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δ -Almost Periodic Functions and Applications to Dynamic Equations

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Abstract: In this paper, by employing matched spaces for time scales, we introduce a δ -almost periodic function and obtain some related properties. Also the hull equation for homogeneous dynamic equation is introduced and results of the existence are presented. In the sense of admitting exponential dichotomy for the homogeneous equation, the expression of a δ -almost periodic solution for a type of nonhomogeneous dynamic equation is obtained and the existence of δ -almost periodic solutions for new delay dynamic equations is considered. The results in this paper are valid for delay q -difference equations and delay dynamic equations whose delays may be completely separated from the time scale \mathbb{T} .

Keywords: matched spaces; almost periodic functions; delay dynamic equations; time scale

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1. Introduction

Almost periodic phenomena are common in real world phenomena and the concept of almost periodic functions was first introduced by H. Bohr and later studied by H. Weyl and V. Stepanov, etc. (see [1–4]). Time scale calculus was first presented by S. Hilger [5] and there were many results concerning almost periodic dynamic equations on time scales (see [6–19]). However, in these papers, the results are only limited to periodic time scales with translation invariance. In fact, $\overline{q^{\mathbb{Z}}} := \{q^n : q > 1, n \in \mathbb{Z}\}$ is not periodic under translations. This time scale leads to q -difference dynamic equations and plays an important role in different fields of engineering and biological science (see [20,21]). However, it was difficult to combine almost periodic problems of q -difference dynamic equations on time scales in the past because $\overline{q^{\mathbb{Z}}}$ has no translation invariance since the graininess function μ is unbounded and there was no concept of relatively dense set defined on it. To consider more general time scales without translation periodicity, the authors in [22,23] studied a type of dynamic equation under a new concept of periodic time scale whose period set is contained in \mathbb{T} . However, the period set of a periodic time scale is not always contained in this time scale, and for this reason, in Sections 3 and 4 of the paper cited in Ref. [24], C. Wang and R.P. Agarwal for the first time proposed a type of delay dynamic equation whose delay function range is from the period set of the time scale and completely separated from the time scale \mathbb{T} . They assumed $\tau(t)$ is a delay function, the set $\Pi_1 := \{\tau(t), t \in \mathbb{T}\}$ and $\Pi_1 \cap \mathbb{T}$ may be an empty set, and considered whether $t \pm \tau(t)$ belongs to

\mathbb{T} . Let \mathbb{T} be a time scale with translation invariance and $\Pi_0 := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \notin \{\{0\}, \emptyset\}$, then $t \pm \tau(t) \in \mathbb{T}$ if $\Pi_1 \subseteq \Pi_0$. Note $\mathbb{T} \cap \Pi_0$ may be an empty set (see Example 1.2 from [15]). We observe that the period set of time scales are not only closely related to introducing special functions but also closely related to delays in delay dynamic equations.

Motivated by the above, under the algebraic structure of matched spaces of time scales, the closedness of non-translational shifts of time scales can be guaranteed. Therefore, by adopting the Definition of matched spaces, the time scale $\overline{q^{\mathbb{Z}}}$ will possess a perfect shift invariance. Moreover, the period set of a time scale under matched spaces may be completely separated from itself. We organize our paper in five sections. In Section 2, we introduce some preliminaries of matched spaces for time scales and the non-translational shift closedness of time scales is investigated. Section 3 is mainly devoted to introducing δ -almost periodic functions and establishing some related properties. In Section 4, under matched spaces, through the basic properties of δ -almost periodic functions, some fundamental results of homogeneous and nonhomogeneous dynamic equations are established in which a sufficient existence condition of solutions for a new delay dynamic equation is derived. Finally, we present a conclusion to end the paper.

2. Matched Spaces for Time Scales

In this section, we introduce some preliminaries of matched spaces for time scales.

Definition 1 ([25]). Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there exists a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$, and we call $f^\Delta(t)$ the delta (or Hilger) derivative of f at t .

For more details of dynamic equations on time scales, one may consult [25,26].

