, there exist $u \in \partial B_{r_2} \cap P$ and $\lambda \geq 0$ such that $u = Au + \lambda\widetilde{\psi}$, and thus $u(t) \geq (Au)(t)$, for $t \in [1, e]$. Now, we consider two cases. □

Case 1. Let $p \geq 2$. Then, we have $\frac{1}{p-1} \in (0, 1]$. Using (21) and (H4), we conclude

$$u(t) \geq \kappa_1\phi(t)\Gamma^{\frac{p-2}{p-1}}(\alpha)\int_1^e \mu(\tau)f^{\frac{1}{p-1}}(\tau, u(\tau))\frac{d\tau}{\tau} \geq \kappa_1\phi(t)\Gamma^{\frac{p-2}{p-1}}(\alpha)\int_1^e \mu(\tau)a_3^{\frac{1}{p-1}}u(\tau)\frac{d\tau}{\tau}.$$

Multiplying by $\mu(t)$ on both sides of the preceding inequalities and integrating over $[1, e]$, we find

$$\int_1^e \mu(t)u(t)\frac{dt}{t} \geq a_3^{\frac{1}{p-1}}\kappa_1\Gamma^{\frac{p-2}{p-1}}(\alpha)\int_1^e \phi(t)\mu(t)\frac{dt}{t}\int_1^e \mu(t)u(t)\frac{dt}{t}.$$

This implies that

$$\int_1^e \mu(t)u(t)\frac{dt}{t} = 0, \text{ and } u(t) \equiv 0, \text{ for } t \in [1, e],$$

since $\mu(t) \not\equiv 0$, for $t \in [1, e]$. This contradicts $u \in \partial B_{r_2} \cap P, r_2 > 0$.

Case 2. Let $p \in (1, 2]$. Then, we have $p - 1 \in (0, 1]$. Using (22) and (H4), we obtain

$$u^{p-1}(t) \geq \kappa_1\kappa_3^{p-2}\phi(t)\int_1^e \mu(\tau)f(\tau, u(\tau))\frac{d\tau}{\tau} \geq \kappa_1\kappa_3^{p-2}\phi(t)\int_1^e \mu(\tau)a_3u^{p-1}(\tau)\frac{d\tau}{\tau}.$$

Multiplying by $\mu(t)$ on both sides of the preceding inequalities and integrating over $[1, e]$, we find

$$\int_1^e \mu(t)u^{p-1}(t)\frac{dt}{t} \geq a_3\kappa_1\kappa_3^{p-2}\int_1^e \phi(t)\mu(t)\frac{dt}{t}\int_1^e \mu(t)u^{p-1}(t)\frac{dt}{t}.$$

This implies that

$$\int_1^e \mu(t)u^{p-1}(t)\frac{dt}{t} = 0, \text{ and } u(t) \equiv 0, \text{ for } t \in [1, e],$$

since $\mu(t) \not\equiv 0$, for $t \in [1, e]$. This contradicts $u \in \partial B_{r_2} \cap P, r_2 > 0$.

As a result, we have that (28) holds, and Lemma 7 implies that

$$i(A, B_{r_2} \cap P, P) = 0. \tag{29}$$

Let $S_2 = \{u \in P : u = \lambda Au, \ \forall \lambda \in [0,1]\}$. Then, we claim that S_2 is bounded in P. Indeed, if $u \in S_2$, then from Lemma 10 we have $u \in P_0$, and $u(t) \le (Au)(t)$, for $t \in [1,e]$. Now, we consider two cases.

Case 1. Let $p \ge 2$. Then, we have $p-1 \ge 1$. Using (25) and (H5), we have

$$u^{p-1}(t) \le \kappa_3^{p-2}\kappa_2 \int_1^e \mu(\tau)(a_4 u^{p-1}(\tau) + c_2)\frac{d\tau}{\tau}.$$

Multiplying by $\mu(t)$ on both sides of the preceding inequalities and integrating over $[1,e]$, we find

$$\int_1^e \mu(t)u^{p-1}(t)\frac{dt}{t} \le \kappa_3^{p-2}\kappa_4 \int_1^e \mu(t)(a_4 u^{p-1}(t) + c_2)\frac{dt}{t}.$$

Solving this inequality, we have

$$\int_1^e \mu(t)u^{p-1}(t)\frac{dt}{t} \le \frac{\kappa_3^{p-2}c_2\kappa_4 \int_1^e \mu(t)\frac{dt}{t}}{1-\kappa_3^{p-2}a_4\kappa_4}.$$

