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The Convergence of Gallego's Iterative Method for Distribution-Free Inventory Models

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Abstract: For inventory models with unknown distribution demand, during shortages, researchers used the first and the second moments to derive an upper bound for the worst case, that is the min-max distribution-free procedure for inventory models. They applied an iterative method to generate a sequence to obtain the optimal order quantity. A researcher developed a three-sequence proof for the convergence of the order quantity sequence. We directly provide proof for the original order quantity sequence. Under our proof, we can construct an increasing sequence and a decreasing sequence that both converge to the optimal order quantity such that we can obtain the optimal solution within the predesigned threshold value.

Keywords: inventory models; minimax distribution-free procedure

1. Introduction

In this paper, we directly prove that the iterative sequence proposed by Gallego [1] is convergent. Gallego [1] provided an upper bound to estimate the shortage during the lead time for stochastic demand inventory models, where the distribution of the demand is unknown but the first and the second moments are known. Gallego [1] focused on proving his upper bound as a good estimation without paying attention to the existence and uniqueness of the optimal solution. His new inventory model is complicated so that no closed formula for the optimal solution could be found. Moreover, his optimal solution is expressed by an implicit relation to imply a sequence solution. Additionally, Gallego [1] did not prove his sequence to converge toward its optimal solution. Since the publication of his approach, it has been cited 47 times. These papers can be classified into the following four categories:

(a) The construction of new inventory models: Qi et al. [2], Şen and Talebian [3], Wang et al. [4], Kumar and Goswami [5], Sarkar et al. [6], Yu and Zhai [7], Qin and Kar [8], Yu and Zhen [9], Moon et al. [10], Tajbakhsh [11], Gallego and ŞAhin [12], Perakis and Roeis [13], Ahmed et al. [14], Levin et al. [15], Mostard et al. [16], Alfares and Elmorra [17], Lin [18], Hariga and Ben-Daya [19], Talluri and Van Ryzin [20], and Gallego [21].

(b) The development of new solution procedures: Fu et al. [22], Postek et al. [23], Zhou et al. [24], Wu and Warsing [25], Popescu [26], Gallego et al. [27], Chung et al. [28], Puerto and Fernández [29], Tyworth and O'Neill [30], Moon and Choi [31], Moon and Yun [32], and Baganha et al. [33].

(c) The application to solve decision-making problems: Du et al. [34], Wright [35], Wang et al. [36], Das and Maiti [37], Puerto and Rodríguez-Chía [38], Fricker and Goodhart [39], Vairaktarakis [40], Hariga [41], Hariga and Ben-Daya [42], and Platt et al. [43].

(d) The improvement of existing solution approaches: Tuan [44], Lin [45], Ruiz-Torres and Mahmoodi [46], Hung et al. [47], Lin et al. [48], and Lin and Chu [49].