Let Π^* be a subset of \mathbb{R} together with a binary operation $\tilde{\delta}$ and a pair $(\Pi^*, \tilde{\delta})$ be an Abelian group and $\tilde{\delta}$ be increasing with respect to its second argument, i.e., Π^* and $\tilde{\delta}$ satisfy the following conditions:

- (1) Π^* is closed with respect to an operation $\tilde{\delta}$, i.e., for any $\tau_1, \tau_2 \in \Pi^*$, we have $\tilde{\delta}(\tau_1, \tau_2) \in \Pi^*$.
- (2) For any $\tau \in \Pi^*$, there exists an identity element $e_{\Pi^*} \in \Pi^*$ such that $\tilde{\delta}(e_{\Pi^*}, \tau) = \tau$.
- (3) For all $\tau_1, \tau_2, \tau_3 \in \Pi^*$, $\tilde{\delta}(\tau_1, \tilde{\delta}(\tau_2, \tau_3)) = \tilde{\delta}(\tilde{\delta}(\tau_1, \tau_2), \tau_3)$ and $\tilde{\delta}(\tau_1, \tau_2) = \tilde{\delta}(\tau_2, \tau_1)$.
- (4) For each $\tau \in \Pi^*$, there exists an element $\tau^{-1} \in \Pi^*$ such that $\tilde{\delta}(\tau, \tau^{-1}) = \tilde{\delta}(\tau^{-1}, \tau) = e_{\Pi^*}$, where e_{Π^*} is the identity element in Π^* .
- (5) If $\tau_1 > \tau_2$, then $\tilde{\delta}(\cdot, \tau_1) > \tilde{\delta}(\cdot, \tau_2)$.

Definition 2 ([27]). A subset S of \mathbb{R} is called relatively dense with respect to the pair $(\Pi^*, \tilde{\delta})$ if there exists a number $L \in \Pi^*$ and $L > e_{\Pi^*}$ such that $[a, \tilde{\delta}(a, L)]_{\Pi^*} \cap S \neq \emptyset$ for all $a \in \Pi^*$. The number L is called the inclusion length with respect to the group $(\Pi^*, \tilde{\delta})$.

To illustrate the Definition of matched spaces for time scales, a notion of an adjoint mapping between \mathbb{T} and Π is needed.

Definition 3 ([27]). Let \mathbb{T} and Π be time scales, where $\mathbb{T} = \bigcup_{i \in I_1} A_i$, $\Pi = \bigcup_{i \in I_2} B_i$. If Π^* is the largest subset of the time scale Π , i.e., $\overline{\Pi^*} = \Pi$, where \overline{A} denote the closure of the set A , and $(\Pi^*, \tilde{\delta})$ is an Abelian group, I_1, I_2 are countable index sets, then we say Π is an adjoint set of \mathbb{T} if there exists a bijective map:

$$F : \quad \mathbb{T} \quad \rightarrow \quad \Pi$$

$$A \in \{A_i, i \in I_1\} \rightarrow B \in \{B_i, i \in I_2\},$$

i.e., $F(A) = B$. Now F is called the adjoint mapping between \mathbb{T} and Π .

Definition 4 ([27]). Let the pair $(\Pi^*, \tilde{\delta})$ be an Abelian group and Π^*, \mathbb{T}^* be the largest subsets of the time scales Π and \mathbb{T} , respectively. Furthermore, let Π be a adjoint set of \mathbb{T} and F the adjoint mapping between \mathbb{T} and Π . The operator $\delta : \Pi^* \times \mathbb{T}^* \rightarrow \mathbb{T}^*$ satisfies the following properties:

(P₁) (Monotonicity) The function δ is strictly increasing with respect to its second argument, i.e., if

$$(T_0, t), (T_0, u) \in \mathcal{D}_\delta := \{(s, t) \in \Pi^* \times \mathbb{T}^* : \delta(s, t) \in \mathbb{T}^*\},$$

then $t < u$ implies $\delta(T_0, t) < \delta(T_0, u)$.

(P₂) (Existence of inverse elements) The operator δ has the inverse operator $\delta^{-1} : \Pi^* \times \mathbb{T}^* \rightarrow \mathbb{T}^*$ and $\delta^{-1}(\tau, t) = \delta(\tau^{-1}, t)$, where $\tau^{-1} \in \Pi^*$ is the inverse element of τ .

(P₃) (Existence of identity element) $e_{\Pi^*} \in \Pi^*$ and $\delta(e_{\Pi^*}, t) = t$ for any $t \in \mathbb{T}^*$, where e_{Π^*} is the identity element in Π^* .

