mathematics

MDPI

Article

Characterization of n-Vertex Graphs of Metric Dimension $n-3$ by Metric Matrix

Juan Wang [1], Lianying Miao [1,*] and Yunlong Liu [2,*]

[1] School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China; d1wj@163.com
[2] College of Information and Control Engineering, Weifang University, Weifang 261061, China
* Correspondence: miaolianying@cumt.edu.cn (L.M.); fhylren@163.com (Y.L.)

Received: 15 April 2019; Accepted: 17 May 2019; Published: 27 May 2019

Abstract: Let $G = (V(G), E(G))$ be a connected graph. An ordered set $W \subset V(G)$ is a resolving set for G if every vertex of G is uniquely determined by its vector of distances to the vertices in W. The metric dimension of G is the minimum cardinality of a resolving set. In this paper, we characterize the graphs of metric dimension $n-3$ by constructing a special distance matrix, called metric matrix. The metric matrix makes it so a class of graph and its twin graph are bijective and the class of graph is obtained from its twin graph, so it provides a basis for the extension of graphs with respect to metric dimension. Further, the metric matrix gives a new idea of the characterization of extremal graphs based on metric dimension.

Keywords: extremal graph; metric dimension; resolving set; metric matrix

MSC: 05C12; 05C35; 05C75

1. Introduction

Let $G = (V(G), V(E))$ be a simple connected graph in this paper. The distance between two vertices $u, v \in V(G)$, denoted by $d(u, v)$, is the length of a shortest path between u and v in G. The diameter of G is denoted by $d(G)$ and $d(G) = max\{d(u, v)|u, v \in V(G)\}$. Let $W = \{w_1, w_2, \cdots, w_m\} \subseteq V(G)$ be an ordered set of G, the representation of $v \in V(G)$ with respect to W is the vector $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_m))$. We say that W is a resolving set of G if $r(v|W) \neq r(u|W)$ for every pair of distinct vertices $u, v \in V(G)$. A resolving set of minimum cardinality is called a metric basis of G. The metric dimension of a graph G, denoted by $\dim(G)$, is the cardinality of a metric basis. For $S, W_0 \subseteq V(G)$, we say that the set W_0 resolves S if $r(v|W_0) \neq r(u|W_0)$ for every pair of distinct vertices $u, v \in S$. Moreover, for distinct vertices $u, v, w \in V(G)$, if $d(w, u) \neq d(w, v)$, then we say that w resolves u and v.

The concepts of resolving set of a graph was first introduced by Slater [1] in 1975 and independently by Harary and Melter [2] in 1976. The metric dimension of a graph has been widely studied and a large number of related concepts have been extended (see [3–11]). As a parameter of a graph, it has been applied to lots of practical problems, such as robot navigation [12], connected joins in graphs and combinatorial optimization [13], and pharmaceutical chemistry [14].

There have been lots of results about the metric dimension of graphs. Some researchers focus on characterizations of metric dimension of graph families. For instance, the metric dimension of trees, cycles and wheels was considered in [14,15], respectively. Moreover, the metric dimension of some constructions of graphs was given. For example, the metric dimension of cartesian products of graphs and corona product of graphs was obtained in [16,17], respectively, the effect of vertex or edge deletion on the metric dimension of graphs was considered in [18] and the metric dimensions of symmetric graphs obtained by rooted product were given in [19].

In addition, some graphs with a fixed value of metric dimension have been characterized. Let G be a graph on n-vertex. In [14], the following conclusions were given: (1) G has metric dimension 1 if and only if $G = P_n$, where P_n denotes a path on n vertices; (2) G has metric dimension $n-1$ if and only if $G = K_n$, where K_n denotes a complete graph on n vertices; and (3) all graphs G of metric dimension $n-2$ were characterized (see Lemma 8). In [20,21], all the graphs of metric dimension $n-3$ and $n-d$ are characterized, respectively, where d is the diameter of G. Some other results on metric dimension of a graph are considered in [22–25].

It is interesting to extend a low-order graph to a high-order graph based on the given rulers. In this paper, we give a novel and effective method on the extension of graphs with respect to metric dimension and characterize the graphs with metric dimension $n-3$ via the method. Hernando et al. [21] gave an idea of using the twin graph (Definition 1) to characterize the graphs with dimension $n-r$, that is, to determine all the twin graphs of these graphs and extend them to corresponding graphs. We define the metric matrix of a graph (Definition 2) to determine and extend the twin graphs, which is different from that used by Jannesari and Omoomi [20] and Hernando et al. [21]. Since the metric matrix makes that a class of graph and its twin graph are bijective, the method makes the proof concise and readable. In addition, it has certain applicability to some other problems of metric dimension. For instance, it can be used to consider the graphs with dimension $n-4$ and even $n-r$ for $r \geq 5$. More importantly, it can be used as an effective basis for the extension of graphs with respect to metric dimension.

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries, including definitions, symbols and results used in this paper. In Section 3, we characterize all extremal graphs of $\dim(G) = n-3$ and diameter 2 by constructing a special distance matrix and discussing the structure of graphs.

2. Preliminaries

Let $n(G)$ and $G[S]$ denote the order and the subgraph induced by a subset $S \subseteq V(G)$ of a graph G, respectively. We say that S is an independent set of G if every pair of vertices in S are nonadjacent in G, and S is a clique of G if every pair of vertices in S are adjacent in G. The neighborhood of $u \in V(G)$ is denoted by $N(u)$ and $N(u) = \{v | uv \in E(G)\}$. Let $N[u] = N(u) \cup \{u\}$. We use $\deg(u)$, $\delta(G)$ to denote the degree of v and the minimum degree of G, respectively, where $\deg(u) = |N(u)|$. A pair of vertices $u, v \in V(G)$ are twins in G if $N(u) = N(v)(uv \notin E(G))$ or $N[u] = N[v](uv \in E(G))$. We say that a subset V_i of vertices is a twin set of G if its vertices are pairwise twins in G, and a maximal twin set is a twin class. Clearly, if V_i is a twin set of a graph G, then it is an independent set or a clique of G. The circumference of a graph G, denoted by $c(G)$, is the length of a longest circle of G.

Definition 1. *The twin graph of a graph G, denoted by G_T, is the subgraph induced by $\{v_1, v_2, \ldots, v_k\}$, that is, $G_T = G[\{v_1, v_2, \ldots, v_k\}]$, where $v_i \in V_i$ for $1 \leq i \leq k$ and $V_1, V_2, \ldots, V_k$ are the all distinct twin classes of G.*

Definition 2. *For a graph G, let $V(G_T) = \{v_1, v_2, \ldots, v_k\}$ and let V_i be the twin class of G with $v_i \in V_i$ for $1 \leq i \leq k$, the metric matrix of G is denoted by $D = [d_{ij}]_{k\times k}$ and*

$$d_{ij} = \begin{cases} d(v_i, v_j), & i \neq j, \\ d(v_i, v_i'), & i = j \text{ and } |V_i| \geq 2, \\ 0, & i = j \text{ and } |V_i| = 1, \end{cases}$$

where $v_i' \in V_i$ and $v_i' \neq v_i$.

Let d_i denote d_{ii} in the following sections. Since V_i is an independent set or a clique of G, we have $d_i = 0$ *or* 1 *or* 2. The metric matrix determines different classes of graphs with the same twin graph.

