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Some Incomplete Hypergeometric Functions and Incomplete Riemann-Liouville Fractional Integral Operators

Mehmet Ali Özarslan ¹ and Ceren Ustaoglu ^{2,*}

¹ Department of Mathematics, Faculty of Arts and Sciences, Eastern Mediterranean University, Famagusta, TRNC, Mersin 10, Turkey; mehmetali.ozarslan@emu.edu.tr

² Department of Computer Engineering, Final International University, Toroslar Caddesi, No. 6, Çatalköy, Girne, TRNC, Mersin 10, Turkey

* Correspondence: ceren.ustaoglu@final.edu.tr

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Abstract: Very recently, the incomplete Pochhammer ratios were defined in terms of the incomplete beta function $B_y(x, z)$. With the help of these incomplete Pochhammer ratios, we introduce new incomplete Gauss, confluent hypergeometric, and Appell's functions and investigate several properties of them such as integral representations, derivative formulas, transformation formulas, and recurrence relations. Furthermore, incomplete Riemann-Liouville fractional integral operators are introduced. This definition helps us to obtain linear and bilinear generating relations for the new incomplete Gauss hypergeometric functions.

Keywords: Gauss hypergeometric function; confluent hypergeometric function; Appell's functions; incomplete fractional calculus; Riemann-Liouville fractional integral; generating functions

1. Introduction and Preliminaries

In recent years, some extensions of the well-known special functions have been considered by several authors (see, for example, [1–9]). The familiar incomplete gamma functions $\gamma(s, x)$ and $\Gamma(s, x)$ are defined by:

$$\gamma(s, x) := \int_0^x t^{s-1} e^{-t} dt \quad (\operatorname{Re}(s) > 0; x \geq 0)$$

and

$$\Gamma(s, x) := \int_x^\infty t^{s-1} e^{-t} dt \quad (x \geq 0; \operatorname{Re}(s) > 0 \text{ when } x = 0),$$

respectively. They satisfy the following decomposition formula:

$$\gamma(s, x) + \Gamma(s, x) = \Gamma(s) \quad (\operatorname{Re}(s) > 0). \quad (1)$$

The function $\Gamma(s)$ and its incomplete versions $\gamma(s, x)$ and $\Gamma(s, x)$ play important roles in the study of analytical solutions of a variety of problems in diverse areas of science and engineering.

The widely-used Pochhammer symbol $(\lambda)_\nu$ ($\lambda, \nu \in \mathbb{C}$) is defined, in general, by:

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \dots (\lambda + \nu - 1) & (\nu \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases} \quad (2)$$

In terms of the incomplete gamma functions $\gamma(s, x)$ and $\Gamma(s, x)$, the incomplete Pochhammer symbols $(\lambda; x)_\nu$ and $[\lambda; x]_\nu$ ($\lambda, \nu \in \mathbb{C}; x \geq 0$) were defined as follows [10]:

$$(\lambda; x)_\nu := \frac{\gamma(\lambda + \nu, x)}{\Gamma(\lambda)} \quad (\lambda, \nu \in \mathbb{C}; x \geq 0) \tag{3}$$

and:

$$[\lambda; x]_\nu := \frac{\Gamma(\lambda + \nu, x)}{\Gamma(\lambda)} \quad (\lambda, \nu \in \mathbb{C}; x \geq 0). \tag{4}$$

In view of (1), these incomplete Pochhammer symbols $(\lambda; x)_\nu$ and $[\lambda; x]_\nu$ satisfy the following decomposition relation:

$$(\lambda; x)_\nu + [\lambda; x]_\nu = (\lambda)_\nu \quad (\lambda, \nu \in \mathbb{C}; x \geq 0), \tag{5}$$

where $(\lambda)_\nu$ is the Pochhammer symbol given by (2).

The incomplete Gauss hypergeometric functions were defined by means of the incomplete gamma functions as follows [10]:

$${}_2\gamma_1 \left[\begin{matrix} (a, x) & \cdot & b & ; & z \\ & & c & ; & \end{matrix} \right] := \sum_{n=0}^{\infty} \frac{(a; x)_n (b)_n z^n}{(c)_n n!} \tag{6}$$

and:

$${}_2\Gamma_1 \left[\begin{matrix} (a, x) & \cdot & b & ; & z \\ & & c & ; & \end{matrix} \right] := \sum_{n=0}^{\infty} \frac{[a; x]_n (b)_n z^n}{(c)_n n!}. \tag{7}$$

After this work, incomplete hypergeometric functions have become a fruitful topic of research in recent years [4,5,9,11–20].

Fractional derivative and integral operators are another important topic of research in recent years. They have found applications in many diverse areas of mathematical, physical, and engineering problems; good summaries of these applications may be found in [21–26] and recently in [27]. The use of fractional derivative operators in obtaining generating relations for some special functions can be found in [6,9,28–30].

In fractional calculus, there are two important differential operators: the Riemann-Liouville and Liouville-Caputo fractional derivatives. In a recent paper [12], which covered work done after the work herein, we introduced incomplete Liouville-Caputo fractional derivative operators and focused on their use in special function theory. For the definitions in [12], we considered the same incomplete Riemann-Liouville integral as in (60) and (61) of this paper, but the operators introduced there were of the Liouville-Caputo type and not of the Riemann-Liouville type like those in the current work. The difference between Liouville-Caputo and Riemann-Liouville is very important for applications to differential equations, because the required initial conditions are of different types between these two cases.

In the present paper, we introduce new incomplete hypergeometric functions with the aid of incomplete Pochhammer ratios and investigate certain properties of them. Moreover, we introduce incomplete Riemann-Liouville fractional integral operators, and we obtain some generating relations for these new incomplete hypergeometric functions with the aid of these new defined operators. The organization of the paper is as follows.

In Section 2, the incomplete Pochhammer ratios are introduced by using the incomplete beta function, and some derivative formulas involving these new incomplete Pochhammer ratios are investigated. In Section 3, new incomplete Gauss hypergeometric functions and confluent hypergeometric functions are introduced with the help of these incomplete Pochhammer ratios, and integral representations, derivative formulas, transformation formulas, and recurrence relations are obtained for them. In Section 4, we define new incomplete Appell’s functions $\mathfrak{F}_1[a, b, c; d; x, z; y]$, $\mathfrak{F}_1\{a, b, c; d; x, z; y\}$, $\mathfrak{F}_2[a, b, c; d, e; x, z; y]$, and

$\mathfrak{F}_2\{a, b, c; d, e; x, z; y\}$ and obtain their integral representations. In Section 5, we introduce incomplete Riemann-Liouville fractional integral operators and show that the incomplete Riemann-Liouville fractional integrals of some elementary functions give the new incomplete functions defined in Sections 3 and 4. Finally, in the last section, we obtain linear and bilinear generating relations for the incomplete hypergeometric functions.