Papers in categories (a), (b) and (c) did not discuss the solution procedure of Gallego [1], and so papers in categories (a), (b) and (c) are beyond the realm of our discussion. Only publications in category (d) are in the scope of our discussion, therefore an outline of their results is provided. There are three papers, Moon and Gallego [50], Wu and Ouyang [51] and Tung et al. [52] that are worthy of mentioning even they did not cite Gallego [1] in their references. For distribution-free inventory models, Gallego [1] used order quantity Q and reorder point R as decision variables. On the other hand, Moon and Gallego [50] applied order quantity Q and safety factor k as decision variables. Wu and Ouyang [51] extended Moon and Gallego [50] with defective items and provided an iterative sequence solution approach. Moreover, Tung et al. [52] showed that the optimal solution for inventory models of Wu and Ouyang [51] exists and is unique. Lin and Chu [49] verified that the optimal solution may not occur on the boundary for inventory models with a service-level constraint. Lin et al. [48] showed the existence and uniqueness of the optimal solution for the first partial derivative system proposed by Gallego [1] and Moon and Gallego [50]. Lin et al. [48] also showed that the iterative sequence solution of Wu and Ouyang [51] cannot be executed. Hung et al. [47] proved that the minimum may not happen on the boundary for inventory models with a service-level constraint. Ruiz-Torres and Mahmoodi [46] used simulation to show that their findings are much closer to the target service level which reduces the total holding cost. Lin [45] constructed a three-sequence method to prove the convergence of the sequence derived by Gallego [1]. We can claim that up to now, Lin [45] is the only paper that had provided proof for the convergence problem of Gallego [1] and Moon and Gallego [50]. However, her approach is too tedious that contains three sequences. In this paper, we will present a direct derivation for the original sequence proposed by Gallego [1]. Tuan [44] studied the restriction proposed by Gallego [1] to find a more general condition such that the term by term examination in the iterative algorithm proposed by Gallego [1] becomes unnecessary. Tuan [44] also pointed out that the direct proof for the original sequence proposed by Gallego [1] for its convergence will be an interesting research topic in the future. Thus, it motivates us to simplify the three-sequence approach of Lin [45] back to the original sequence generated by the expression of Gallego [1].

From the above discussion, we know that none of the aforementioned papers directly discussed the convergence for the iterative method proposed by Gallego [1]. The purpose of this paper is directly to verify that Gallego's iterative method is convergent.

2. Notation and Assumptions

To be compatible with Gallego [1], since our main concern is to prove the convergence of the sequence generated by Gallego [1], we adopted the same notation and assumptions of Gallego [1].

Notation

D = average demand per unit of time.
h = inventory carrying cost per item per unit of time.
K = the fixed ordering cost per order.
Q = order quantities per order.
R = reorder point.
π = unit shortage cost.
μ = mean of the lead time demand.
σ^2 = the variance of the lead time demand.

Assumptions

1. Backorder cost is proportional to the number of items back ordered and not to the time for which they are outstanding.
2. F is the cumulative distribution of the lead time demand. F has known finite first and second moments with mean μ and variance σ^2 and then makes no assumptions on the distribution form of F.

3. The inventory model is continuously reviewed. Replenishments are made when the inventory level drops to the reorder point R.

3. Recap of Three Related Papers

We recall the objective function for the stochastic inventory model with distribution-free demand in Gallego [1] as follows

$$C(Q,\Delta)=\frac{KD}{Q}+h\left(\frac{Q}{2}+\Delta\right)+\frac{\pi D}{2Q}\left(\sqrt{\Delta^2+\sigma^2}-\Delta\right). \tag{1}$$

Remark 1. *In Lin et al. [48], they considered the same objective function, however, there was a typo in Lin et al. [48], and so the denominator of the third term in Equation (1) was typed as "Q" which should be revised to "2Q".*

Gallego [1] took the partial derivatives with respect to Q and Δ then he canceled out Δ to merge $\frac{\partial}{\partial Q}C(Q,\Delta)=0$ and $\frac{\partial}{\partial \Delta}C(Q,\Delta)=0$ into one Equation with a single variable, Q, to derive the next formula:

$$Q=\sqrt{\frac{2KD}{h}+\frac{D\pi\sigma}{h}\sqrt{\frac{hQ}{\pi D-hQ}}} \tag{2}$$

under the condition $\pi D \geq 2hQ$.

Gallego [1] had mentioned that based on Equation (2) an iterative method will generate a convergent sequence which will converge to the optimal solution. However, up to now, except for Lin [45], none had discussed the convergence problem proposed by Gallego [1].

Lin et al. [48] pointed out that the restriction should be revised from $\pi D \geq 2hQ$ to $\pi D > 2hQ$, which they then used to prove that there is a unique root for the partial derivative system. However, they did not discuss theconvergence of the iterative method of Gallego [1].