(P₄) (Bridge condition) For any $\tau_1, \tau_2 \in \Pi^*$ and $t \in \mathbb{T}^*$, $\delta(\tilde{\delta}(\tau_1, \tau_2), t) = \delta(\tau_1, \delta(\tau_2, t)) = \delta(\tau_2, \delta(\tau_1, t))$.

Then the operator δ associated with $e_{\Pi^*} \in \Pi^*$ is said to be shift operator on the set \mathbb{T}^* . The variable $s \in \Pi^*$ in δ is called the shift size. The value $\delta(s, t)$ in \mathbb{T}^* indicates s units shift of the term $t \in \mathbb{T}^*$. The sets \mathcal{D}_δ is the domain of the shift operator δ .

Next we present the concept of matched spaces for time scales.

Definition 5 ([27]). Let the pair $(\Pi^*, \tilde{\delta})$ be an Abelian group and Π^*, \mathbb{T}^* be the largest subsets of the time scales Π and \mathbb{T} , respectively. Furthermore, let Π be a adjoint set of \mathbb{T} and F the adjoint mapping between \mathbb{T} and Π . If there exists the shift operator δ satisfying Definition 4, then the group $(\mathbb{T}, \Pi, F, \delta)$ is said to be a matched space for \mathbb{T} .

Remark 1. Definition 5 demonstrates that a matched space of a time scale is the group $(\mathbb{T}, \Pi, F, \delta)$.

Definition 6 ([27]). If $\mathbb{T} = \Pi$, then we say the group $(\mathbb{T}, \Pi, F, \delta)$ is a standard matched space.

Remark 2. Please note that the adjoint mapping F is a unit operator I for a standard matched space, denote it by $(\mathbb{T}, \Pi, F, \delta) := (\mathbb{T}, \delta)$. It is worth noticing that one can use the algebraic structure of a standard matched space (\mathbb{T}, δ) to obtain all the results from the literature [22].

Example 1. Let $\mathbb{T} = \overline{\bigcup_{n \in \mathbb{Z}} [2^{2n}, 2^{2n+1}]}$ and $\Pi = \overline{\{\tau : \tau = 2^{2n}, n \in \mathbb{Z}\}}$. $\delta(\tau, t) = \tau t$, $\tau = 2^{2n_0}$, $\forall n_0 \in \mathbb{Z}$. Thus, $\forall \tau_1, \tau_2 \in \Pi^* = \Pi \setminus \{0\}$, $\tilde{\delta}(\tau_1, \tau_2) = \tau_1 \cdot \tau_2$. Obviously, $\tilde{\delta}(\tau_1, \tau_2) \in \Pi^*$. There exists a bijective map

$$F : \quad \mathbb{T} \quad \rightarrow \quad \Pi$$

$$A \in \overline{\bigcup_{n \in \mathbb{Z}} [2^{2n}, 2^{2n+1}]} \rightarrow B \in \overline{\{\tau : \tau = 2^{2n}, n \in \mathbb{Z}\}}$$

$$A \quad \rightarrow \quad \text{the left point of the interval } A$$

$$A = 0 \quad \rightarrow \quad B = 0,$$

i.e., $F(A) = B$. Therefore, $(\mathbb{T}, \Pi, F, \delta)$ is a matched space for \mathbb{T} .

Example 2. Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ and $\Pi = \overline{q^{\mathbb{Z}}}$. $\delta(\tau, t) = \tau t, \tau = q^{n_0}, \forall n_0 \in \mathbb{Z}$. Thus, $\forall \tau_1, \tau_2 \in \Pi^* = \Pi \setminus \{0\}$, $\tilde{\delta}(\tau_1, \tau_2) = \tau_1 \cdot \tau_2$. Obviously, $\tilde{\delta}(\tau_1, \tau_2) \in \Pi^*$. There exists a bijective map

$$F : \begin{array}{ccc} \mathbb{T} & \rightarrow & \Pi \\ A \in \overline{\{q^n, n \in \mathbb{Z}\}} & \rightarrow & B \in \overline{\{q^n, n \in \mathbb{Z}\}} \\ A & \rightarrow & A = B, \end{array}$$

i.e., $F(A) = A$. Therefore, $(\mathbb{T}, \Pi, F, \delta)$ is a matched space for \mathbb{T} .