2. The Incomplete Pochhammer Ratio

The incomplete beta function is defined by:

$$B_y(x, z) := \int_0^y t^{x-1}(1-t)^{z-1} dt, \operatorname{Re}(x) > \operatorname{Re}(z) > 0, 0 \leq y < 1 \tag{8}$$

and can be expressed in terms of the Gauss hypergeometric function:

$$B_y(x, z) := \frac{y^x}{x} {}_2F_1(x, 1-z; 1+x; y). \tag{9}$$

The incomplete Pochhammer ratios $[b, c; y]_n$ and $\{b, c; y\}_n$ are introduced in terms of the incomplete beta function $B_y(x, z)$ as follows [12]:

$$[b, c; y]_n := \frac{B_y(b+n, c-b)}{B(b, c-b)} \tag{10}$$

and:

$$\{b, c; y\}_n := \frac{B_{1-y}(c-b, b+n)}{B(b, c-b)} \tag{11}$$

where $0 \leq y < 1$. They satisfy the following relation:

$$[b, c; y]_n + \{b, c; y\}_n = \frac{(b)_n}{(c)_n}. \tag{12}$$

In view of (9), we have the following relations:

$$[b, c; y]_n := \frac{1}{B(b, c-b)} \frac{y^{b+n}}{b+n} {}_2F_1(b+n, 1-c+b; b+n+1; y) \tag{13}$$

and:

$$\{b, c; y\}_n := \frac{1}{B(b, c-b)} \frac{(1-y)^{c-b}}{c-b} {}_2F_1(c-b, 1-b-n; 1+c-b; 1-y). \tag{14}$$

In the following theorem, we investigate the n^{th} derivatives of the incomplete beta function by means of incomplete Pochhammer ratios.

Theorem 1. *The following derivative formulas hold true:*

$$[b, c; y]_n = \frac{(-1)^n \Gamma(c)}{\Gamma(c-b+n) \Gamma(b)} y^{b+n} \frac{d^n}{dy^n} \left[y^{-b} B_y(b, c-b+n) \right], \tag{15}$$

and:

$$\{b, c; y\}_n = \frac{\Gamma(b+n)}{\Gamma(b+2n)} \frac{1}{B(b, c-b)} (1-y)^{c-b} \frac{d^n}{dy^n} \left((1-y)^{-c+b+n} B_{1-y}(c-b-n, b+2n) \right). \tag{16}$$

Proof. Using (8) and (10), we immediately obtain the following equation:

$$[b, c; y]_n = \frac{y^{b+n}}{B(b, c-b)} \int_0^1 u^{b+n-1} (1-uy)^{c-b-1} du.$$

On the other hand, we have:

$$y^{-b} B_y(b, c-b+n) = \int_0^1 u^{b-1} (1-uy)^{c-b+n-1} du. \tag{17}$$

Taking derivatives n times on both sides of (17) with respect to y , we can obtain a derivative formula for the incomplete beta function $[b, c; y]_n$ asserted by (15). Formula (16) can be proven in a similar way. \square

3. The New Incomplete Gauss and Confluent Hypergeometric Functions

In this section, we introduce new incomplete Gauss and confluent hypergeometric functions by:

$${}_2\tilde{\mathfrak{F}}_1(a, [b, c; y]; x) := \sum_{n=0}^{\infty} (a)_n [b, c; y]_n \frac{x^n}{n!}, \tag{18}$$

$${}_2\tilde{\mathfrak{F}}_1(a, \{b, c; y\}; x) := \sum_{n=0}^{\infty} (a)_n \{b, c; y\}_n \frac{x^n}{n!}, \tag{19}$$

$${}_1\tilde{\mathfrak{F}}_1([a, b; y]; x) := \sum_{n=0}^{\infty} [a, b; y]_n \frac{x^n}{n!}, \tag{20}$$

and:

$${}_1\tilde{\mathfrak{F}}_1(\{a, b; y\}; x) := \sum_{n=0}^{\infty} \{a, b; y\}_n \frac{x^n}{n!} \tag{21}$$

where $0 \leq y < 1$.

An immediate consequence of (12) and the definitions (18), (19), (20), and (21) is the following decomposition formulas:

$${}_2\tilde{\mathfrak{F}}_1(a, [b, c; y]; x) + {}_2\tilde{\mathfrak{F}}_1(a, \{b, c; y\}; x) = {}_2F_1(a, b; c; x) \tag{22}$$

and:

$${}_1\tilde{\mathfrak{F}}_1([a, b; y]; x) + {}_1\tilde{\mathfrak{F}}_1(\{a, b; y\}; x) = {}_1F_1(a; b; x). \tag{23}$$

Theorem 2. *The following integral representation holds true:*

$${}_2\tilde{\mathfrak{F}}_1(a, [b, c; y], x) = \frac{y^b}{B(b, c-b)} \int_0^1 u^{b-1} (1-uy)^{c-b-1} (1-xuy)^{-a} du, \tag{24}$$

$$Re(c) > Re(b) > 0, \quad |\arg(1-x)| < \pi.$$

Proof. Replacing the incomplete Pochhammer ratio $[b, c; y]_n$ in the definition (18) by its integral representation given by (8) and interchanging the order of summation and integral, which is permissible under the conditions given in the hypothesis of the Theorem, we find:

$${}_2\tilde{\mathfrak{F}}_1(a, [b, c; y], x) = \frac{1}{B(b, c-b)} \int_0^y t^{b-1} (1-t)^{c-b-1} (1-xt)^{-a} dt, \tag{25}$$

which can be written as follows:

$${}_2\mathfrak{F}_1(a, [b, c; y], x) = \frac{y^b}{B(b, c - b)} \int_0^1 u^{b-1} (1 - uy)^{c-b-1} (1 - xuy)^{-a} du. \tag{26}$$

□

In a similar way, we have the following theorem:

Theorem 3. *The following integral representation holds true:*

$$\begin{aligned} {}_2\mathfrak{F}_1(a, \{b, c; y\}, x) &= \frac{(1 - y)^{c-b}}{B(b, c - b)} \int_0^1 u^{c-b-1} (1 - u(1 - y))^{b-1} (1 - x + xu(1 - y))^{-a} du, \\ \operatorname{Re}(c) > \operatorname{Re}(b) > 0, |\arg(1 - x)| < \pi. \end{aligned} \tag{27}$$

Theorem 4. *The following result holds true:*

$${}_2\mathfrak{F}_1(a, [b, c; y], 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} - \frac{(1 - y)^{c-b-a}y^b}{B(b, c - b)(c - a - b)} {}_2F_1(c - a, 1; 1 + c - b - a; 1 - y). \tag{28}$$

Proof. Putting $x = 1$ in (22), we obtain:

$$\begin{aligned} {}_2\mathfrak{F}_1(a, [b, c; y], 1) &= {}_2F_1(a, b; c; 1) - {}_2\mathfrak{F}_1(a, \{b, c; 1 - y\}, 1) \\ &= \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} - \frac{(1 - y)^{c-b-a}}{B(b, c - b)} \int_0^1 u^{c-b-a-1} (1 - u(1 - y))^{b-1} du. \end{aligned} \tag{29}$$