The graph $G_1 + G_2$ is obtained from G_1 and G_2 by adding the edges from every vertex of G_1 to every vertex of G_2, which is represented as in Figure 1 in this paper. The union $G_1 \cup G_2$ of G_1 and G_2 is the graph whose vertex set and edge set are $V(G_1) \cup V(G_2)$ and $E(G_1) \cup E(G_2)$, respectively.

The graph $G - v$ is obtained from G by deleting the vertex v. The graph $G - e$ is obtained from G by deleting the edge e. Let $\overline{K_s}$ be the complement of K_s and let $K_{1,m}$ denote a star with $m + 1$ vertices.

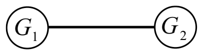

Figure 1. $G_1 + G_2$.

Unless explicitly noted during the rest of this paper, let V_i be the twin classes of G with $v_i \in V_i$, and if $|V_i| \geq 2$, then assume that $v_i' \in V_i$ and $v_i' \neq v_i$.

Lemma 1 ([20]). *For a graph G, we have $n(G_T) - dim(G_T) \leq n(G) - dim(G)$.*

Lemma 2. *For a graph G, if v_i and v_j are twins in G_T, then at least one of V_i and V_j have cardinality at least 2. Moreover, if $|V_i| = s \geq 2$, then $G[V_i] = \overline{K_s}$ when $d(v_i, v_j) = 1$ and $G[V_i] = K_s$ when $d(v_i, v_j) = 2$.*

Proof. Since v_i and v_j are twins in G_T, we have $d(u, v_i) = d(u, v_j)$ for every $u \in V(G) \setminus (V_i \cup V_j)$. Moreover, by the definition of G_T, we obtain that v_i and v_j are not twins in G, so there is $x \notin \{v_i, v_j\}$ and $x \in V_i$ or $x \in V_j$ such that $d(x, v_i) \neq d(x, v_j)$. Thus, at least one of V_i and V_j have cardinality at least 2. Assume that $x \in V_i$, that is, $|V_i| = s \geq 2$. If $d(v_i, v_j) = 1$, then $d(x, v_j) = 1$, so $d(x, v_i) = 2$, that is, $G[V_i] = \overline{K_s}$. Otherwise, $d(v_i, v_j) = 2$, then $d(x, v_i) = 1$, that is, $G[V_i] = K_s$. □

Corollary 1. *For a graph G, if $S = \{v_{i_1}, v_{i_2}, \ldots, v_{i_t}\}$ is a twin set in G_T, then we have that if $G[S] = K_t$, then all but one of the sets V_{i_1}, Vi_2,. . . ,V_{i_t} have cardinality at least 2 and induce an empty graph, and if $G[S] = \overline{K_t}$, then all but one of the sets V_{i_1}, Vi_2,. . . ,V_{i_t} have cardinality at least 2 and induce an complete graph.*

Lemma 3 ([21]). *Let W be a metric basis of G and let $S \subseteq V$ be a nonempty subset. If S is a twin set of G, then at most one of the vertices in S is not in W and the set $(W \setminus S) \cup \{s\}$ resolves the set $V(G) \backslash S$ for each $s \in S$.*

Let G_T and D be the twin graph and the metric matrix of G, respectively. Let $S_1 = \{v_i' \,|\, v_i \in V(G_T)$ and $d_i > 0\}$, $S_2 = \{v_i \in V(G_T) \,|\, d_i = 0\}$, and U be an arbitrary subset of S_2. The matrix D_U is obtained from D by deleting the corresponding rows of all vertices in U and the corresponding columns of all vertices in $S_2 \backslash U$. We can get that $r(v \,|\, S_1 \cup U) = D_U(v)$ for each $v \in V(G_T) \setminus U$, where $D_U(v)$ is the row vectors corresponding to v in D_U. Moreover, each subset of S_2 corresponds to a matrix D_U. Let $\mathcal{D} = \{D_U : U \subseteq S_2\}$ be the set constituting of all matrix D_U. We have the following result.

Lemma 4. *For a graph G, $dim(G) = n(G) - r$ if and only if for each $U \subseteq S_2$ the matrix D_U has at most r different row vectors, and there exists a subset $U_0 \subseteq S_2$ such that D_{U_0} has exactly r different row vectors.*

Proof. Suppose that W and V_i are a metric basis and a twin class of G, respectively. By Lemma 3, at most one of the vertices in V_i is not in W. Let W_0 be a set obtained from W by replacing v_i with v for each vertex $v \notin W$, where $v \in V_i$. Then W_0 is a metric basis of G, and we get that $V(G) \backslash W_0 \subseteq V(G_T)$ and $S_1 \subseteq W_0$. Moreover, we have that $(W_0 \cap S_2) \cup S_1$ resolves $V(G) \setminus W_0$ by Lemma 3.

Necessity. Since $\dim(G) = n(G) - r$, there are exactly r vertices in $V(G_T)$ not in W_0. Thus the set $S_1 \cup U$ resolves at most r vertices in $V(G_T) \backslash U$ for each subset U, which implies that each $D_U \in \mathcal{D}$ has at most r different row vectors. Let $U_0 = W_0 \cap S_2$, then $S_1 \cup U_0$ resolves exactly r vertices in $V(G_T) \backslash U_0$. Thus, D_{U_0} has exactly r different row vectors.

Sufficiency. Since each $D_U \in \mathcal{D}$ has at most r different row vectors, $S_1 \cup U$ resolves at most r vertices in $V(G_T) \backslash U$ for each subset $U \subseteq S_2$, then $S_1 \cup (W_0 \cap S_2)$ resolves at most r vertices in $V(G_T) \backslash (W_0 \cap S_2)$. Thus, $\dim(G) \geq n - r$. In addition, since there exists $D_{U_0} \in \mathcal{D}$ with exactly r different row vectors, $S_1 \cup (W_0 \cap S_2)$ resolves at least r vertices in $V(G_T) \backslash (W_0 \cap S_2)$, so $\dim(G) \leq n - r$. Thus, $\dim(G) = n - r$. □

Corollary 2. *For a graph G, we have $n(G) - dim(G) \leq n(G_T)$.*

Let $D_{i_1 i_2 \cdots i_k}$ be the matrix consisting of the i_1, i_2,$\cdots$,i_k columns of the metric matrix D of G. Let $p_{i_1 i_2 \cdots i_k}$ be the number of different row vectors without zero element in $D_{i_1 i_2 \cdots i_k}$.

Corollary 3. *If $p_{i_1 i_2 \cdots i_k} = r$, then $dim(G) \leq n(G) - r$.*

Lemma 5. *Let G be a graph of $dim(G) = n(G) - r$. If the set T is a twin class of G_T with $|T| = t \geq 2$, then $r = t$ if $n(G_T) = t$, and $r \geq t + 1$ if $n(G_T) > t$.*

Proof. Suppose that $T = \{v_1, v_2, \cdots, v_t\}$. Then $G[T] \cong K_t$ or $\overline{K_t}$. By Corollary 1, assume that $|V_i| \geq 2$ for $1 \leq i \leq t - 1$, then we have that $d(v_i, v_i') = 2$, $d(v_j, v_i') = 1$ when $G[T] \cong K_t$, and $d(v_i, v_i') = 1$, $d(v_j, v_i') = 2$ when $G[T] \cong \overline{K_t}$ for $1 \leq i < j \leq t$. Let $W_0 = \{v_1', v_2', \ldots, v_{t-1}'\}$, then W_0 resolves the set $\{v_1, v_2, \cdots, v_t\}$. Thus, we have $r \geq t$. Since $r = n(G) - \dim(G) \leq n(G_T)$ by Corollary 2, if $n(G_T) = t$, then $r = t$. If $n(G_T) > t$, then there exists $v_{t+1} \in V(G_T) \setminus T$. Since $v_{t+1} \notin T$, there is $x \in V(G_T)$ such that $d(v_{t+1}, x) \neq d(v_i, x)$ for all $i \in \{1, 2, \cdots, t\}$. If $x \in T$, then W_0 resolves the set $\{v_1, v_2, \cdots, v_{t+1}\}$. Otherwise, $W_0 \cup \{x\}$ resolves the set $\{v_1, v_2, \cdots, v_{t+1}\}$. Therefore, $r \geq t + 1$. □