Remark 3. For a time scale \mathbb{T} with translations invariance, we can easily observe that the translation number set $\Pi = \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \notin \{\emptyset, \{0\}\}$ is a matched space for \mathbb{T} .

Using the algebraic structure of matched spaces, the following new concept of time scales with shift closedness can be introduced.

Definition 7 ([27]). A time scale \mathbb{T} is called a periodic time scale under a matched space $(\mathbb{T}, \Pi, F, \delta)$ if

$$\tilde{\Pi} := \{\tau \in \Pi^* : (\tau^{\pm 1}, t) \in \mathcal{D}_\delta, \forall t \in \mathbb{T}^*\} \notin \{e_{\Pi^*}, \emptyset\}. \tag{1}$$

Remark 4. From Definition 7, one should note that $\tilde{\Pi} \subseteq \Pi^* \subseteq \Pi$ and for every $\tau \in \tilde{\Pi}$, there exists $\tau^{-1} \in \tilde{\Pi}$, i.e., there exists an inverse element for every element in $\tilde{\Pi}$.

Remark 5. For any $t \in \mathbb{T}^*$, if $(\tau_2, t) \in \mathcal{D}_\delta, (\tau_1, t) \in \mathcal{D}_\delta$, then by Definition 4, there exists a function $\tilde{\delta} : \Pi^* \times \Pi^* \rightarrow \Pi^*$ such that $\tilde{\delta}(\tau_1, \tau_2) \in \Pi^*$. Obviously, if \mathbb{T} is a periodic time scale under Definition 1.1 from [28], i.e., $\tilde{\Pi} = \Pi^* = \Pi = \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \notin \{0, \emptyset\}$, we have $\tau_1, \tau_2 \in \tilde{\Pi}$, then $\tilde{\delta}(\tau_1, \tau_2) = \tau_1 + \tau_2 \in \tilde{\Pi}$.

Remark 6. From (1), we can observe $(\tau^{\pm 1}, t) \in \mathcal{D}_\delta$ implies that $\delta(\tau^{-1}, t) = \delta^{-1}(\tau, t)$ exists.

Example 3. Under matched spaces for time scales, the following time scales are periodic:

- (1) $\mathbb{T} = \overline{q^{\mathbb{Z}}}, \delta(\tau, t) = \tau t, \tau = q^{\pm 1}$.
- (2) $\mathbb{T} = \bigcup_{n \in \mathbb{Z}} [2^{2n}, 2^{2n+1}], \delta(\tau, t) = \tau t, \tau = 4^{\pm 1}$.
- (3) $\delta(\tau, t) = t + \tau, \tau = \pm(a + b)$.

For any $\tau \in \Pi^*$, we define a function $A : \Pi^* \rightarrow \Pi^*$,

$$A(\tau) = \begin{cases} \tilde{\delta}(\tau, e_{\Pi^*}), & \tau > e_{\Pi^*}, \\ \tilde{\delta}(\tau^{-1}, e_{\Pi^*}), & \tau < e_{\Pi^*}. \end{cases}$$

Under the matched space $(\mathbb{T}, F, \Pi, \delta)$, for convenience, the sub-timescale that the variable t belongs to is denoted by A_{i_t} . Let I_1 be an index set satisfying $\mathbb{T} = \bigcup_{i \in I_1} A_i$, obviously, $i_t \in I_1$. If $\delta_t^\Delta(t)$ exists and is bounded, then its upper bound is denoted by δ_B^Δ .

Definition 8 ([27]). Suppose the adjoint mapping $F : \mathbb{T} \rightarrow \Pi$ is continuous and satisfies:

- (1) for any $\tau \in \Pi^*, t_0 \in \mathbb{T}, F(\delta_\tau(A_{i_{t_0}})) = \tilde{\delta}(\tau, F(A_{i_{t_0}}))$ holds;
- (2) if $t_1, t_2 \in \mathbb{T}$ and $t_1 \leq t_2$, then $F(A_{i_{t_1}}) \leq F(A_{i_{t_2}})$.

We say $(\mathbb{T}, F, \Pi, \delta)$ a regular matched space for the time scale \mathbb{T} .