Using Euler’s integral representation for (29), we have:

$${}_2\mathfrak{F}_1(a, [b, c; y], 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} - \frac{(1 - y)^{c-b-a}}{B(b, c - b)(c - b - a)} {}_2F_1(1 - b, c - b - a; 1 + c - b - a; 1 - y). \tag{30}$$

Using transformation formula:

$${}_2F_1(\alpha, \beta; \gamma; z) = (1 - z)^{\gamma - \beta - \alpha} {}_2F_1(\gamma - \alpha, \gamma - \beta; \gamma; z), \tag{31}$$

in (30), we obtain:

$${}_2F_1(1 - b, c - b - a; 1 + c - b - a; 1 - y) = y^b {}_2F_1(c - a, 1; 1 + c - b - a; 1 - y). \tag{32}$$

Considering (32) in (30), we get:

$${}_2\mathfrak{F}_1(a, [b, c; y], 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} - \frac{(1 - y)^{c-b-a}y^b}{B(b, c - b)(c - b - a)} {}_2F_1(c - a, 1; 1 + c - b - a; 1 - y). \tag{33}$$

□

Theorem 5. *The following result holds true:*

$${}_2\mathfrak{F}_1(a, \{b, c; y\}, 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} - \frac{(1 - y)^{c-b-a}y^b}{B(b, c - b)b} {}_2F_1(c - a, 1; b + 1; y). \tag{34}$$

Theorem 6. The following integral representations hold true:

$${}_1\mathfrak{F}_1([a, b; y], x) = \frac{y^a}{B(a, b - a)} \int_0^1 u^{a-1} (1 - uy)^{b-a-1} e^{xuy} du, \quad \operatorname{Re}(b) > \operatorname{Re}(a) > 0 \quad (35)$$

and:

$${}_1\mathfrak{F}_1(\{a, b; y\}, x) = \frac{(1 - y)^{b-a}}{B(a, b - a)} \int_0^1 u^{b-a-1} (1 - u(1 - y))^{a-1} e^{(1-u(1-y))x} du, \quad \operatorname{Re}(b) > \operatorname{Re}(a) > 0. \quad (36)$$

Proof. Replacing the incomplete Pochhammer ratio $[a, b; y]_n$ in the definition (20) by its integral representation given by (8), we are led to the desired result (35). Formula (36) can be proven in a similar way. \square

Theorem 7. The following integral representation holds true:

$$\int_0^1 y^{k-1} {}_2\mathfrak{F}_1(a, [b, c - k; y]; x) dy = \frac{1}{k} \left[{}_2F_1(a, b; c - k; x) - \frac{\Gamma(c - k)\Gamma(b + k)}{\Gamma(b)\Gamma(c)} {}_2F_1(a, b + k; c; x) \right], \quad k \in \mathbb{N}. \quad (37)$$

Proof. It is known that from Euler’s formula that:

$${}_2F_1(a, b + k; c; x) = \frac{1}{B(b + k, c - b - k)} \int_0^1 y^{b+k-1} (1 - y)^{c-b-k-1} (1 - xy)^{-a} dy, \quad k \in \mathbb{N}.$$

Taking $u = y^k$ and the remaining part as dv and applying the integration by parts, we get:

$${}_2F_1(a, b + k; c; x) = \frac{\Gamma(b)\Gamma(c)}{\Gamma(c - k)\Gamma(b + k)} \left[{}_2F_1(a, b; c - k; x) - k \int_0^1 y^{k-1} {}_2\mathfrak{F}_1(a, [b, c - k; y], x) dy \right].$$

By rearranging the terms, we get the result. \square

Corollary 1. Taking $k = 1$ in Theorem 7, we get the following result:

$$\int_0^1 {}_2\mathfrak{F}_1(a, [b, c - 1; y], x) dy = {}_2F_1(a, b; c - 1; x) - \frac{b}{c - 1} {}_2F_1(a, b + 1; c; x). \quad (38)$$

Theorem 8. The following integral representation holds true:

$$\int_0^1 y^{k-1} {}_2\mathfrak{F}_1(a, [b, c; y], x) dy = \frac{1}{k} \frac{\Gamma(c)\Gamma(c - b + k)}{\Gamma(c - b)\Gamma(c + k)} {}_2F_1(a, b; c + k; x). \quad (39)$$

Proof. It is known that:

$${}_2F_1(a, b; c + k; x) = \frac{1}{B(b, c - b + k)} \int_0^1 y^{b-1} (1 - y)^{c-b+k-1} (1 - xy)^{-a} dy.$$

Taking $u = (1 - y)^k$ and the rest as dv and using integration by parts, we get the result. \square

Corollary 2. Taking $k = 1$ in Theorem 9, we get the following result:

$${}_2F_1(a, b; c + 1; x) = \frac{c}{c - b} \int_0^1 {}_2\mathfrak{F}_1(a, [b, c; y], x) dy. \quad (40)$$

Theorem 9. The following derivative formula holds true:

$$\frac{d^n}{dx^n} ({}_2\mathfrak{F}_1(a, [b, c; y]; x)) = \frac{(a)_n (b)_n}{(c)_n} {}_2\mathfrak{F}_1(a + n, [b + n, c + n; y]; x). \tag{41}$$

Proof. Using (25), differentiating on both sides with respect to x , we obtain:

$$\begin{aligned} \frac{d}{dx} ({}_2\mathfrak{F}_1(a, [b, c; y]; x)) &= \frac{a}{B(b, c - b)} \int_0^y t^b (1 - t)^{c-b-1} (1 - xt)^{-a-1} dt \\ &= \frac{a}{B(b, c - b)} \int_0^y t^{(b+1)-1} (1 - t)^{(c+1)-(b+1)-1} (1 - xt)^{-(a+1)} dt \\ &= \frac{ab}{c} \frac{1}{B(b + 1, c - b)} \int_0^y t^{(b+1)-1} (1 - t)^{(c+1)-(b+1)-1} (1 - xt)^{-(a+1)} dt \\ &= \frac{ab}{c} {}_2\mathfrak{F}_1(a + 1, [b + 1, c + 1; y]; x \end{aligned}$$

which is (41) for $n = 1$. The general result follows by the principle of mathematical induction on n . \square

Theorem 10. The following derivative formula holds true:

$$\frac{d^n}{dx^n} ({}_1\mathfrak{F}_1([a, b; y]; x)) = \frac{(a)_n}{(b)_n} {}_1\mathfrak{F}_1([a + n, b + n; y]; x). \tag{42}$$