Note that $u \in P_0$, and we have

$$\|u\|^{p-1} \le \frac{\kappa_3^{p-2}\kappa_2^{p-2}\kappa_1^{1-p}c_2\kappa_4^2}{1-\kappa_3^{p-2}a_4\kappa_4}\left(\int_1^e \mu(t)\phi^{p-1}(t)\frac{dt}{t}\right)^{-1}.$$

Case 2. Let $p \in (1,2]$. Then, we have $\frac{1}{p-1} \ge 1$. Using (26) and (H5), we obtain

$$u(t) \le \Gamma^{\frac{p-2}{p-1}}(\alpha)\kappa_2 \int_1^e \mu(\tau)(a_4 u^{p-1}(\tau) + c_2)^{\frac{1}{p-1}}\frac{d\tau}{\tau} \le \Gamma^{\frac{p-2}{p-1}}(\alpha)2^{\frac{2-p}{p-1}}\kappa_2 \int_1^e \mu(t)(a_4^{\frac{1}{p-1}}u(t) + c_2^{\frac{1}{p-1}})\frac{dt}{t}.$$

Multiplying by $\mu(t)$ on both sides of the preceding inequalities and integrating over $[1,e]$, we find

$$\int_1^e \mu(t)u(t)\frac{dt}{t} \le \Gamma^{\frac{p-2}{p-1}}(\alpha)2^{\frac{2-p}{p-1}}\kappa_4 \int_1^e \mu(t)(a_4^{\frac{1}{p-1}}u(t) + c_2^{\frac{1}{p-1}})\frac{dt}{t}.$$

Solving this inequality, we have

$$\int_1^e \mu(t)u(t)\frac{dt}{t} \le \frac{\Gamma^{\frac{p-2}{p-1}}(\alpha)2^{\frac{2-p}{p-1}}c_2^{\frac{1}{p-1}}\kappa_4^2}{\kappa_2(1-\Gamma^{\frac{p-2}{p-1}}(\alpha)2^{\frac{2-p}{p-1}}a_4^{\frac{1}{p-1}}\kappa_4)}.$$

Noting that $u \in P_0$, we have

$$\|u\| \le \frac{\Gamma^{\frac{p-2}{p-1}}(\alpha)2^{\frac{2-p}{p-1}}c_2^{\frac{1}{p-1}}\kappa_4^2}{\kappa_3(1-\Gamma^{\frac{p-2}{p-1}}(\alpha)2^{\frac{2-p}{p-1}}a_4^{\frac{1}{p-1}}\kappa_4)}.$$

Combining the above two cases, we have proved that S_2 is bounded in P. Then, we can choose $R_2 > r_2$ and

$$R_2 > \begin{cases} \dfrac{\Gamma^{\frac{p-2}{p-1}}(\alpha)2^{\frac{2-p}{p-1}}c_2^{\frac{1}{p-1}}\kappa_4^2}{\kappa_3(1-\Gamma^{\frac{p-2}{p-1}}(\alpha)2^{\frac{2-p}{p-1}}a_4^{\frac{1}{p-1}}\kappa_4)}, & 1 < p \leq 2, \\ \sqrt[p-1]{\dfrac{\kappa_3^{p-2}\kappa_2^{p-2}\kappa_1^{1-p}c_2\kappa_4^2}{1-\kappa_3^{p-2}a_4\kappa_4}\left(\int_1^e \mu(t)\phi^{p-1}(t)\frac{dt}{t}\right)^{-1}}, & p \geq 2, \end{cases}$$

such that

$$u \neq \lambda Au, \ \text{ for } u \in \partial B_{R_2} \cap P, \forall \lambda \in [0,1]. \tag{30}$$

Then, from Lemma 8, we have

$$i(A, B_{R_2} \cap P, P) = 1. \tag{31}$$

Thus, from (29) and (31), we have

$$i(A, (B_{R_2} \backslash \overline{B}_{r_2}) \cap P, P) = i(A, B_{R_2} \cap P, P) - i(A, B_{r_2} \cap P, P) = 1,$$

and hence A has at least one fixed point in $(B_{R_2} \backslash \overline{B}_{r_2}) \cap P$, i.e., (1) has at least one positive solution. This completes the proof.

In what follows, we consider the uniqueness of positive solutions for (1) with the boundary conditions $D^\beta u(1) = D^\beta u(e) = 0, u(1) = u'(1) = u(e) = 0$. This problem is equivalent to the Hammerstein type integral equation

$$u(t) = \int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau, u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}, \tag{32}$$

where $G_\beta(t,s) = G_{1\beta}(t,s)$ for $t, s \in [1,e]$. Note that here we still use the operator A as in (16).