Based on Moon and Gallego [50] and Tung et al. [52], Lin [45] developed three sequences: (k_i), (Q_i) and (d_i), with the initial point $k_0=0$, and three relations:

$$Q_{n+1}=\left(B_1+B_2\left(\sqrt{1+k_n^2}-k_n\right)\right)^{1/2} \tag{3}$$

with $B_1=\frac{2D}{h\delta}\left[A+c_i(L_{i-1}-L)+\sum_{j=1}^{i-1}c_j\left(b_j-a_j\right)\right]$ and $B_2=\frac{D\sigma}{h\delta}(\pi+\pi_0(1-\beta))\sqrt{L}$,

$$d_{n+1}=\frac{1}{2}\left(1-\beta+\frac{D(\pi+\pi_0(1-\beta))}{hQ_{n+1}(1-E(p))}\right) \tag{4}$$

and

$$k_{n+1}=\frac{d_{n+1}-1}{\sqrt{2d_{n+1}-1}}, \tag{5}$$

with $\delta = 1-2E(p)+E(p^2)+2(h_1/h)E(p(1-p))$,where h_1 is the holding cost for non-defective items, h is the holding cost for defective items, and p is the defective rate in an order lot. In Gallego [1], there are no defective items, then $p = 0$ to imply that $(p) = 0$, $(p^2) = 0$ and $E(p(1-p)) = 0$. Hence, $\delta = 1$.

Lin [45] proved that (d_n) and (k_n) are increasing sequences and (Q_n) is a decreasing sequence bounded below by zero, so (Q_n) converges and then both (d_n) and (k_n) are convergent.

4. Our Improvement

The purpose of this paper is to provide proof that the following sequence

$$Q_{n+1} = \sqrt{\frac{2KD}{h} + \frac{D\pi\sigma}{h}\sqrt{\frac{hQ_n}{\pi D - hQ_n}}}, \tag{6}$$

under the condition (Q_n) proposed by Gallego [1] that (Q_n) indeed converges, and the value it converges to is the optimal solution. Hence, we can derive the convergence of Gallego's iterative method by the original sequence to simplify the three-sequence approach proposed by Lin [45]. We mention our main result in the following theorem.

Theorem 1. *The ordering quantity sequence, proposed by Gallego [1], converges to its interior optimal solution.*

Proof of Theorem 1. From the restriction of $\pi D > 2hQ_n$, we know that

$$\frac{hQ_n}{D\pi - hQ_n} < 1. \tag{7}$$

Using the restriction of $\pi D > 2hQ_n$, we derive an upper bound for the sequence (Q_n) as,

$$Q_n < \frac{D\pi}{2h}. \tag{8}$$

On the other hand, we know that there is a natural lower bound for (Q_n) as,

$$0 < Q_n. \tag{9}$$

From Equation (6), we derive that,

$$Q_{n+1}^2 - Q_n^2 = \frac{D\pi\sigma}{h}\left(\frac{1}{\sqrt{(D\pi/hQ_n) - 1}} - \frac{1}{\sqrt{(D\pi/hQ_{n-1}) - 1}}\right). \tag{10}$$

By Equations (9) and (10), $Q_{n+1} > Q_n$ is equivalent to:

$$\sqrt{\frac{D\pi}{hQ_{n-1}} - 1} > \sqrt{\frac{D\pi}{hQ_n} - 1} \tag{11}$$

that is,

$$Q_n > Q_{n-1} \tag{12}$$

to derive that (Q_n) is increasing, if $Q_1 > Q_0$.

Similarly, we know that $Q_{n+1} < Q_n$, if $Q_1 < Q_0$ to imply that (Q_n) is decreasing.

For a selected initial point Q_0, there are three conditions: (a) $Q_0 > Q_1$, (b) $Q_0 < Q_1$, and (c) $Q_0 = Q_1$ that may occur.

For condition (a), from $Q_1 < Q_0$, (Q_n) is a decreasing sequence bound below by zero so it converges to its greatest lower bound.