Theorem 11. We have the following difference formula for ${}_2\mathfrak{F}_1(a, [b, b + h; y]; x)$:

$$\begin{aligned} \frac{b + h - 1}{B(b, h)} y^{b-1} (1 - y)^{h-1} (1 - xy)^{-a} &= {}_2\mathfrak{F}_1(a, [b, b + h - 1; y]; x) + \\ {}_2\mathfrak{F}_1(a, [b - 1, b + h - 1; y]; x) - ax(b + h - 1) &{}_2\mathfrak{F}_1(a + 1, [b, b + h; y]; x). \end{aligned} \tag{43}$$

Proof. Recalling that the Mellin transform operator is defined by:

$$\mathfrak{M} \{f(t) : s\} := \int_0^\infty t^{s-1} f(t) dt, \text{ Re}(s) > 0,$$

we observe that ${}_2F_1(a, [b, b + h; y]; x)$ is the Mellin transform of the function:

$$f(t : x; y, a; h) = H(y - t)(1 - t)^{h-1} (1 - xt)^{-a},$$

where:

$$H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases},$$

is the Heaviside unit function. Observing the fact that:

$${}_2\mathfrak{F}_1(a, [b, b + h; y]; x) := \frac{\mathfrak{M} \{f(t : x; y, a; h) : b\}}{B(b, h)}, \tag{44}$$

we can write that:

$$\begin{aligned} \frac{\partial}{\partial t} (f(t : x; y, a; h)) &= -[(y - t)(1 - t)^{h-1} (1 - xt)^{-a} + (h - 1)H(y - t)(1 - t)^{h-2} (1 - xt)^{-a}] \\ &\quad + ax(1 - xt)^{-a-1} H(y - t)(1 - t)^{h-1}, \end{aligned} \tag{45}$$

where $\frac{\partial}{\partial t}(H(t)) = \delta(t - t_0)$,

$$\delta(t - t_0) = \begin{cases} \infty & \text{if } t = t_0 \\ 0 & \text{if } t \neq t_0 \end{cases} ,$$

is the Dirac delta function. Applying the Mellin transform on both sides (45) and using (44) and the fact that:

$$\mathfrak{M} \{f'(t) : x\} = (1 - x)\mathfrak{M} \{f(t) : x - 1\} ,$$

we have:

$$\begin{aligned} \frac{b + h - 1}{B(b, h)} y^{b-1} (1 - y)^{h-1} (1 - xy)^{-a} &= {}_2\mathfrak{F}_1(a, [b, b + h - 1; y]; x) \\ + {}_2\mathfrak{F}_1(a, [b - 1, b + h - 1; y]; x) - ax(b + h - 1) {}_2\mathfrak{F}_1(a + 1, [b, b + h; y]; x). \end{aligned}$$

This completes the proof. \square

In the following theorems, we give transformation formulas:

Theorem 12. *The following transformation formula holds true:*

$${}_2\mathfrak{F}_1(a, [\beta, \gamma; y]; z) = (1 - z)^{-a} {}_2\mathfrak{F}_1(a, \{\gamma - \beta, \gamma; 1 - y\}; \frac{z}{z - 1}), \quad |\arg(1 - z)| < \pi. \tag{46}$$

Proof. Using (25), we obtain:

$${}_2\mathfrak{F}_1(a, [\beta, \gamma; y]; z) = \frac{(1 - z)^{-a}}{B(\beta, \gamma - \beta)} \int_{1-y}^1 (1 - s)^{\beta-1} s^{\gamma-\beta-1} \left(1 - \frac{z}{z - 1}s\right)^{-a} ds. \tag{47}$$

The substitution $s = 1 - t$ in (47) leads to:

$$\begin{aligned} {}_2\mathfrak{F}_1(a, [\beta, \gamma; y]; z) &= \frac{(1 - z)^{-a}}{B(\beta, \gamma - \beta)} \int_0^y t^{\beta-1} (1 - t)^{\gamma-\beta-1} \left(1 - \frac{z(1 - t)}{z - 1}\right)^{-a} dt \\ &= (1 - z)^{-a} {}_2\mathfrak{F}_1(a, \{\gamma - \beta, \gamma; 1 - y\}; \frac{z}{z - 1}). \end{aligned}$$

\square

Theorem 13. *The following transformation formula holds true:*

$${}_2\mathfrak{F}_1(a, \{\beta, \gamma; y\}; z) = (1 - z)^{-a} {}_2\mathfrak{F}_1(a, [\gamma - \beta, \gamma; 1 - y]; \frac{z}{z - 1}), \quad |\arg(1 - z)| < \pi. \tag{48}$$

Theorem 14. *The following transformation formulas hold true:*

$${}_1\mathfrak{F}_1(\{\alpha, \beta; 1 - y\}; z) = e^z {}_1\mathfrak{F}_1([\beta - \alpha, \beta; y]; -z) \tag{49}$$

and:

$${}_1\mathfrak{F}_1([\alpha, \beta; y]; z) = e^z {}_1\mathfrak{F}_1(\{\beta - \alpha, \beta; 1 - y\}; -z). \tag{50}$$

Proof. The proofs of (49) and (50) are direct consequences of Theorem 6. \square

4. The Incomplete Appell’s Functions

In this section, we introduce the incomplete Appell’s functions $\mathfrak{F}_1[a, b, c; d; x, z; y]$, $\mathfrak{F}_1\{a, b, c; d; x, z; y\}$, $\mathfrak{F}_2[a, b, c; d, e; x, z; y]$, and $\mathfrak{F}_2\{a, b, c; d, e; x, z; y\}$ by:

$$\mathfrak{F}_1[a, b, c; d; x, z; y] := \sum_{m,n=0}^{\infty} [a, d; y]_{m+n} (b)_m (c)_n \frac{x^m z^n}{m! n!}, \quad \max\{|x|, |z|\} < 1 \tag{51}$$

and:

$$\mathfrak{F}_1\{a, b, c; d; x, z; y\} := \sum_{m,n=0}^{\infty} \{a, d; y\}_{m+n} (b)_m (c)_n \frac{x^m z^n}{m! n!}, \quad \max\{|x|, |z|\} < 1 \tag{52}$$

and:

$$\mathfrak{F}_2[a, b, c; d, e; x, z; y] := \sum_{m,n=0}^{\infty} (a)_{m+n} [b, d; y]_m [c, e; y]_n \frac{x^m z^n}{m! n!}, \quad |x| + |z| < 1 \tag{53}$$

and:

$$\mathfrak{F}_2\{a, b, c; d, e; x, z; y\} := \sum_{m,n=0}^{\infty} (a)_{m+n} \{b, d; y\}_m \{c, e; y\}_n \frac{x^m z^n}{m! n!}, \quad |x| + |z| < 1. \tag{54}$$