Corollary 4. *For a graph G, if $G_T = K_{1,m}$ or $G_T = K_m$, then $dim(G) = n(G) - n(G_T)$.*

Lemma 6 ([14])**.** *Let G be a graph of order $n(G) > 2$ and diameter d, then $dim(G) \leq n(G) - d$.*

Lemma 7 ([14])**.** *Let G be graph of order $n(G) > 2$. Then $dim(G) = n(G) - 1$ if and only if $G = K_n$.*

Lemma 8 ([14])**.** *Let G be graph of order $n(G) \geq 4$. Then $dim(G) = n(G) - 2$ if and only if $G = K_{s,t}$ $(s, t \geq 1)$, $G = K_s + \overline{K_t}$ $(s \geq 1, t \geq 2)$, or $G = K_s + (K_1 \cup K_t)$ $(s, t \geq 1)$.*

Lemma 9. *Let G be a graph with $dim(G) = n(G) - 3$ and a metric basis W. Then there exists a set $W_0 \subseteq W$ and $|W_0| \leq 2$ such that it resolves $V(G) \setminus W$.*

Proof. Suppose that $V(G) \setminus W = \{v_1, v_2, v_3\}$. Then there is a vertex $w_1 \in W$ such that $d(w_1, v_1) \neq d(w_1, v_2)$. If $d(w_1, v_i) \neq d(w_1, v_3)$ for $i \in \{1, 2\}$, then w_1 resolves $\{v_1, v_2, v_3\}$. Let $W_0 = \{w_1\}$, then we are done. Otherwise, assume that $d(w_1, v_1) = d(w_1, v_3)$, then there is $w_2 \in W$ such that $d(w_2, v_1) \neq d(w_2, v_3)$. Thus, $\{w_1, w_2\}$ resolves $\{v_1, v_2, v_3\}$. Let $W_0 = \{w_1, w_2\}$, then we are done. □

3. Extremal Graphs G of $\dim(G) = n(G) - 3$

By Lemma 6, if $\dim(G) = n(G) - 3$, then $d(G) \leq 3$. Moreover, if $d(G) = 1$, then $\dim(G) = n(G) - 1$. Thus, $\dim(G) = n(G) - 3$ only if $d(G) = 2$ *or* 3. Since all the graphs of $\dim(G) = n(G) - d(G)$ were characterized in [21], we only need to consider the graphs of $d(G) = 2$.

In the following, unless noted otherwise, let D, W be the metric matrix and a resolving set of G, respectively. Let $e_{ij} = v_i v_j$ and $B_m = K_m$ *or* $\overline{K_m}$.

Lemma 10. *Let G be a graph of $d(G) = 2$ and $n(G_T) = 3$. Then $dim(G) = n(G) - 3$ if and only if $G = (K_s \cup B_r) + K_t$ $(s, r \geq 2,\ t \geq 1)$ or $G = (K_s \cup B_r) + \overline{K_t}$ $(s, t \geq 2,\ r \geq 1)$ or $G = (\overline{K_s} + \overline{K_t}) + B_r$ $(s, t \geq 2,\ r \geq 1)$.*

Proof. Suppose that $V(G_T) = \{v_1, v_2, v_3\}$, there are two cases to be considered as follows.

Case 1. $G_T \cong P_3 = v_1v_2v_3$. Since v_1 and v_3 are twins in G_T and $e_{13} \notin E(G_T)$, by Lemma 2, we may assume that $d_1 = 1$, then

$$D = \begin{bmatrix} 1 & 1 & 2 \\ 1 & d_2 & 1 \\ 2 & 1 & d_3 \end{bmatrix}.$$

Since there are at most three different row vectors in D, by Lemma 4, $\dim(G) = n(G) - 3$ if and only if there are exactly three different row vectors in some D_U, which implies that (1) $d_2 = 0$ *or* 1, $d_3 = 1$ *or* 2, or (2) $d_2 = 2$, $d_3 = 0$ *or* 1 *or* 2. Therefore, $G = (K_s \cup B_r) + K_t$ $(s, r \geq 2,\ t \geq 1)$ or $(K_s \cup B_r) + \overline{K_t}$ $(s, t \geq 2,\ r \geq 1)$.

Case 2. $G_T \cong C_3$. Since any two vertices in $\{v_1, v_2, v_3\}$ are twins in G_T, by Corollary 1, we may assume that $d_1 = d_2 = 2$, then

$$D = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & d_3 \end{bmatrix}.$$

Since there are at most three different row vectors in D and $p_{12} = 3$, by Lemma 4, $\dim(G) = n(G) - 3$ if and only if $d_3 = 0$ *or* 1 *or* 2, which implies that $G = (\overline{K_s} + \overline{K_t}) + B_r$ $(s, t \geq 2,\ r \geq 1)$. □

Lemma 11. *Let G be a graph of $d(G) = 2$ and $n(G_T) = 4$. Then $\dim(G) = n(G) - 3$ if and only if G is one of the graphs in Figure 2, where a small circle denotes K_1 (similarly in the following figures).*

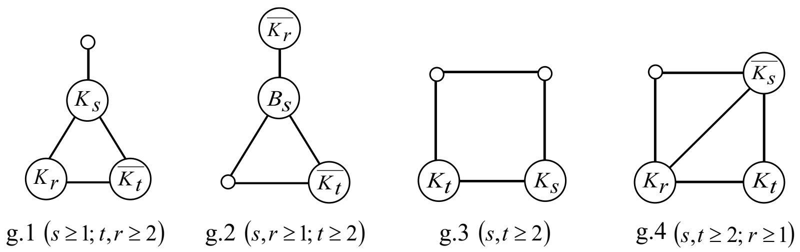

Figure 2. All graphs of $d(G) = 2$, $n(G_T) = 4$ and $\dim(G) = n(G) - 3$.

Proof. Let $V(G_T) = \{v_1, v_2, v_3, v_4\}$. There are the following cases to be considered depending on the circumference of G_T.

Case 1. G_T is acyclic. In such a case, $G_T \cong K_{1,3}$. There is no graph G with $\dim(G) = n(G) - 3$ by Corollary 4.