3. δ -Almost Periodic Functions under Matched Spaces

In this section, we will study δ -almost periodic functions under the matched space $(\mathbb{T}, \Pi, F, \delta)$.

For convenience, \mathbb{E}^n denotes \mathbb{R}^n or \mathbb{C}^n , an open set in \mathbb{E}^n is denoted by D or $D = \mathbb{E}^n$, and an arbitrary compact subset of D is denoted by S .

Definition 9 ([12]). Let \mathbb{T} be a periodic time scale under the matched space $(\mathbb{T}, \Pi, F, \delta)$. A function $f \in C(\mathbb{T} \times D, \mathbb{X})$ is called uniformly δ -almost periodic function with shift operators if the ε -shift set of f

$$E\{\varepsilon, f, S\} = \{\tau \in \tilde{\Pi} : \|f(\delta_{\tau \pm 1}(t), x) - f(t, x)\| < \varepsilon, \text{ for all } t \in \mathbb{T}^* \text{ and } x \in S\}$$

is a relatively dense set with respect to the pair $(\Pi^*, \tilde{\delta})$ for all $\varepsilon > 0$ and for each compact subset S of D ; that is, for any given $\varepsilon > 0$ and each compact subset S of D , there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that

$$\|f(\delta_{\tau}(t), x) - f(t, x)\| < \varepsilon, \text{ for all } t \in \mathbb{T}^* \text{ and } x \in S.$$

Now τ is called the ε -shift number of f and $l(\varepsilon, S)$ is called the inclusion length of $E\{\varepsilon, f, S\}$.

Definition 10 ([12]). Let \mathbb{T} be a periodic time scale under the matched space $(\mathbb{T}, \Pi, F, \delta)$ and $n_0 \in \mathbb{N}$, the shift $\delta_{\tau}(t)$ is Δ -differentiable with rd-continuous bounded derivatives $\delta_{\tau}^{\Delta}(t) := \delta^{\Delta}(\tau, t)$ for all $t \in \mathbb{T}^*$. A function $f \in C(\mathbb{T} \times D, \mathbb{X})$ is called an **n_0 -order Δ -almost periodic function** ($\Delta_{n_0}^{\delta}$ -almost periodic function) with shift operators in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε -shift set of f

$$E\{\varepsilon, f, S\} = \{\tau \in \tilde{\Pi} : \|f(\delta_{\tau}(t), x) (\delta_{\tau}^{\Delta}(t))^{n_0} - f(t, x)\| < \varepsilon, \text{ for all } t \in \mathbb{T}^* \text{ and } x \in S\}$$

is a relatively dense set with respect to the pair $(\Pi^*, \tilde{\delta})$ for all $\varepsilon > 0$ and for each compact subset S of D ; that is, for any given $\varepsilon > 0$ and each compact subset S of D , there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that

$$\|f(\delta_{\tau}(t), x) (\delta_{\tau}^{\Delta}(t))^{n_0} - f(t, x)\| < \varepsilon, \text{ for all } t \in \mathbb{T}^* \text{ and } x \in S. \tag{2}$$

Now τ is called the ε -shift number of f and $l(\varepsilon, S)$ is called the inclusion length of $E\{\varepsilon, f, S\}$.

An example of an Δ_1^{δ} -almost periodic function is provided below.

Example 4. Let $a \in \mathbb{R} \setminus \{0\}$, $f_1(t) = a/t$ and $\mathbb{T} = (\sqrt{5})^{\mathbb{Z}} = \{(\sqrt{5})^n, n \in \mathbb{Z}\}$, then $f_1(t)$ is Δ_1^{δ} -periodic under the matched space $(\mathbb{T}, \Pi, F, \delta)$ and the period $\tau = \sqrt{5}$. In fact,

$$f_1(\delta_{(\sqrt{5})^{\pm 1}}(t)) \delta_{(\sqrt{5})^{\pm 1}}^{\Delta}(t) = \frac{a}{(\sqrt{5})^{\pm 1}t} (\sqrt{5})^{\pm 1} = \frac{a}{t} = f_1(t).$$