Remark 1. For the reader’s convenience, we show how the convergence domains are obtained for the functions defined in (51)–(54). We just give the proof of (51). The other three definitions can be proven in a similar manner. Considering the absolute value:

$$\begin{aligned} \left| \mathfrak{F}_1[a, b, c; d; x, z; y] \right| &\leq \sum_{m,n=0}^{\infty} \left| [a, d; y]_{m+n} (b)_m (c)_n \frac{x^m z^n}{m! n!} \right| \\ &\leq \sum_{m,n=0}^{\infty} \left| [a, d; y]_{m+n} \right| \left| (b)_m (c)_n \frac{x^m z^n}{m! n!} \right| \\ &= \sum_{m,n=0}^{\infty} \left| \frac{1}{B(a, d - a)} \int_0^y t^{a+m+n-1} (1 - t)^{d-a-1} dt \right| \left| (b)_m (c)_n \frac{x^m z^n}{m! n!} \right| \\ &\leq \sum_{m,n=0}^{\infty} \frac{1}{B(a, d - a)} \int_0^1 \left| t^{a+m+n-1} (1 - t)^{d-a-1} dt \right| \left| (b)_m (c)_n \frac{x^m z^n}{m! n!} \right| \\ &= \sum_{m,n=0}^{\infty} \frac{B(a + m + n, d - a)}{B(a, d - a)} \left| (b)_m (c)_n \frac{x^m z^n}{m! n!} \right|, \end{aligned}$$

where the final series is the one corresponding to absolute convergence of the series for $F_1(a, b, c; d; x, z)$. Therefore, the series for $\mathfrak{F}_1[a, b, c; d; x, z; y]$ is absolutely convergent under the same conditions as the one for $F_1(a, b, c; d; x, z)$.

We proceed by obtaining the integral representations of the functions $\mathfrak{F}_1[a, b, c; d; x, z; y]$, $\mathfrak{F}_1\{a, b, c; d; x, z; y\}$, $\mathfrak{F}_2[a, b, c; d, e; x, z; y]$, and $\mathfrak{F}_2\{a, b, c; d, e; x, z; y\}$.

Theorem 15. For the incomplete Appell’s functions $\mathfrak{F}_1[a, b, c; d; x, z; y]$ and $\mathfrak{F}_1\{a, b, c; d; x, z; y\}$, we have the following integral representation:

$$\begin{aligned} \mathfrak{F}_1[a, b, c; d; x, z; y] &= \frac{y^a}{B(a, d - a)} \int_0^1 u^{a-1} (1 - uy)^{d-a-1} (1 - xuy)^{-b} (1 - zuy)^{-c} du, \tag{55} \\ \operatorname{Re}(d) &> 0, \operatorname{Re}(a) > 0, \operatorname{Re}(b) > 0, \operatorname{Re}(c) > 0, |\arg(1 - x)| < \pi, |\arg(1 - z)| < \pi. \end{aligned}$$

and:

$$\begin{aligned} \mathfrak{F}_1\{a, b, c; d; x, z; y\} &= \frac{(1-y)^{d-a}}{B(a, d-a)} \times \int_0^1 u^{d-a-1} (1-u(1-y))^{a-1} \\ &\quad (1-x(1-u(1-y)))^{-b} (1-z(1-u(1-y)))^{-c} du, \operatorname{Re}(d) > 0, \\ \operatorname{Re}(a) > 0, \operatorname{Re}(b) > 0, \operatorname{Re}(c) > 0, |\arg(1-x)| < \pi, |\arg(1-z)| < \pi. \end{aligned} \tag{56}$$

Proof. Replacing the integral representation for the incomplete beta function, which is given by (8), we find that:

$$\mathfrak{F}_1[a, b, c; d; x, z; y] = \frac{1}{B(a, d-a)} \int_0^y t^{a-1} (1-t)^{d-a-1} (1-xt)^{-b} (1-zt)^{-c} dt,$$

which can be written as:

$$\mathfrak{F}_1[a, b, c; d; x, z; y] = \frac{y^a}{B(a, d-a)} \int_0^1 u^{a-1} (1-uy)^{d-a-1} (1-xuy)^{-b} (1-zuy)^{-c} du;$$

whence the result. Formula (56) can be proven in a similar way. \square

Theorem 16. For the incomplete Appell’s functions $\mathfrak{F}_2[a, b, c; d, e; x, z; y]$ and $\mathfrak{F}_2\{a, b, c; d, e; x, z; y\}$, we have the following integral representation:

$$\begin{aligned} \mathfrak{F}_2[a, b, c; d, e; x, z; y] &= \frac{y^{b+c}}{B(b, d-b)B(c, e-c)} \\ &\quad \times \int_0^1 \int_0^1 u^{b-1} (1-uy)^{d-b-1} v^{c-1} (1-vy)^{e-c-1} (1-xuy-zvy)^{-a} dudv, \\ \operatorname{Re}(d) > \operatorname{Re}(a) > \operatorname{Re}(b) > \operatorname{Re}(c) > \operatorname{Re}(m) > 0, |\arg(1-x-z)| < \pi. \end{aligned} \tag{57}$$

and:

$$\begin{aligned} \mathfrak{F}_2\{a, b, c; d, e; x, z; y\} &= \frac{(1-y)^{d-b+e-c}}{B(b, d-b)B(c, e-c)} \int_0^1 \int_0^1 u^{d-b-1} (1-u(1-y))^{b-1} v^{e-c-1} (1-v(1-y))^{c-1} \\ &\quad (1-x(1-u(1-y))-z(1-v(1-y)))^{-a} dudv, \\ \operatorname{Re}(d) > 0, \operatorname{Re}(a) > 0, \operatorname{Re}(b) > 0, \operatorname{Re}(c) > 0, \operatorname{Re}(e) > 0, |\arg(1-x-z)| < \pi. \end{aligned} \tag{58}$$

Proof. Replacing the integral representation for the incomplete beta function, which is given by (8), we get:

$$\begin{aligned} \mathfrak{F}_2[a, b, c; d, e; x, z; y] &= \frac{1}{B(b, d-b)B(c, e-c)} \\ &\quad \times \sum_{m,n=0}^{\infty} \int_0^y \int_0^y (a)_{m+n} t^{b+m-1} (1-t)^{d-b-1} s^{c+n-1} (1-s)^{e-c-1} \frac{x^m}{m!} \frac{z^n}{n!} dt ds. \end{aligned}$$

Considering the fact that the series involved are uniformly convergent and we have a right to interchange the order of summation and integration, we get:

$$\begin{aligned} \mathfrak{F}_2[a, b, c; d, e; x, z; y] &= \frac{1}{B(b, d - b)B(c, e - c)} \\ &\times \int_0^y \int_0^y t^{b-1}(1 - t)^{d-b-1}s^{c-1}(1 - s)^{e-c-1}(1 - xt - zs)^{-a} dt ds, \\ &= \frac{y^{b+c}}{B(b, d - b)B(c, e - c)} \\ &\times \int_0^1 \int_0^1 u^{b-1}(1 - uy)^{d-b-1}v^{c-1}(1 - vy)^{e-c-1}(1 - xuy - zvy)^{-a} dudv. \end{aligned}$$