Case 2. $c(G_T) = 3$. Let $C_3 = v_1v_2v_3v_1$, we may assume that $G_T \cong C_3 \cup e_{14}$. Since v_2 and v_3 are twins in G_T and $e_{23} \in E(G_T)$, by Lemma 2, we may assume that $d_2 = 2$, then

$$D = \begin{bmatrix} d_1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & d_3 & 2 \\ 1 & 2 & 2 & d_4 \end{bmatrix}.$$

By Corollary 3, we obtain that $d_3 \neq 2$ and $d_4 \neq 1$; if not, $p_{23} = 4$ or $p_{24} = 4$, which contradicts $\dim(G) = n(G) - 3$. If $d_3 = 1$, then $d_1 \neq 2$ and $d_4 = 0$; otherwise, $p_{123} = 4$ or $p_{234} = 4$. By Lemma 4, $\dim(G) = n(G) - 3$ when $d_3 = 1$, $d_1 = 0$ *or* 1, $d_4 = 0$; or $d_3 = 0$, $d_1 = 0$ *or* 1 *or* 2, $d_4 = 0$ *or* 2. Thus $\dim(G) = n(G) - 3$ if and only if (1) $d_1 = 0$ *or* 1, $d_2 = 2$, $d_3 = 1$, $d_4 = 0$, or (2) $d_1 = 0$ *or* 1 *or* 2, $d_2 = 2$, $d_3 = 0$, $d_4 = 0$ *or* 2, which implies that G is the graph g.1 or g.2 in Figure 2.

Case 3. $c(G_T) = 4$. Let $C_4 = v_1v_2v_3v_4v_1$ is a longest circle of G_T.

Case 3.1. $G_T \cong C_4$. Since the pairs of vertices v_1, v_3 and v_2, v_4 are twins in G_T, respectively, we may suppose that $d_1 = d_2 = 1$ by Lemma 2, then

$$D = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & d_3 & 1 \\ 1 & 2 & 1 & d_4 \end{bmatrix}.$$

By Corollary 3, we have that $d_3 = d_4 = 0$; if not, $P_{123} = 4$ or $P_{124} = 4$. Since $\dim(G) = n(G) - 3$ when $d_3 = d_4 = 0$ by Lemma 4, $\dim(G) = n(G) - 3$ if and only if $d_1 = d_2 = 1$ and $d_3 = d_4 = 0$, which implies that G is the graph g.3 in Figure 2.

Case 3.2. $G_T \cong C_4 \cup e_{13}$. Since the pairs of vertices v_1, v_3 and v_2, v_4 are twins in G_T, respectively, we may assume that $d_1 = 2$, $d_2 = 1$ by Lemma 2, then

$$D = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & d_3 & 1 \\ 1 & 2 & 1 & d_4 \end{bmatrix}.$$

By Corollary 3, $d_3 \neq 2$ and $d_4 = 0$; if not, $P_{123} = 4$ or $P_{124} = 4$. Since $\dim(G) = n(G) - 3$ when $d_3 = 0$ *or* 1, $d_4 = 0$ by Lemma 4, $\dim(G) = n(G) - 3$ if and only of $d_1 = 2$, $d_2 = 1$, $d_3 = 0$ *or* 1 and $d_4 = 0$, which implies that G is the graph g.4 in Figure 2.

Case 3.3. $G_T \cong K_4$. There is no graph G with $\dim(G) = n(G) - 3$ by Corollary 4. □

Lemma 12. *Let G be a graph of $d(G) = 2$ and $n(G_T) = 5$. Then $\dim(G) = n(G) - 3$ if and only if G is C_5 or one of graphs in Figure 3.*

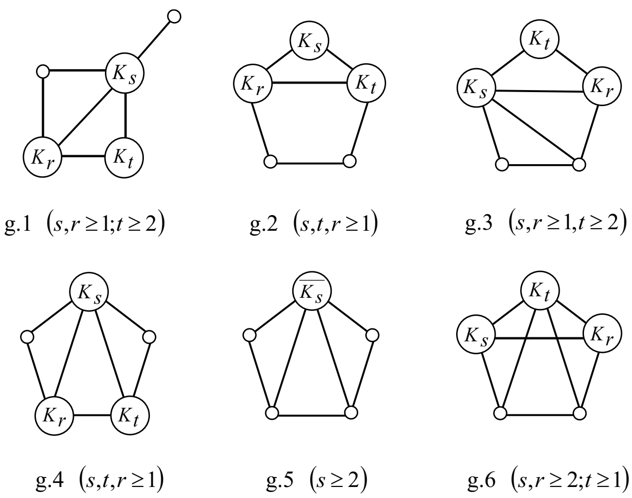

Figure 3. All graphs of $d(G) = 2$, $n(G_T) = 5$ and $\dim(G) = n(G) - 3$.

Proof. Let $V(G_T) = \{v_1, v_2, v_3, v_4, v_5\}$. There are the following cases to be considered depending on the circumference of G_T.

Case 1. G_T is acyclic. In such a case, $G_T \cong K_{1,4}$. There is no graph G with $\dim(G) = n(G) - 3$ by Corollary 4.

Case 2. $c(G_T) = 3$. Let $C_3 = v_1 v_2 v_3 v_1$ be a longest circle of G_T, then G_T is isomorphic to the graph (a) or (b) in Figure 4. Since the pairs of vertices v_2, v_3 and v_4, v_5 are twins in G_T, respectively, we may assume that $|V_2| \geq 2$ and $|V_4| \geq 2$, then, for graph (a), $\{v_2', v_4'\}$ resolves $V(G_T)$, and for graph (b), $\{v_2, v_4'\}$ resolves $\{v_1, v_3, v_4, v_5\}$, a contradiction.

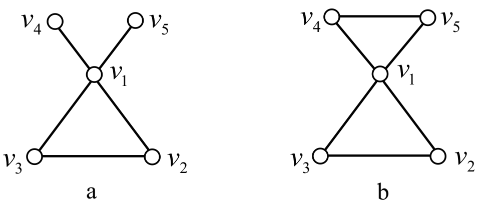

Figure 4. Graphs of $d(G) = 2$, $n(G) = 5$ and $c(G) = 3$. For **a**, v_4 and v_5 are nonadjacent; for **b**, v_4 and v_5 are adjacent.

Case 3. $c(G_T) = 4$. Let $C_4 = v_1v_2v_3v_4v_1$ be a longest circle of G_T. Then v_5 is nonadjacent to two adjacent vertices of C_4; otherwise, $c(G_T) = 5$.

Case 3.1. The vertex v_5 is exactly adjacent to one vertex of C_4. Then G_T is isomorphic to the graph (a) or (b) in Figure 5.

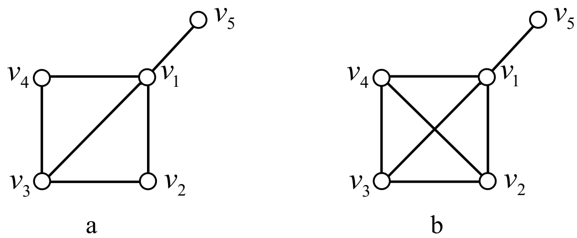

Figure 5. Vertex v_5 is exactly adjacent to one vertex of C_4. For **a**, v_2 and v_4 are nonadjacent; for **b**, v_2 and v_4 are adjacent.

For Figure 5a, since v_2 and v_4 are twins in G_T, we may assume that $d_2 = 1$, then

$$D = \begin{bmatrix} d_1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & d_3 & 1 & 2 \\ 1 & 2 & 1 & d_4 & 2 \\ 1 & 2 & 2 & 2 & d_5 \end{bmatrix}.$$

By Corollary 3, we have that $d_1 \neq 2$, $d_3 \neq 2$ and $d_4 = d_5 = 0$. Otherwise, if $d_1 = 2$, then $p_{123} \geq 4$; if $d_3 = 2$, then $p_{23} = 4$; and if $d_4 \neq 0$ or $d_5 \neq 0$, then $p_{245} \geq 4$, which is a contradiction. Since $\dim(G) = n(G) - 3$ when $d_1 = 0$ *or* 1, $d_3 = 0$ *or* 1 and $d_4 = d_5 = 0$ by Lemma 4, $\dim(G) = n(G) - 3$ if and only if G is the graph g.1 in Figure 3.