Lemma 11. *Let $w_0(t) = \int_1^e G_\beta(t,s)\frac{ds}{s}$ for $t \in [1,e]$. Then, for all nonnegative functions $w \in C[1,e](\not\equiv 0)$, there exist two positive $a_w, b_w (a_w \leq b_w)$ such that*

$$a_w w_0(t) \leq \int_1^e G_\beta(t,s)w(s)\frac{ds}{s} \leq b_w w_0(t), \ \textit{for } t \in [1,e]. \tag{33}$$

Proof. We first calculate w_0. From (14), we have

$$\begin{aligned} \int_1^e G_\beta(t,s)\frac{ds}{s} &= \frac{1}{\Gamma(\beta)}\int_1^t \left[(\log t)^{\beta-1}(1-\log s)^{\beta-1} - (\log t - \log s)^{\beta-1}\right]\frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(\beta)}\int_t^e (\log t)^{\beta-1}(1-\log s)^{\beta-1}\frac{ds}{s} \\ &= \frac{1}{\Gamma(\beta)}\int_1^e (\log t)^{\beta-1}(1-\log s)^{\beta-1}\frac{ds}{s} - \frac{1}{\Gamma(\beta)}\int_1^t (\log t - \log s)^{\beta-1}\frac{ds}{s} \\ &= \frac{(\log t)^{\beta-1}(1-\log t)}{\beta\Gamma(\beta)}. \end{aligned}$$

Using (17) and (18), we have

$$\int_1^e \Gamma(\beta)\phi(t)\eta(s)w(s)\frac{ds}{s} \leq \int_1^e G_\beta(t,s)w(s)\frac{ds}{s} \leq \int_1^e (\beta-1)\phi(t)w(s)\frac{ds}{s}.$$

Therefore, let $a_w = \beta\Gamma(\beta)\int_1^e \eta(s)w(s)\frac{ds}{s}$, and $b_w = \beta(\beta-1)\int_1^e w(s)\frac{ds}{s}$; then, we have that (33) holds. This completes the proof. □

Theorem 3. *Suppose that (H1), (H6)–(H7) hold. Then, (1) has a unique positive solution.*

Proof. Note that (H7) implies that A is an increasing operator, and 0 isn't a fixed point for A. Next, we shall prove that A has a subsolution and a supersolution. Let

$$\xi(t)=\int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,\rho(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s},$$

where

$$\rho(t)=\int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s},\ \text{for } t\in[1,e].$$

From Lemma 11, there exist $a_\rho>0, b_\rho>0$ such that

$$a_\rho\rho(t)\le\xi(t)\le b_\rho\rho(t),\ \text{for } t\in[1,e].$$

Take $\xi_1(t)=\delta_1\xi(t), \xi_2(t)=\delta_2\xi(t)$, where $0<\delta_1<\min\left\{\frac{1}{b_\rho}, a_\rho^{\frac{k}{1-k}}\right\}$, $\delta_2>\max\left\{\frac{1}{a_\rho}, b_\rho^{\frac{k}{1-k}}\right\}$. Then, we have

$$\begin{aligned}
(A\xi_1)(t)&=\int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,\xi_1(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&=\int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,\delta_1\xi(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&=\int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f\left(\tau,\frac{\delta_1\xi(\tau)}{\rho(\tau)}\rho(\tau)\right)\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&\ge\int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)\left(\frac{\delta_1\xi(\tau)}{\rho(\tau)}\right)^{k(p-1)}f(\tau,\rho(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&\ge\int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)\left(\delta_1 a_\rho\right)^{k(p-1)}f(\tau,\rho(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&=\left(\delta_1 a_\rho\right)^k\int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,\rho(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&\ge\delta_1\xi(t),
\end{aligned}$$

and

$$A\xi_1\ge\xi_1,\ \text{i.e., } \xi_1 \text{ is a subsolution of } A.$$

In addition, we have

$$\begin{aligned}
(A\xi_2)(t)&=\int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,\xi_2(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&=\int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)\delta_2^{p-1}\delta_2^{1-p}f(\tau,\xi_2(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&\le\int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)\delta_2^{p-1}\left(\frac{1}{\delta_2 b_\rho}\right)^{k(p-1)}f(\tau,\xi_2(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&\le\int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)\delta_2^{p-1}\left(\frac{\rho(\tau)}{\delta_2\xi(\tau)}\right)^{k(p-1)}f(\tau,\xi_2(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&\le\int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)\delta_2^{p-1}f\left(\tau,\frac{\rho(\tau)}{\delta_2\xi(\tau)}\xi_2(\tau)\right)\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
&=\delta_2\int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,\rho(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s},
\end{aligned}$$

and

$$A\xi_2\le\xi_2,\ \text{i.e., } \xi_2 \text{ is a supersolution of } A.$$

As a result, from Lemma 9, A has the smallest fixed point u_* and the largest fixed point u^* in $[\xi_1, \xi_2]$, respectively. Moreover, $u_* = \lim_{n\to\infty} A^n\xi_1$ and $u^* = \lim_{n\to\infty} A^n\xi_2$.