Formula (58) can be proven in a similar way. □

5. Incomplete Riemann-Liouville Fractional Integral Operators

In this section, we introduce and investigate the incomplete Riemann-Liouville fractional integral operators. The Riemann-Liouville fractional integral of order μ is defined by:

$$D_z^\mu \{f(z)\} := \frac{1}{\Gamma(-\mu)} \int_0^z f(t)(z - t)^{-\mu-1} dt, \quad \text{Re}(\mu) < 0. \tag{59}$$

Now, we define the incomplete Riemann-Liouville fractional integral operators $D_z^\mu [f(z); y]$ and $D_z^\mu \{f(z); y\}$ by:

$$\begin{aligned} D_z^\mu [f(z); y] &: = \frac{z^{-\mu}}{\Gamma(-\mu)} \int_0^y f(uz)(1 - u)^{-\mu-1} du \\ &: = \frac{z^{-\mu}y}{\Gamma(-\mu)} \int_0^1 f(ywz)(1 - wy)^{-\mu-1} dw, \quad \text{Re}(\mu) < 0. \end{aligned} \tag{60}$$

and its counterpart is by:

$$\begin{aligned} D_z^\mu \{f(z); y\} &: = \frac{z^{-\mu}}{\Gamma(-\mu)} \int_y^1 f(uz)(1 - u)^{-\mu-1} du \\ &: = \frac{z^{-\mu}}{\Gamma(-\mu)} \int_0^{1-y} f((1 - t)z)t^{-\mu-1} dt, \quad \text{Re}(\mu) < 0. \end{aligned} \tag{61}$$

Remark 2. If $y = 1$, then (60) is equivalent to the standard Riemann-Liouville fractional integral (59). If $y = 0$, then (61) is equivalent to the standard Riemann-Liouville fractional integral (59). Thus, the original definition (59) is a particular case of both types of the incomplete Riemann-Liouville fractional integral.

We start our investigation by calculating the incomplete fractional integrals of some elementary functions.

Theorem 17. Let $\text{Re}(\lambda) > -1, \text{Re}(\mu) < 0$. Then:

$$D_z^\mu [z^\lambda; y] = \frac{B_y(\lambda + 1, -\mu)}{\Gamma(-\mu)} z^{\lambda-\mu}. \tag{62}$$

Proof. Using (60) and (8), we get:

$$\begin{aligned} D_z^\mu [z^\lambda; y] &= \frac{z^{-\mu}}{\Gamma(-\mu)} \int_0^y (uz)^\lambda (1-u)^{-\mu-1} du \\ &= \frac{B_y(\lambda+1, -\mu)}{\Gamma(-\mu)} z^{\lambda-\mu}; \end{aligned}$$

whence the result. \square

Theorem 18. Let $Re(\lambda) > -1, Re(\mu) < 0$. Then:

$$D_z^\mu \{z^\lambda; y\} = \frac{B_{1-y}(-\mu, \lambda+1)}{\Gamma(-\mu)} z^{-\mu+\lambda}. \tag{63}$$

Theorem 19. Let $Re(\lambda) > 0, Re(\alpha) > 0, Re(\mu) < 0$ and $|z| < 1$. Then:

$$D_z^{\lambda-\mu} [z^{\lambda-1}(1-z)^{-\alpha}; y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} {}_2\mathfrak{F}_1(\alpha, [\lambda, \mu; y]; z), \tag{64}$$

and:

$$D_z^{\lambda-\mu} \{z^{\lambda-1}(1-z)^{-\alpha}; y\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} {}_2\mathfrak{F}_1(\alpha, \{\lambda, \mu; y\}; z). \tag{65}$$

Proof. Direct calculations yield:

$$\begin{aligned} D_z^{\lambda-\mu} [z^{\lambda-1}(1-z)^{-\alpha}; y] &= \frac{z^{\mu-\lambda}}{\Gamma(\mu-\lambda)} \int_0^y (uz)^{\lambda-1} (1-uz)^{-\alpha} (1-u)^{\mu-\lambda-1} du \\ &= \frac{z^{\mu-\lambda} y}{\Gamma(\mu-\lambda)} \int_0^1 (yz)^{\lambda-1} w^{\lambda-1} (1-ywz)^{-\alpha} (1-wy)^{\mu-\lambda-1} dw \\ &= \frac{z^{\mu-1} y^\lambda}{\Gamma(\mu-\lambda)} \int_0^1 w^{\lambda-1} (1-ywz)^{-\alpha} (1-wy)^{\mu-\lambda-1} dw. \end{aligned}$$

By (24), we can write:

$$\begin{aligned} D_z^{\lambda-\mu} [z^{\lambda-1}(1-z)^{-\alpha}; y] &= \frac{z^{\mu-1}}{\Gamma(\mu-\lambda)} B(\lambda, \mu-\lambda) {}_2\mathfrak{F}_1(\alpha, [\lambda, \mu; y]; z) \\ &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} {}_2\mathfrak{F}_1(\alpha, [\lambda, \mu; y]; z). \end{aligned}$$

Hence, the proof is completed. Formula (65) can be proven in a similar way. \square

Theorem 20. Let $Re(\lambda) > Re(\mu) > 0, Re(\alpha) > 0, Re(\beta) > 0; |az| < 1$ and $|bz| < 1$. Then:

$$D_z^{\lambda-\mu} [z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}; y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \mathfrak{F}_1[\lambda, \alpha, \beta; \mu; az, bz; y], \tag{66}$$

and:

$$D_z^{\lambda-\mu} \{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}; y\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \mathfrak{F}_1\{\lambda, \alpha, \beta; \mu; az, bz; y\}. \tag{67}$$

Proof. We have:

$$\begin{aligned}
 & D_z^{\lambda-\mu} [z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}; y] \\
 &= \frac{z^{\mu-\lambda}}{\Gamma(\mu-\lambda)} \int_0^y (uz)^{\lambda-1}(1-auz)^{-\alpha}(1-buz)^{-\beta}(1-u)^{\mu-\lambda-1} du \\
 &= \frac{z^{\mu-\lambda}y}{\Gamma(\mu-\lambda)} \int_0^1 (yw)^{\lambda-1}(z)^{\lambda-1}(1-aywz)^{-\alpha}(1-bywz)^{-\beta}(1-wy)^{\mu-\lambda-1} dw \\
 &= \frac{z^{\mu-1}y^\lambda}{\Gamma(\mu-\lambda)} \int_0^1 w^{\lambda-1}(1-aywz)^{-\alpha}(1-bywz)^{-\beta}(1-wy)^{\mu-\lambda-1} dw.
 \end{aligned}$$