For Figure 5b, since $\{v_2, v_3, v_4\}$ is a twin set of G_T, by Lemma 5, $\dim(G) < n(G) - 3$. Thus, there is no G with $\dim(G) = n(G) - 3$.

Case 3.2. The vertex v_5 is exactly adjacent to two vertices of C_4. Then $c(G) = 4$ if and only if G_T is isomorphic to the graph (a) or (b) in Figure 6. Since $\{v_1, v_3, v_5\}$ is a twin set of G_T, $\dim(G) < n(G) - 3$ by Lemma 5.

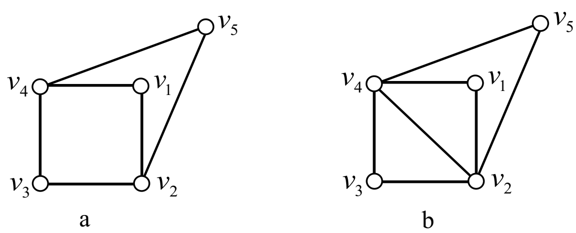

Figure 6. Vertex v_5 is exactly adjacent to two vertices of C_4. For **a**, v_2 and v_4 are nonadjacent; for **b**, v_2 and v_4 are adjacent.

Case 4. $c(G_{\mathrm{T}}) = 5$. Let $C_5 = v_1v_2v_3v_4v_5v_1$ be a longest circle in G_{T}. Since $5 \le |E(G_{\mathrm{T}})| \le 10$, we consider the five subcases as follows.

Case 4.1. $|E(G_{\mathrm{T}})| = 5$. We obtain that $G_{\mathrm{T}} \cong C_5$, then

$$D = \begin{bmatrix} d_1 & 1 & 2 & 2 & 1 \\ 1 & d_2 & 1 & 2 & 2 \\ 2 & 1 & d_3 & 1 & 2 \\ 2 & 2 & 1 & d_4 & 1 \\ 1 & 2 & 2 & 1 & d_5 \end{bmatrix}.$$

By Corollary 3, $d_1 = 0$. Otherwise, if $d_1 = 1$, then $p_{12} \ge 4$; and if $d_1 = 2$, then $p_{13} \ge 4$. By the symmetry of G_{T}, $d_i = 0\,(2 \le i \le 5)$. Since $\dim(C_5) = 2$, $\dim(G) = n(G) - 3$ if and only if $G \cong C_5$.

Case 4.2. $|E(G_{\mathrm{T}})| = 6$. We obtain that $c(G_{\mathrm{T}}) = 5$ if and only if G_{T} is isomorphic to Figure 7, then

$$D = \begin{bmatrix} d_1 & 1 & 2 & 2 & 1 \\ 1 & d_2 & 1 & 2 & 1 \\ 2 & 1 & d_3 & 1 & 2 \\ 2 & 2 & 1 & d_4 & 1 \\ 1 & 1 & 2 & 1 & d_5 \end{bmatrix}.$$

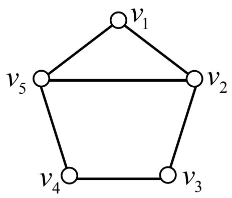

Figure 7. Graph G of $d(G) = 2$, $n(G) = c(G) = 5$ and $|E(G)| = 6$.

By Corollary 3, we have that $d_i \ne 2\,(1 \le i \le 5)$ and d_3, $d_4 \ne 1$. Otherwise, if $d_1 = 2$ or $d_3 = 2$, then $p_{13} \ge 4$; if $d_2 = 2$, then $p_{12} \ge 4$; if $d_3 = 1$, then $p_{23} \ge 4$. By the symmetry of G_{T}, we get that $d_4 \ne 1$ *or* 2 and $d_5 \ne 2$. Since $\dim(G) = n(G) - 3$ when $d_i = 0$ *or* 1 $(i = 1, 2, 5)$ and $d_3 = d_4 = 0$ by Lemma 4, $\dim(G) = n(G) - 3$ if and only if G is the graph g.2 in Figure 3.

Case 4.3. $|E(G_{\mathrm{T}})| = 7$. We obtain that $c(G_{\mathrm{T}}) = 5$ if and only if G_{T} is isomorphic to the graph (a) or (b) in Figure 8.

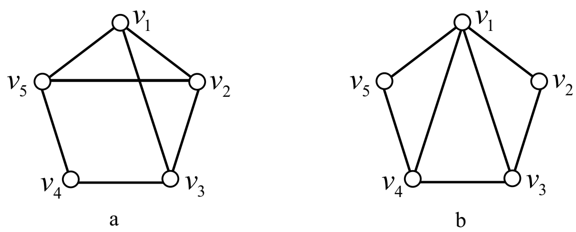

Figure 8. Graphs of $d(G) = 2$, $n(G) = c(G) = 5$ and $|E(G)| = 7$. For **a**, there are two edges without common vertices in C_5; for **b**, there are two edges with a common vertex in C_5.

For Figure 8a, since the pairs of vertices v_1, v_2 and v_3, v_5 are twins in G_{T}, respectively, we may assume that $d_1 = 2$ and $d_3 = 1$, then $\{v_1', v_2, v_3'\}$ resolves $\{v_1, v_3, v_4, v_5\}$, which is a contradiction.

For Figure 8b, since

$$D = \begin{bmatrix} d_1 & 1 & 1 & 1 & 1 \\ 1 & d_2 & 1 & 2 & 2 \\ 1 & 1 & d_3 & 1 & 2 \\ 1 & 2 & 1 & d_4 & 1 \\ 1 & 2 & 2 & 1 & d_5 \end{bmatrix},$$

by Corollary 3, $d_i \neq 2$ $(2 \leq i \leq 5)$. Otherwise, if $d_2 = 2$ or $d_3 = 2$, then $p_{25} \geq 4$ or $p_{23} \geq 4$. By the symmetry of G_T, we get that $d_4, d_5 \neq 2$.

If $d_1 = 0$ *or* 1, then $d_2 = 0$ or 1. Assume $d_2 = 1$, then $d_4 = d_5 = 0$, if not, $p_{245} = 4$. In this case, $d_3 = 0$ *or* 1 by Lemma 4. Now we assume that $d_2 = 0$. By the symmetry of G_T, we may assume that $d_5 = 0$; if not, it is the same as $d_2 = 1$. By Lemma 4, we have that $d_3 = d_4 = 0$ *or* 1.

If $d_1 = 2$, then $d_i = 0$ for $2 \leq i \leq 5$. Otherwise, if $d_2 = 1$, then $p_{124} = 4$; if $d_3 = 1$, then $p_{123} = 4$; if $d_4 = 1$, then $p_{134} = 4$; and if $d_5 = 1$, then $p_{145} = 4$.

Thus, we have that $\dim(G) = n(G) - 3$ when (1) $d_2 = 1$, $d_i = 0$ *or* 1 $(i = 1, 3)$, $d_4 = d_5 = 0$, (2) $d_i = 0$ *or* 1 $(i = 1, 3, 4)$, $d_2 = d_5 = 0$, or (3) $d_1 = 2$, $d_i = 0$ $(2 \leq i \leq 5)$, which implies that $\dim(G) = n(G) - 3$ if and only if G is one of the graphs g.3, g.4 and g.5 in Figure 3.