In the following, we will prove that the greatest lower bound is greater than zero. Owing to $Q_n > 0$ so we have $\lim_{n\to\infty} Q_n \geq 0$. By the way of contradiction, we assume that $\lim_{n\to\infty} Q_n = 0$.

It implies that $\lim_{n\to\infty} Q_{n+1} = 0$ and we compute the limit of Equation (6) for $n \to \infty$, then it yields that $0 = \sqrt{\frac{2KD}{h}}$ which is a contradiction so our assumption of $\lim_{n\to\infty} Q_n = 0$ is not valid. Hence, we obtain that $\lim_{n\to\infty} Q_n > 0$.

Owing to $\lim_{n\to\infty} Q_n$ satisfies Equation (2), $\lim_{n\to\infty} Q_n$ is the optimal solution since Lin et al. [48] showed the existence and uniqueness of the optimal solution.

For condition (b), because $Q_0 < Q_1$, (Q_n) is an increasing sequence bound above by $\sqrt{\frac{2}{h}(KD + \pi\sigma)}$, therefore, it converges to its least upper bound. By the same reasoning for the decreasing sequence, $\lim_{n\to\infty} Q_n$ is the optimal solution.

For condition (c), because $Q_0 = Q_1$, this yields that (Q_n) is a constant sequence so it converges to Q_0 which satisfies Equation (2) so it is the optimal solution.

Now, we combine the above findings to derive that for all three conditions (Q_n) converges. and converges to the optimal solution.

By an analytical approach with calculus, Lin et al. [48] have already proved that there is a unique solution for the interior minimum.

Because of the limit point, $\lim_{n\to\infty} Q_n$, satisfies the partial derivative system so it must be the optimal solution. Therefore, (Q_n) converges to the optimal solution, for all three conditions. □

5. Numerical Examples

We will present two numerical examples to illustrate that the sequence of ordering quantity, (Q_n), depending on the initial point, can be a decreasing sequence or an increasing sequence. However, Gallego [1] and Lin et al. [48] did not provide numerical examples in their paper, so that we refer to Lin [45] to decide the following data: $K = 200$, $D = 600$, $h = 20$, $\pi = 50$ and $\sigma = 7$ for our numerical examples. We know that $\pi D/2h$ is an upper bound and 0 is a lower bound for the ordering quantity.

From Table 1, we construct a decreasing sequence that is a support for our analytical result as $Q_1 < Q_0$, then we generate a decreasing sequence.

Table 1. A decreasing sequence.

Q_0	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
750	150	124.499	123.122	123.044	123.040	123.039	123.039

For our second numerical example with respect to condition (b), we assume that $Q_0 = 0$ and using Equation (6) to derive the iterative sequence. The computation results are listed in the following Table 2.

Table 2. An increasing sequence.

Q_0	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
0	109.545	122.259	122.995	123.037	123.039	123.039

From Table 2, we construct an increasing sequence that is numerical evidence for our analytical result as $Q_1 > Q_0$ then we derive an increasing sequence.

From Table 1, the decreasing sequence, we derive that the limit is less than or equal to 123.039. From Table 2, the increasing sequence, we obtain that the limit is greater than or equal to 123.039. Hence, we find that the optimal order quantity, Q = 123.039 up to the third decimal place.

Let us recall the order quantity sequence derived by Lin [45], where $Q_1 = 271.444019$, $Q_2 = 179.201796$, $Q_3 = 171.872873$, $Q_4 = 171.206597$, $Q_5 = 171.145237$, $Q_6 = 171.139579$, $Q_7 = 171.139060$, $Q_8 = 171.139014$, and $Q_9 = 171.139014$.

From the above sequence, usually researchers have observed that the sequence decreases very slowly until $Q_8 = Q_9$ such that they have accepted the optimal order quantity is 171.139014, up to the sixth decimal place.