By (55), we can write:

$$\begin{aligned}
 D_z^{\lambda-\mu} [z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}; y] &= \frac{z^{\mu-1}}{\Gamma(\mu-\lambda)} B(\lambda, \mu-\lambda) \mathfrak{F}_1[\lambda, \alpha, \beta; \mu; az, bz; y] \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \mathfrak{F}_1[\lambda, \alpha, \beta; \mu; az, bz; y];
 \end{aligned}$$

whence the result. Formula (67) can be proven in a similar way. \square

Theorem 21. Let $Re(\lambda) > Re(\mu) > 0, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0; \left| \frac{t}{1-z} \right| < 1$ and $|t| + |z| < 1$. We have:

$$D_z^{\lambda-\mu} [z^{\lambda-1}(1-z)^{-\alpha} {}_2F_1(\alpha, [\beta, \gamma; y]; \frac{t}{1-z}); y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \mathfrak{F}_2[\alpha, \beta, \lambda; \gamma, \mu; t, z; y], \tag{68}$$

and:

$$D_z^{\lambda-\mu} \{z^{\lambda-1}(1-z)^{-\alpha} {}_2F_1(\alpha, [\beta, \gamma; y]; \frac{t}{1-z}); y\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \mathfrak{F}_2\{\alpha, \beta, \lambda; \gamma, \mu; t, z; y\}. \tag{69}$$

Proof. Using Theorem 19 and (53), we get:

$$\begin{aligned}
 & D_z^{\lambda-\mu} [z^{\lambda-1}(1-z)^{-\alpha} {}_2F_1(\alpha, [\beta, \gamma; y]; \frac{t}{1-z}); y] \\
 &= D_z^{\lambda-\mu} [z^{\lambda-1}(1-z)^{-\alpha} \frac{1}{B(\beta, \gamma-\beta)} \sum_{n=0}^{\infty} \frac{(\alpha)_n B_y(\beta+n, \gamma-\beta)}{n!} \left(\frac{t}{1-z}\right)^n; y] \\
 &= \frac{1}{B(\beta, \gamma-\beta)} D_z^{\lambda-\mu} [z^{\lambda-1} \sum_{n=0}^{\infty} (\alpha)_n B_y(\beta+n, \gamma-\beta) \frac{t^n}{n!} (1-z)^{-\alpha-n}; y] \\
 &= \frac{1}{B(\beta, \gamma-\beta)} \sum_{m,n=0}^{\infty} B_y(\beta+n, \gamma-\beta) \frac{t^n}{n!} \frac{(\alpha)_n (\alpha+n)_m}{m!} D_z^{\lambda-\mu} [z^{\lambda-1+m}; y] \\
 &= \frac{1}{B(\beta, \gamma-\beta)} \sum_{m,n=0}^{\infty} B_y(\beta+n, \gamma-\beta) \frac{t^n}{n!} \frac{(\alpha)_{n+m} B_y(\lambda+m, \mu-\lambda)}{m! \Gamma(\mu-\lambda)} z^{\mu+m-1} \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \mathfrak{F}_2[\alpha, \beta, \lambda; \gamma, \mu; t, z; y].
 \end{aligned}$$

Hence, the proof is complete. Formula (69) can be proven in a similar way. \square

6. Generating Functions

Now, we obtain linear and bilinear generating relations for the incomplete hypergeometric functions ${}_2\mathfrak{F}_1(a, [b, c; y]; x)$ by following the methods described in [2]. We start with the following theorem:

Theorem 22. For the incomplete hypergeometric functions, we have:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2\mathfrak{F}_1(\lambda + n, [\alpha, \beta; y]; z)t^n = (1 - t)^{-\lambda} {}_2\mathfrak{F}_1(\lambda, [\alpha, \beta; y]; \frac{z}{1 - t}) \tag{70}$$

and:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2\mathfrak{F}_1(\lambda + n, \{\alpha, \beta; y\}; z)t^n = (1 - t)^{-\lambda} {}_2\mathfrak{F}_1(\lambda, \{\alpha, \beta; y\}; \frac{z}{1 - t}) \tag{71}$$

where $|z| < \min\{1, |1 - t|\}$ and $Re(\lambda) > 0, Re(\beta) > Re(\alpha) > 0$.

Proof. Considering the elementary identity:

$$[(1 - z) - t]^{-\lambda} = (1 - t)^{-\lambda} \left[1 - \frac{z}{1 - t}\right]^{-\lambda}$$

and expanding the left-hand side, we have for $|t| < |1 - z|$ that:

$$(1 - z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left(\frac{t}{1 - z}\right)^n = (1 - t)^{-\lambda} \left[1 - \frac{z}{1 - t}\right]^{-\lambda}.$$

Now, multiplying both sides of the above equality by $z^{\alpha-1}$ and applying the incomplete fractional integral operator $D_z^{\alpha-\beta}[f(z); y]$ on both sides, we can write:

$$D_z^{\alpha-\beta} \left[\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1 - z)^{-\lambda} \left(\frac{t}{1 - z}\right)^n z^{\alpha-1}; y \right] = (1 - t)^{-\lambda} D_z^{\alpha-\beta} \left[z^{\alpha-1} \left[1 - \frac{z}{1 - t}\right]^{-\lambda}; y \right].$$

Interchanging the order, which is valid for $Re(\alpha) > 0$ and $|t| < |1 - z|$, we get:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} D_z^{\alpha-\beta} \left[z^{\alpha-1} (1 - z)^{-\lambda-n}; y \right] t^n = (1 - t)^{-\lambda} D_z^{\alpha-\beta} \left[z^{\alpha-1} \left[1 - \frac{z}{1 - t}\right]^{-\lambda}; y \right].$$

Using Theorem 21, we get the desired result. Formula (71) can be proven in a similar way. \square

The following theorem gives another linear generating relation for the incomplete hypergeometric functions.

Theorem 23. For the incomplete hypergeometric functions, we have:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2\mathfrak{F}_1(\rho - n, [\alpha, \beta; y]; z)t^n = (1 - t)^{-\lambda} \mathfrak{F}_1[\alpha, \rho, \lambda; \beta; z; \frac{-zt}{1 - t}; y] \tag{72}$$

and:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2\mathfrak{F}_1(\rho - n, \{\alpha, \beta; y\}; z)t^n = (1 - t)^{-\lambda} \mathfrak{F}_1\{\alpha, \rho, \lambda; \beta; z; \frac{-zt}{1 - t}; y\} \tag{73}$$

where $Re(\lambda) > 0, Re(\rho) > 0, Re(\beta) > Re(\alpha) > 0; |t| < \frac{1}{1+|z|}$.