Case 4.4. $|E(G_T)| = 8$. We obtain that $c(G_T) = 5$ if and only if G_T is isomorphic to the graph (a) or (b) in Figure 9.

For Figure 9a, since the pairs of vertices v_1, v_2 and v_3, v_5 are twins in G_T, respectively, we may assume that $d_1 = d_3 = 1$, then $\{v_1', v_3', v_4\}$ resolves $\{v_1, v_2, v_3, v_5\}$, which contradicts $\dim(G) = n(G) - 3$.

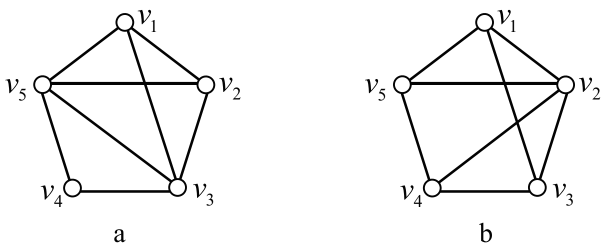

Figure 9. Graphs of $d(G) = 2$, $n(G) = c(G) = 5$ and $|E(G)| = 8$. For **a**, v_3 and v_5 are adjacent; for **b**, v_2 and v_4 are adjacent.

For Figure 9b, the pairs of vertices v_1, v_2 and v_3, v_5 are twins in G_T, respectively, we may assume that $d_1 = d_3 = 2$, then

$$D = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ 1 & d_2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & d_4 & 1 \\ 1 & 1 & 2 & 1 & d_5 \end{bmatrix}.$$

By Corollary 3, we obtain that $d_4 = d_5 = 0$ and $d_2 \neq 2$. Otherwise, if $d_4 \neq 0$, then $p_{134} = 4$, by the symmetry of G_T, $d_5 = 0$; and if $d_2 = 2$, then $p_{125} = 4$. By Lemma 4, we get that $\dim(G) = n(G) - 3$ when $d_1 = d_3 = 1$, $d_2 = 0$ *or* 1, $d_4 = d_5 = 0$, which implies that G is the graph g.6 in Figure 3.

Case 4.5. $|E(G_T)| \geq 9$. We obtain that G_T is isomorphic to $K_5 - e$ or K_5. By Lemma 5 and Corollary 4, there is no graph G with $\dim(G) = n(G) - 3$. □

Lemma 13. *Let G be a graph with $d(G) = 2$ and $\dim(G) = n(G) - 3$. If $G_T = G$, then $n(G) \leq 5$.*

Proof. Since $G_T = G$, there are no twins in G. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$. We consider the both cases $n(G) > 6$ and $n(G) = 6$.

Case 1. $n(G) > 6$. By Lemma 9, we may assume that $W_0 = \{v_1, v_2\}$ resolves $\{v_3, v_4, v_5\}$. Since $\dim(G) = n(G) - 3$, $r(v_6|W_0), r(v_7|W_0) \in \{r(v_3|W_0), r(v_4|W_0), r(v_5|W_0)\}$.

Case 1.1. $r(v_6|W_0) \neq r(v_7|W_0)$. We may assume that $r(v_6|W_0) = r(v_3|W_0)$ and $r(v_7|W_0) = r(v_4|W_0)$. Since there are no twins in G, there exists $x \in V(G)$ such that $d(v_6, x) \neq d(v_3, x)$. Then $x = v_5$; if not, $\{v_1, v_2, x\}$ resolves $\{v_3, v_4, v_5, v_6\}$ or $\{v_3, v_5, v_6, v_7\}$. Similarly, $d(v_4, v_5) \neq d(v_7, v_5)$. Thus $\{v_1, v_2, v_5\}$ resolves $\{v_3, v_4, v_6, v_7\}$, which contradicts $\dim(G) = n(G) - 3$.

Case 1.2. $r(v_6|W_0) = r(v_7|W_0)$. We may assume that $r(v_6|W_0) = r(v_7|W_0) = r(v_3|W_0)$. Since there are no twins in G, there exists $x \in V(G)$ such that x resolves two vertices in $\{v_3, v_6, v_7\}$. Assume that $d(v_6, x) \neq d(v_3, x)$, then $x \in \{v_4, v_5\}$; if not, $\{v_1, v_2, x\}$ resolves $\{v_3, v_4, v_5, v_6\}$, which is a contradiction. Thus $\{v_4, v_5\}$ resolves both $\{v_3, v_6, v_7\}$ and $\{v_1, v_2\}$, it becomes case 1.1.

Case 2. $n(G) = 6$. We first prove that $\delta(G) \geq 2$. Assume for a contradiction that $\deg(v_1) = 1$. We may assume that $e_{12} \in E(G)$. Since $d(G) = 2$, $N(v_2) = \{v_1, v_3, v_4, v_5, v_6\}$. Since v_1 and v_3 are not twins, we may assume that $e_{34} \in E(G)$. Similarly, assume that $e_{35} \notin E(G)$, $e_{45} \in E(G)$, $e_{36} \notin E(G)$ and $e_{56} \in E(G)$. In such a case, $e_{46} \notin E(G)$, otherwise v_5 and v_6 are twins. Thus, $\{v_3, v_6\}$ is a resolving set of G, then $\dim(G) = n(G) - 4$, which is a contradiction.

Now we construct G. Let $P_3 = v_1v_2v_3$ be a shortest path of length 2 of G. Since v_1 and v_3 are not twins, we may assume that $e_{14} \notin E(G)$ and $e_{34} \in E(G)$. There are two subcases to be considered as follows.

Case 2.1. $e_{24} \in E(G)$. Since v_3 and v_4 are not twins, we may assume that $e_{35} \notin E(G)$ and $e_{45} \in E(G)$. There are four subcases to be considered by the adjacency relationship between v_5 and v_1, v_2 as fallows.

Case 2.1.1. $e_{15} \in E(G)$, $e_{25} \notin E(G)$. We obtain that the metric matrix of $G - v_6$ is

$$D = \begin{bmatrix} 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 \end{bmatrix}.$$

The distance from v_6 to v_i and v_j $(1 \leq i < j \leq 5)$ is denoted by $(v_i, v_j|d(v_i, v_6), d(v_j, v_6))$. For $\dim(G) = n(G) - 3 = 3$, by Corollary 3, any one of $(v_1, v_3|2, 2)$, $(v_1, v_5|1, 1)$, $(v_2, v_3|2, 1)$, $(v_2, v_4|2, 2)$, $(v_3, v_4|1, 2)$ and $(v_3, v_5|2, 2)$ does not hold. Hence there is at least one of v_1 and v_3 adjacent to v_6. If $e_{16} \in E(G)$, then $e_{56} \notin E(G)$, $e_{36} \in E(G)$ and e_{26}, $e_{46} \in E(G)$, which implies that v_2 and v_6 are twins. If $e_{36} \in E(G)$, then $e_{16} \notin E(G)$, e_{26}, $e_{46} \in E(G)$. If $e_{56} \in E(G)$ then v_4 and v_6 are twins; and if $e_{56} \notin E(G)$, then v_3 and v_6 are twins. Therefore, there is no graph G with $\dim(G) = n(G) - 3$.