We may hypothetically assume that Q_{10} = 170.142368, Q_{11} = 170.141352, Q_{12} = 170.141351, and Q_{13} = 170.141351, that is the decreasing rate becomes very slow from Q_6 to Q_9, and then decreased again from Q_{10} to Q_{13}. Hence, only apply a decreasing sequence, researchers did not know where the optimal solution is.

On the other hand, we can construct two sequences: in Table 1, a decreasing sequence, and in Table 2, an increasing sequence, such that we can derive the limit, that is, the optimal order quantity without any doubts.

In [45], the numerical data is different from our results, because Lin [45] considered inventory models with partial back order. In Gallego [1] and this paper, shortages are fully back order.

In the following, we develop an algorithm to decide a sequence solution, denoted as Q^{Δ}, to approximate the optimal order quantity, Q^{*}, within the pre-designed threshold value. We assume that $Q_0 = 0$ and $Q_1 = \sqrt{\pi D/2h}$, and then for $n = 0,\ 1,\ 2, \ldots,$

$$Q_{2n+2} = \sqrt{\frac{2KD}{h} + \frac{D\pi\sigma}{h}\sqrt{\frac{hQ_{2n}}{\pi D - hQ_{2n}}}}, \tag{13}$$

and

$$Q_{2n+3} = \sqrt{\frac{2KD}{h} + \frac{D\pi\sigma}{h}\sqrt{\frac{hQ_{2n+1}}{\pi D - hQ_{2n+1}}}} \tag{14}$$

Owing to $Q_0 = 0 < Q_2 = \sqrt{2KD/h}$, we derive that (Q_{2n}) is an increasing sequence. Next, to check $Q_1 > Q_3$, we need the following lemma.

Lemma 1. *For inventory models proposed by Gallego [1], we claim that,*

$$8hK + 4h\pi\sigma < \pi^2 D \tag{15}$$

Proof of Lemma 1. If $Q_1 = \sqrt{\pi D/2h}$, then we compute that,

$$\frac{4h^2}{D}\left(Q_1^2 - Q_3^2\right) = \pi^2 D - (8hK + 4h\pi\sigma). \tag{16}$$

With the data K = 200, D = 600, h = 20, π = 50 and σ = 7 from Lin [45], we derive that,

$$8hK + 4h\pi\sigma = 6 \times 10^4, \tag{17}$$

and

$$\pi^2 D = 15 \times 10^7 \tag{18}$$

such that the inequality in Equation (15) is reasonable. □

Based on our Lemma, we imply that $Q_1 > Q_3$ and then (Q_{2n+1}) is a decreasing sequence.

We construct our algorithm in the following.

Step 1: For a given threshold value, ε, we assume that $Q_0 = 0$ and $Q_1 = \sqrt{\pi D/2h}$.

Step 2: We check the distance between Q_{2n} and Q_{2n+1}, and assume

$$m = \min\{n : Q_{2n+1} - Q_{2n} < \varepsilon\} \tag{19}$$

Step 3: Our approximated solution, denoted as Q^{Δ}, is defined as

$$Q^{\Delta} = \frac{1}{2}(Q_{2m} + Q_{2m+1}). \tag{20}$$

We demonstrate our algorithm for a threshold value $\varepsilon = 10^{-6}$. We list the decreasing sequence (Q_{2n+1}) in Table 3 and the increasing sequence (Q_{2n}) in Table 4.

Table 3. A decreasing sequence to the sixth decimal place.

Q_1	Q_3	Q_5	Q_7	Q_9	Q_{11}	Q_{13}	Q_{15}
750	150	124.498996	123.121659	123.044093	123.039467	123.039453	123.039452

Table 4. An increasing sequence to the sixth decimal place.