Proof. Considering:

$$[1 - (1 - z)t]^{-\lambda} = (1 - t)^{-\lambda} \left[1 + \frac{zt}{1 - t} \right]^{-\lambda}$$

and expanding the left-hand side, we have for $|t| < |1 - z|$ that:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1 - z)^n t^n = (1 - t)^{-\lambda} \left[1 - \frac{-zt}{1 - t} \right]^{-\lambda}.$$

Now, multiplying both sides of the above equality by $z^{\alpha-1}(1 - z)^{-\rho}$ and applying the fractional integral operator $D_z^{\alpha-\beta}[f(z); y]$ on both sides, we get:

$$D_z^{\alpha-\beta} \left[\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^{\alpha-1} (1 - z)^{-\rho+n} t^n; y \right] = (1 - t)^{-\lambda} D_z^{\alpha-\beta} \left[z^{\alpha-1} (1 - z)^{-\rho} \left[1 - \frac{-zt}{1 - t} \right]^{-\lambda}; y \right].$$

Interchanging the order, which is valid for $Re(\alpha) > 0$ and $|zt| < |1 - t|$, we get:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} D_z^{\alpha-\beta} \left[z^{\alpha-1} (1 - z)^{-(\rho-n)}; y \right] t^n = (1 - t)^{-\lambda} D_z^{\alpha-\beta} \left[z^{\alpha-1} (1 - z)^{-\rho} \left[1 - \frac{-zt}{1 - t} \right]^{-\lambda}; y \right].$$

Using Theorem 21 and 22, we get the desired result. The generating relation (73) can be proven in a similar way. \square

Finally, we have the following bilinear generating relation for the incomplete hypergeometric functions.

Theorem 24. For the incomplete hypergeometric functions, we have:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2\mathfrak{F}_1(\gamma, [-n, \delta; y]; x) {}_2\mathfrak{F}_1(\gamma, [\lambda + n, \beta; y]; z) t^n = (1 - t)^{-\lambda} \mathfrak{F}_2\left[\lambda, \alpha, \gamma; \beta, \delta; \frac{z}{1 - t}; \frac{-xt}{1 - t}; y\right] \quad (74)$$

and:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2\mathfrak{F}_1(\gamma, \{-n, \delta; y\}; x) {}_2\mathfrak{F}_1(\gamma, \{\lambda + n, \beta; y\}; z) t^n = (1 - t)^{-\lambda} \mathfrak{F}_2\left\{\lambda, \alpha, \gamma; \beta, \delta; \frac{z}{1 - t}; \frac{-xt}{1 - t}; y\right\} \quad (75)$$

where $Re(\lambda) > 0, Re(\gamma) > 0, Re(\beta) > 0, Re(\delta) > 0, Re(\alpha) > 0; |t| < \frac{1-|z|}{1+|x|}$, and $|z| < 1$.

Proof. Replacing t by $(1 - x)t$ in (70), multiplying the resulting equality by $x^{\gamma-1}$, and then applying the incomplete fractional integral operator $D_x^{\gamma-\delta}[f(x); y]$, we get:

$$\begin{aligned} & D_x^{\gamma-\delta} \left[\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} x^{\gamma-1} {}_2\mathfrak{F}_1(\lambda + n, [\alpha, \beta; y]; z) (1 - x)^n t^n; y \right] \\ &= D_x^{\gamma-\delta} \left[(1 - (1 - x)t)^{-\lambda} x^{\gamma-1} {}_2\mathfrak{F}_1(\lambda, [\alpha, \beta; y]; \frac{z}{1 - (1 - x)t}); y \right]. \end{aligned}$$

Interchanging the order, which is valid for $|z| < 1$, $|\frac{1-x}{1-z}t| < 1$ and $|\frac{z}{1-t}| + |\frac{xt}{1-t}| < 1$, we can write that:

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} D_x^{\gamma-\delta} \left[x^{\gamma-1}(1-x)^n; y \right] {}_2\mathfrak{F}_1(\lambda+n, [\alpha, \beta; y]; z) \\ &= (1-t)^{-\lambda} D_x^{\gamma-\delta} \left[x^{\gamma-1} \left(1 - \frac{-xt}{1-t}\right) {}_2\mathfrak{F}_1\left(\lambda, [\alpha, \beta; y]; \frac{\frac{z}{1-t}}{1 - \frac{-xt}{1-t}}\right); y \right]. \end{aligned}$$

Using Theorems 21 and 23, we get (74). The generating relation (75) can be proven in a similar way. □

In the following remark, first of all, we obtained a series formula for the Gauss hypergeometric functions as an application of Theorem 22. Similar results can be obtained for Theorem 23 and 24. Furthermore, we showed that the result obtained in (70) coincides with usual case when $y \rightarrow 1^-$.

Remark 3. Using the relation that is given by (13) in Equation (70), we have:

$$\frac{1}{B(\alpha, \beta - \alpha)} \sum_{n,k=0}^{\infty} (\lambda)_n (\lambda+n)_k \frac{y^{\alpha+k}}{\alpha+k} {}_2F_1(\alpha+k, 1-\beta+\alpha; \alpha+k+1; y) \frac{z^k t^n}{k! n!} \tag{76}$$

$$= \frac{(1-t)^{-\lambda}}{B(\alpha, \beta - \alpha)} \sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!} \left(\frac{z}{1-t}\right)^m \frac{y^{\alpha+m}}{\alpha+m} {}_2F_1(\alpha+m, 1-\beta+\alpha; \alpha+m+1; y) \tag{77}$$

which is a series identity between the Gauss hypergeometric functions. If we take $y = 1$ in the above identity and use the following relation:

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

we obtain:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2F_1(\lambda+n, \alpha, \beta; z) t^n = (1-t)^{-\lambda} {}_2F_1(\lambda, \alpha, \beta; \frac{z}{1-t}) \tag{78}$$

7. Conclusions

Recently, in [27], various applications of fractional calculus were exhibited in areas ranging from engineering to life sciences. For the applications of fractional calculus, we should also recommend the references of the paper [27] and, in particular, the book [21].

In the present paper, we introduced the incomplete versions of Riemann-Liouville integral operators. Approaching the problems mentioned in [27] using these incomplete operators may give rise to interesting perspectives on solving these problems. For instance, in a nonlocal fractional process, which occurs on an interval, but whose behavior changes in the middle, it may be useful to consider splitting the domain into subintervals and integrating from both sides separately using incomplete fractional operators.

These operators have already been used to define Liouville-Caputo-type incomplete fractional derivatives in [12]. Furthermore, for the incomplete Riemann-Liouville fractional integrals defined here, their analyticity properties have been investigated in [31]. Some of these, such as a transformation property on the domains of the functions concerned, may also lend themselves well to applications.

Incomplete Pochhammer ratios were defined in (10) and (11) by using the incomplete beta functions. Several properties of these functions were obtained. Incomplete hypergeometric functions were introduced with the help of these incomplete Pochhammer ratios, and certain properties such as integral representations, derivative formulas, transformation formulas, and recurrence relations were investigated.

Furthermore, incomplete Riemann-Liouville fractional integral operators were defined. The incomplete Riemann-Liouville fractional integrals for the some elementary functions were given. Linear and bilinear generating relations for incomplete hypergeometric functions were obtained.

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