Case 2.1.2. $e_{15} \notin E(G)$, $e_{25} \in E(G)$. Since v_3 and v_5 are not twins, $d(v_3, v_6) \neq d(v_5, v_6)$. We may assume that $e_{36} \notin E(G)$, $e_{56} \in E(G)$. Since $\delta(G) \geq 2$, $\deg(v_1) \geq 2$, and we get that $e_{16} \in E(G)$. Then $\{v_1, v_3\}$ is a resolving set of G. Thus $\dim(G) = 4$, which is a contradiction.

Case 2.1.3. $e_{15}, e_{25} \in E(G)$. The metric matrix of $G - v_6$ is

$$D = \begin{bmatrix} 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{bmatrix}.$$

By Corollary 3, we obtain that any one of $(v_1, v_3|2, 2)$, $(v_1, v_5|1, 2)$, $(v_3, v_4|1, 2)$ and $(v_4, v_5|2, 2)$ does not holds. Hence there is at least one of v_1 and v_3 adjacent to v_6. Assume that $e_{16} \in E(G)$, which implies that $e_{56} \in E(G)$. For e_{36}, if $e_{36} \in E(G)$, then $e_{46} \in E(G)$, which implies that v_2 and v_6 are twins, therefore $e_{36} \notin E(G)$. For e_{26} and e_{46}, if $e_{26} \in E(G)$, $e_{46} \notin E(G)$, then v_1 and v_6 are twins; if $e_{26}, e_{46} \in E(G)$, then v_5 and v_6 are twins; if $e_{26} \notin E(G)$, $e_{46} \in E(G)$, then $\{v_1, v_6\}$ is a resolving set of G, and if $e_{26}, e_{46} \notin E(G)$, then $d(v_3, v_6) = 3$. Thus, there is no graph G with $\dim(G) = n(G) - 3$.

Case 2.1.4. $e_{15}, e_{25} \notin E(G)$. Since $d(v_1, v_5) \leq 2$, $e_{16}, e_{56} \in E(G)$. Then $\{v_1, v_5\}$ is a resolving set of G, which contradicts $\dim(G) = 3$.

Case 2.2. $e_{24} \notin E(G)$. Since $d(v_1, v_4) \leq 2$, we may assume that $e_{15}, e_{45} \in E(G)$. There are three subcases to be considered as follows.

Case 2.2.1. $e_{25} \in E(G), e_{35} \notin E(G)$ or $e_{25} \notin E(G), e_{35} \in E(G)$. Then the graph $G - v_6$ is isomorphic to that of case 2.1.1.

Case 2.2.2. $e_{25}, e_{35} \in E(G)$. Then the graph $G - v_6$ is isomorphic to that of case 2.1.3.

Case 2.2.3. $e_{25}, e_{35} \notin E(G)$. Since there are no twins in G and $\delta(G) \geq 2$, v_6 must be adjacent to two adjacent vertices. Assume that $e_{16}, e_{26} \in E(G)$, then $\{v_1, v_2\}$ is a resolving set of G, which contradicts $\dim(G) = 3$.

Thus, there is no graph G with $\dim(G) = n(G) - 3$ and we are done. □

Lemma 14. *Let G be a graph with $d(G) = 2$ and $\dim(G) = n(G) - 3$, then $n(G_T) \leq 5$.*

Proof. Suppose instead that G is a graph with $d(G) = 2$, $\dim(G) = n(G) - 3$ and $n(G_{\mathrm{T}}) \geq 6$. Now we prove that $\dim(G_{\mathrm{T}}) = n(G_{\mathrm{T}}) - 3$. Since $\dim(G) = n(G) - 3$, $\dim(G_{\mathrm{T}}) \geq n(G_{\mathrm{T}}) - 3$ by Lemma 1. If $\dim(G_{\mathrm{T}}) = n(G_{\mathrm{T}}) - 1$, then by Lemma 7, we have $G_{\mathrm{T}} = K_{n(G_{\mathrm{T}})}$. By Corollary 4, we get that $\dim(G) = n(G) - n(G_{\mathrm{T}})$, which is a contradiction. If $\dim(G_{\mathrm{T}}) = n(G_{\mathrm{T}}) - 2$, then $G_{\mathrm{T}} = K_{s,t}\,(s, t \geq 1)$ or $K_s + \overline{K_t}\,(s \geq 1, t \geq 2)$ or $K_s + (K_1 \cup K_t)\,(s, t \geq 1)$ by Lemma 8. Since $n(G_{\mathrm{T}}) \geq 6$, $s + t \geq 5$, which contradicts $\dim(G) = n(G) - 3$ by Lemma 5. Thus, $\dim(G_{\mathrm{T}}) = n(G_{\mathrm{T}}) - 3$.

For $n(G_{\mathrm{T}}) \geq 6$, by Lemma 13, there exist twins in G_{T}. Assume that v_1, v_2 are twins in G_{T} and $|V_1| \geq 2$, then v_1' resolves v_1 and v_2 by Lemma 2. Moreover, by Lemma 5, the size of every twin set of G_{T} is no more than 2. Thus, for each $v \in V(G_{\mathrm{T}})$ and $v \notin \{v_1, v_2\}$, v_1, v_2 and v are not twins. Let $W(G_{\mathrm{T}})$ be a metric basis of G_{T} and $V(G_{\mathrm{T}}) \setminus W(G_{\mathrm{T}}) = \{u_1, u_2, u_3\}$. By Lemma 8, there exists $W_0 = \{w_1, w_2\} \subset W(G_{\mathrm{T}})$ such that W_0 resolves $\{u_1, u_2, u_3\}$. Since v_1 and v_2 are twins in G_{T}, there is at most one of v_1 and v_2 in $\{u_1, u_2, u_3\}$. Similarly, there is at most one of v_1 and v_2 in W_0.

Then we prove that one of v_1 and v_2 is in $\{u_1, u_2, u_3\}$ and the other is in W_0. Assume that $v_1, v_2 \notin \{u_1, u_2, u_3\}$, then $v_1, v_2 \in W(G_{\mathrm{T}})$. If $v_1, v_2 \notin W_0$, then $r(v_1|W_0) = r(v_2|W_0) \in \{r(u_1|W_0), r(u_2|W_0), r(u_3|W_0)\}$. We may assume that $r(v_1|W_0) = r(v_2|W_0) = r(u_1|W_0)$, then $\{w_1, w_2, v_1'\}$ resolves $\{u_2, u_3, v_1, v_2\}$, which is a contradiction. Otherwise, we may assume that $v_1 = w_1$, then $\{v_1', w_2\}$ resolves $\{u_1, u_2, u_3\}$ and $\{v_1, v_2\}$. Without loss of generality suppose that $r(v_i|\{v_1', w_2\}) = r(u_i|\{v_1', w_2\})$ for $i \in \{1, 2\}$. Since the pairs u_1, v_1 and u_2, v_2 are not twins in G_{T}, then $\{v_1', w_2, u_3\}$ resolves $\{u_1, u_2, v_1, v_2\}$, the argument is similar to that of the case 1.1 of Lemma 13, which is a contradiction. Thus, there is one of v_1 and v_2 in $\{u_1, u_2, u_3\}$. We may assume that $v_1 \in \{u_1, u_2, u_3\}$, then $v_2 \in W_0$; otherwise, $\{v_1', w_1, w_2\}$ resolves $\{u_1, u_2, u_3, v_2\}$, which is a contradiction.