Q_0	Q_2	Q_4	Q_6	Q_8	Q_{10}	Q_{12}	Q_{14}
0	109.544512	122.258644	122.995305	123.036959	123.039311	123.039444	123.039452

We compute $Q_1 - Q_0$, $Q_3 - Q_2, \ldots, Q_{13} - Q_{12} = 9 \times 10^{-6} > \varepsilon$, until $Q_{15} - Q_{14} = 0 < \varepsilon$ such that we derive our approximated order quantity,

$$Q^{\Delta} = \frac{1}{2}(Q_{14} + Q_{15}) = 123.039452. \tag{21}$$

We consider the problem of what factor or parameters affect the speed of the convergence rate for the sequence generated by the iterative procedure proposed by Gallego [1].

We compute that:

$$\begin{aligned} Q_{n+2}^2 - Q_{n+1}^2 &= \frac{\pi D\sigma}{h}\left(\sqrt{\frac{hQ_{n+1}}{\pi D - hQ_{n+1}}} - \sqrt{\frac{hQ_n}{\pi D - hQ_n}}\right) \\ &= \frac{\pi D\sigma}{h}\left(\sqrt{\frac{hQ_{n+1}}{\pi D - hQ_{n+1}}} + \sqrt{\frac{hQ_n}{\pi D - hQ_n}}\right)^{-1}\left(\frac{hQ_{n+1}}{\pi D - hQ_{n+1}} - \frac{hQ_n}{\pi D - hQ_n}\right) \\ &= \frac{\pi D\sigma}{h}\left(\sqrt{\frac{hQ_{n+1}}{\pi D - hQ_{n+1}}} + \sqrt{\frac{hQ_n}{\pi D - hQ_n}}\right)^{-1}\left(\frac{\pi D h(Q_{n+1} - Q_n)}{(\pi D - hQ_{n+1})(\pi D - hQ_n)}\right). \end{aligned} \tag{22}$$

Based on Equation (22), when n is big enough, we derive that,

$$\left|Q_{n+2} - Q_{n+1}\right| \approx \frac{\pi^2 D^2 \sigma}{4((\pi D - hQ^*)Q^*)^{1.5} h^{0.5}}\left|Q_{n+1} - Q_n\right|. \tag{23}$$

First, we derive

$$\frac{\pi^2 D^2 \sigma}{4((\pi D - hQ^*)Q^*)^{1.5} h^{0.5}} = 0.056464. \tag{24}$$

Next, we refer to Table 4, and find that,

$$\frac{Q_{10} - Q_8}{Q_8 - Q_6} = 0.056469, \tag{25}$$

$$\frac{Q_{12} - Q_{10}}{Q_{10} - Q_8} = 0.056464, \tag{26}$$

and

$$\frac{Q_{14} - Q_{12}}{Q_{12} - Q_{10}} = 0.056464. \tag{27}$$

If we compare Equations (24)–(27) to show that our estimation of the converge rate of Equation (24) is very accurate when n is big enough after five iterations.

To estimate which factor influences the converge rate, we consider $\pi D - hQ^* \approx \pi D$, and then we simplify our result of Equation (24) as,

$$\sqrt{\frac{\pi D/h}{Q^*}}\frac{\sigma}{4Q^*} = 0.049685 \tag{28}$$

with

$$\left(\sqrt{\frac{\pi D/h}{Q^*}}\frac{\sigma}{4Q^*}\right)\Bigg/\frac{\pi^2 D^2 \sigma}{4((\pi D - hQ^*)Q^*)^{1.5}h^{0.5}} = 0.049685/0.056464 = 0.88, \tag{29}$$

to indicate $\sqrt{\frac{\pi D/h}{Q^*}}\frac{\sigma}{4Q^*}$ can interpret 88% of the converge rate such that we claim that the main factors for the convergence rate are (i) σ, (ii) $\sqrt{(\pi D/h)/Q^*}$ and (iii) Q^*.

Finally, three related papers of Braglia et al. [53,54] and Castellano et al. [55] are worth mentioning.

6. Conclusions

Our paper provides a patchwork for the Gallego's iterative method since its convergence has not yet been proven. Moreover, we showed that there are three possible conditions for convergence which is a generalization of Lin [45].

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