Now, let $f_2(t) = \frac{b}{(-1)^{\log_{\sqrt{5}} t}}$. Similarly, take $\tau = (\sqrt{5})^2$, we can obtain

$$\begin{aligned} f_2(\delta_{(\sqrt{5})^{\pm 2}}(t))(\delta_{(\sqrt{5})^{\pm 2}})^{\Delta}(t) &= \frac{b}{(-1)^{\log_{\sqrt{5}}(\sqrt{5})^{\pm 2}t} \cdot (\sqrt{5})^{\pm 2}t} \cdot (\sqrt{5})^{\pm 2} = \frac{b}{(-1)^{\pm 2 + \log_{\sqrt{5}} t} \cdot t} \\ &= \frac{b}{(-1)^{\log_{\sqrt{5}} t}} = f_2(t). \end{aligned}$$

Hence, we can obtain $\tilde{F}(t) = f_1(t) + f_2(t)$ which is an Δ_1^δ -almost periodic function under the matched space $(\mathbb{T}, \Pi, F, \delta)$. Please note that the periods of f_1 and f_2 are completely different.

In what follows, we introduce some notations. The set of all δ -almost periodic functions in shifts on \mathbb{T} is denoted by $AP^\delta(\mathbb{T})$. Let $\alpha = \{\alpha_n\} \subset \tilde{\Pi}$ and $\beta = \{\beta_n\} \subset \tilde{\Pi}$ be two sequences. Then $\beta \subset \alpha$ means that β is a subsequence of α ; $\tilde{\delta}(\alpha, \beta) = \{\tilde{\delta}(\alpha_n, \beta_n)\}$; $\alpha^{-1} = \{\alpha_n^{-1}\}$, where $\tilde{\delta}(\alpha, \alpha^{-1}) = e_{\mathbb{T}^*}$; $\alpha \subset \alpha'$ and $\beta \subset \beta'$ are common subsequences, implies that there is a function $n(k)$ such that $\alpha_n = \alpha'_{n(k)}$ and $\beta_n = \beta'_{n(k)}$.

When the following limit exists, we introduce the moving-operator $T^\delta, T_\alpha^\delta f(t, x) = g(t, x)$ by

$$g(t, x) = \lim_{n \rightarrow +\infty} f(\delta_{\alpha_n}(t), x).$$

The convergent modes such as pointwise, uniform, etc., will be stressed at each use of this symbol. Next, some fundamental properties of δ -almost periodic functions will be demonstrated.

Theorem 1. Assume that $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$ is uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$, and $\delta_\tau(t)$ is continuous in t . Then it is uniformly continuous and bounded on $\mathbb{T}^* \times S$.

Proof. For a given $\varepsilon \leq 1$ and some compact set $S \subset D$, there exists an inclusion length $l(\varepsilon, S)$, for any interval with length l , there exists $\tau \in E\{\varepsilon, f, S\}$ such that

$$|f(\delta_\tau(t), x) - f(t, x)| < \varepsilon \leq 1, \quad \text{for all } (t, x) \in \mathbb{T}^* \times S.$$

Since $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$, for any $(t, x) \in ([t_0, \delta(l, t_0)]_{\mathbb{T}}) \times S, t_0 \in \mathbb{T}^*$, there exists an $M > 0$ such that $|f(t, x)| < M$. For any given $t \in \mathbb{T}^*$, take $\tau \in E(\varepsilon, f, S) \cap [F(A_{i_t}), \tilde{\delta}(l, F(A_{i_t}))]_{\tilde{\Pi}'}$ then $\delta_\tau(t) \in [\delta(F(A_{i_t}), t), \delta(\tilde{\delta}(l, F(A_{i_t})), t)]_{\mathbb{T}}$, i.e., $\delta_\tau(t) \in [\delta(F(A_{i_t}), t), \delta(l, \delta(F(A_{i_t}), t))]_{\mathbb{T}}$. Hence, for $x \in S$, we have

$$|f(\delta_\tau(t), x)| < M \quad \text{and} \quad |f(\delta_\tau(t), x) - f(t, x)| < 1.$$

Thus for all $(t, x) \in \mathbb{T}^* \times S$, we have $|f(t, x)| < M + 1$.