Next, we claim that $u_*(t) = u^*(t)$, for $t \in [1, e]$. We only prove that $u_*(t) \geq u^*(t)$. Note that they are fixed points for A, so

$$u_*(t) = \int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,u_*(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s},$$
$$u^*(t) = \int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,u^*(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}.$$

Then, from Lemma 11, there exists $b_i \geq a_i (i = 1, 2)$ such that

$$a_1 w_0 \leq u_* \leq b_1 w_0, \quad a_2 w_0 \leq u^* \leq b_2 w_0.$$

Hence, $u_* \geq \frac{a_1}{b_2} u^*$. Let $\mu_0 := \sup\{\mu > 0 : u_* \geq \mu u^*\}$. Then, $\mu_0 > 0$, and $u_* \geq \mu_0 u^*$. Next, we claim that $\mu_0 \geq 1$. If it is not true, then $\mu_0 \in (0, 1)$. Using (H6), (H7), we have

$$\begin{aligned} u_*(t) &= \int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,u_*(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\geq \int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,\mu_0 u^*(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\geq \int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)\mu_0^{k(p-1)}f(\tau,u^*(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &= \mu_0^k \int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,u^*(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}. \end{aligned}$$

Let

$$g(t) = \varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,\mu_0 u^*(\tau))\frac{d\tau}{\tau}\right) - \mu_0^k\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,u^*(\tau))\frac{d\tau}{\tau}\right).$$

Then, from (H6) and Lemma 11, we have

$$a_3 w_0(t) \leq \int_1^e G_\beta(t,s)g(s)\frac{ds}{s} \leq b_3 w_0(t).$$

Consequently,

$$\begin{aligned} u_*(t) &\geq \int_1^e G_\beta(t,s)g(s)\frac{ds}{s} + \mu_0^k\int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,u^*(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\geq \frac{a_3}{b_2}u^*(t) + \mu_0^k u^*(t) \\ &\geq \left(\frac{a_3}{b_2} + \mu_0\right)u^*(t). \end{aligned}$$

This contradicts the definition of μ_0, and $u_*(t) \geq \mu_0 u^*(t) \geq u^*(t)$. Therefore, A has a unique positive fixed point in $[\xi_1, \xi_2]$, and (1) has also a unique positive solution in $[\xi_1, \xi_2]$. This completes the proof. □

Theorem 4. *Suppose all the assumptions in Theorem 3 hold. Let $\widetilde{u}$ is a unique positive solution in $[\xi_1, \xi_2]$. Then, for any $u_0 \in [\xi_1, \xi_2]$ with $f(t, u_0(t)) \not\equiv 0$, the sequence*

$$u_n(t) = \int_1^e G_\beta(t,s)\varphi_q\left(\int_1^e G_\alpha(s,\tau)f(\tau,u_{n-1}(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}, n = 1, 2, ...,$$

uniformly converges to $\widetilde{u}(t)$*, for* $t \in [1, e]$.

Proof. From Theorem 3, we have $\widetilde{u} = \lim_{n\to\infty} A^n\xi_1 = \lim_{n\to\infty} A^n\xi_2$. Note that A is increasing, so, if $u_0 \in [\xi_1, \xi_2]$, we have

$$A^n\xi_1 \le A^n u_0 \le A^n\xi_2, \ \forall n \in \mathbb{N}_+.$$

This implies that $A^n u_0 \to \widetilde{u}$ as $n \to \infty$. From the definition of A, we have $u_n(t) = (Au_{n-1})(t) = A(Au_{n-2})(t) = (A^2u_{n-2})(t) = \cdots = (A^n u_0)(t)$, and thus $u_n(t) \to \widetilde{u}(t)$ uniformly on $t \in [1, e]$. This completes the proof. □

4. Conclusions

In this paper we investigate the existence and uniqueness of positive solutions for the Hadamard fractional p-Laplacian three-point boundary value problem (1). We first establish some relations from the corresponding problem without the p-Laplacian operator, and use some $(p-1)-$superlinearly and $(p-1)-$sublinearly conditions for the nonlinearity to obtain positive solutions to problem (1). After, using an increasing operator fixed-point theorem, we obtain the unique solution to problem (1), and establish uniform converged sequences for this solution.

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