Thus, we have that at most two pairs of vertices are twins in G_{T} and $\dim(G_{\mathrm{T}} - v_1) = n(G_{\mathrm{T}} - v_1) - 2$. Moreover, for $x_1, x_2 \notin \{v_1, v_2\}$, if x_1 and x_2 are twins in $G_{\mathrm{T}} - v_1$, it easy to see that they are twins in G_{T}. Thus, there is at most one pair of vertices that are twins in $G_{\mathrm{T}} - v_1$. By Lemma 8, $G_{\mathrm{T}} - v_1 = K_{s,t}\,(s, t \geq 1)$ or $K_s + \overline{K_t}\,(s \geq 1, t \geq 2)$ or $K_s + (K_1 \cup K_t)\,(s, t \geq 1)$. Since $n(G_{\mathrm{T}}) \geq 6$, $n(G_{\mathrm{T}} - v_1) \geq 5$. Thus, $s + t \geq 4$, there are at most two pairs of vertices are twins in $G_{\mathrm{T}} - v_1$, which is a contradiction. Therefore, the assumption $n(G_{\mathrm{T}}) \geq 6$ does not hold and we are done. □

Theorem 1. *For a graph G, $\dim(G) = n(G) - 3$ and $d(G) = 2$ if and only if G is $(K_s \cup B_r) + K_t\ (s, r \geq 2,\ t \geq 1)$, $(K_s \cup B_r) + \overline{K_t}\ (s, t \geq 2,\ r \geq 1)$, $G = (\overline{K_s} + \overline{K_t}) + B_r\ (s, t \geq 2,\ r \geq 1)$, C_5 or one of the graphs in Figures 2 and 3.*

Proof. It holds by Lemmas 10, 11, 12 and 14. □

Remark 1. *This method can help us to address the extension problem of a given graph with respect to metric dimension. It is theoretically realized the characterization of extremal graphs with $\dim(G) = n(G) - r$ for any*

$r > 0$. In addition, we also find that the problem will become more and more difficult with the increase of r based on the proof of the case $r = 3$.

4. Conclusions

In this paper, by constructing the metric matrix of G, we make a necessary and sufficient condition of $\dim(G) = n(G) - r$ and characterize the graphs of $\dim(G) = n(G) - 3$ via this condition. Moreover, we give a new idea for the extension of graphs based on metric dimension.

Author Contributions: J.W. designed the problem and wrote the paper. L.M. and Y.L. validate the results.

Funding: This research was funded by National Natural Science Foundation of China grant number 11771443 and Shandong Province Natural Science Foundation grant number ZR2017QF011.

Acknowledgments: The authors would like to thank the referees for careful reading of the manuscript and for valuable comments and thoughtful suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Slater, P.J. Leaves of trees. *Congr. Numer.* **1975**, *14*, 549–559.
2. Harary, F.; Melter, R.A. On the metric dimension of a graph. *Ars Combin.* **1976**, *2*, 191–195.
3. Liu, J.B.; Kashif, A.; Rashid, T.; Javaid, M. Fractional metric dimension of generalized jahangir graph. *Mathematics* **2019**, *7*, 100. [CrossRef]
4. Barrágan-Ramírez, G.A.; Rodríguez-Velázquez, J.A. The local metric dimension of strong product graphs. *Graphs Combin.* **2016**, *32*, 1263–1278. [CrossRef]
5. DasGupta, B.; Mobasheri, N. On optimal approximability results for computing the strong metric dimension. *Discrete Appl. Math.* **2017**, *221*, 18–24. [CrossRef]
6. Estrada-Moreno, A.; Yero, I.G.; Rodríguez-Velázquez, J.A. The *k*-metric dimension of the lexicographic product of graphs. *Discrete Math.* **2016**, *339*, 1924–1934. [CrossRef]
7. Liu, J.B.; Zahid, Z.; Nasir, R.; Nazeer, W. Edge version of metric dimension and doubly resolving sets of the necklace graph. *Mathematics* **2018**, *6*, 243. [CrossRef]
8. Kelenc, A.; Kuziak, D.; Taranenko, A.; Yero, I.G. Mixed metric dimension of graphs. *Appl. Math. Comput.* **2017**, *314*, 429–438.
9. Ma, X.; Feng, M.; Wang, K. The strong metric dimension of the power graph of a finite group. *Discrete Appl. Math.* **2018**, *239*, 159–164. [CrossRef]
10. Ramírez-Cruz, Y.; Estrada-Moreno, A.; Rodríguez-Velázquez, J.A. The simultaneous metric dimension of families composed by lexicographic product graphs. *Graphs Combin.* **2016**, *32*, 2093–2120. [CrossRef]
11. Raza, H.; Hayat, S.; Imran, M.; Pan, X. Fault-tolerant resolvability and extremal structures of graphs. *Mathematics* **2019**, *7*, 78. [CrossRef]
12. Khuller, S.; Raghavachari, B.; Rosenfeld, A. Landmarks in graphs. *Discrete Appl. Math.* **1996**, *70*, 217–229. [CrossRef]
13. Sebő, A.; Tannier, E. On metric generators of graphs. *Math. Oper. Res.* **2004**, *29*, 383–393. [CrossRef]
14. Chartrand, G.; Eroh, L.; Johnson, M.A.; Oellermannn, O.R. Resolvability in graphs and the metric dimension of a graph. *Discrete Appl. Math.* **2000**, *105*, 99–113. [CrossRef]
15. Shanmukha, B.; Sooryanarayana, B.; Harinath, K.S. Metric dimension of wheels. *Far East J. Appl. Math.* **2002**, *8*, 217–229.
16. Cáceres J.; Hernando, C.; Mora, M.; Pelayo, I.M.; Puertas, M.L.; Seara, C.; Wood, D.R. On the metric dimension of cartesian products of graphs. *SIAM J. Discrete Math.* **2007**, *21*, 423–441. [CrossRef]
17. Yero, I.G.; Kuziak, D.; Rodríguez-Velázquez, J.A. On the metric dimension of corona product graphs. *Comput. Math. Appl.* **2011**, *61*, 2793–2798. [CrossRef]
18. Eroh, L.; Feit, P.; Kang, C.X.; Yi, E. The effect of vertex or edge deletion on the metric dimension of graphs. *J. Combin.* **2015**, *6*, 433–444. [CrossRef]
19. Imran, S.; Siddiqui, M.K.; Imran, M.; Hussain, M. On metric dimensions of symmetric graphs obtained by rooted product. *Mathematics* **2018**, *6*, 191. [CrossRef]

20. Jannesari, M.; Omoomi, B. Characterization of n-vertex graphs with metric dimension $n-3$. *Math. Bohem.* **2014**, *139*, 1–23.
21. Hernando, C.; Mora, M.; Pelayo, I.M.; Seara, C.; Wood, D.R. Extremal graph theory for metric dimension and diameter. *Electron. J. Combin.* **2010**, *17*, 30. [CrossRef]
22. Chartrand, G.; Poisson, C.; Zhang, P. Resolvability and the upper dimension of graphs. *Comput. Math. Appl.* **2000**, *39*, 19–28. [CrossRef]
23. Garijo, D.; González, A.; Márquez, A. On the metric dimension, the upper dimension and the resolving number of graphs. *Discrete Appl. Math.* **2013**, *161*, 1440–1447. [CrossRef]
24. Imran, M.; Bokhary, S.A.H.; Ahmad, A.; Semaničová-Feňovčiková, A. On classes of regular graphs with constant metric dimension. *Acta Math. Sci.* **2013**, *33*, 187–206. [CrossRef]
25. Laihonen, T. The metric dimension for resolving several objects. *Inf. Process. Lett.* **2016**, *116*, 694–700. [CrossRef]