Moreover, for any $\varepsilon > 0$, let $l_1 = l_1\left(\frac{\varepsilon}{3}, S\right)$ be an inclusion length of $E\left(\frac{\varepsilon}{3}, f, S\right)$. There is a proper point $t_0 \in \mathbb{T}^*$ such that $f(t, x)$ is uniformly continuous on $([t_0, \delta(l_1, t_0)]_{\mathbb{T}}) \times S$. Hence, there exists a positive constant $\delta^* = \delta^*\left(\frac{\varepsilon}{3}, S\right)$, for any $t_1, t_2 \in [t_0, \delta(l_1, t_0)]_{\mathbb{T}}$ and $|t_1 - t_2| < \delta^*$,

$$|f(t_1, x) - f(t_2, x)| < \frac{\varepsilon}{3} \quad \text{for all } x \in S.$$

Next, one can select an arbitrary $v, t \in \mathbb{T}^*$ with $|t - v| < \delta^*$, and we choose

$$\tau \in E\left(\frac{\varepsilon}{3}, f, S\right) \cap [F(A_{i_t}), \tilde{\delta}(l_1, F(A_{i_t}))]_{\tilde{\Pi}},$$

then $\delta_\tau(t), \delta_\tau(v) \in [\delta(F(A_{i_t}), t), \delta(l_1, \delta(F(A_{i_t}), t))]_{\mathbb{T}}$. Since $\delta_\tau(t)$ is continuous in t , then there exists $\delta^{**} > 0$ such that $|t - v| < \delta^{**}$ implies

$$|\delta_\tau(t) - \delta_\tau(v)| < \delta^*,$$

Now, we take $\delta_{**} = \min\{\delta^*, \delta^{**}\}$, and then $|t - v| < \delta_{**}$ implies

$$|f(\delta_\tau(t), x) - f(\delta_\tau(v), x)| < \frac{\varepsilon}{3} \quad \text{for all } x \in S.$$

Therefore, for $(t, x) \in \mathbb{T}^* \times S$, we have

$$|f(t, x) - f(v, x)| \leq |f(t, x) - f(\delta_\tau(t), x)| + |f(\delta_\tau(t), x) - f(\delta_\tau(v), x)| + |f(\delta_\tau(v), x) - f(v, x)| < \varepsilon.$$

The proof is complete. \square

The following theorem is about shift-convergence of δ -almost periodic functions.

Theorem 2. Assume that $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$ is uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$. Then for an arbitrary sequence $\alpha' \subset \tilde{\Pi}$, there is a subsequence $\beta \subset \alpha'$ and $g \in C(\mathbb{T} \times D, \mathbb{E}^n)$ such that $T_{\beta}^{\delta} f(t, x) = g(t, x)$ holds uniformly on $\mathbb{T}^* \times S$. Furthermore, $g(t, x)$ is uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$.

Proof. For any given $\varepsilon > 0$ and $S \subset D$, there exists a positive constant $l = l\left(\frac{\varepsilon}{4}, S\right)$ as an inclusion length of $E\left\{\frac{\varepsilon}{4}, f, S\right\}$. Since there exists $\tau'_n \in E\left\{\frac{\varepsilon}{4}, f, S\right\}$ in any interval with length of l , then for any subsequence $\alpha' = \{\alpha'_n\} \subset \tilde{\Pi}$, one can select a suitable interval with length of l such that $e_{\Pi^*} \leq \tilde{\delta}(\alpha'_n, (\tau'_n)^{-1}) \leq l$ and $\gamma'_n = \tilde{\delta}(\alpha'_n, (\tau'_n)^{-1}) \in \tilde{\Pi}$. Hence, there exists $\tau'_n \in E\left\{\frac{\varepsilon}{4}, f, S\right\}$ and $\gamma'_n \in \tilde{\Pi}$ with $e_{\Pi^*} \leq \gamma'_n \leq l, n = 1, 2, \dots$ such that $\alpha'_n = \tilde{\delta}(\tau'_n, \gamma'_n)$. Moreover, because $\{\gamma'_n\}$ is bounded, there exists a subsequence $\gamma = \{\gamma_n\} \subset \gamma' = \{\gamma'_n\}$ such that $\gamma_n \rightarrow s$ as $n \rightarrow \infty$ and $e_{\Pi^*} \leq s \leq l$.

According to Theorem 1, one has $f(t, x)$ is uniformly continuous on $\mathbb{T}^* \times S$. Hence, there exists $\delta^*(\varepsilon, S) > 0$, and when $|t_1 - t_2| < \delta^*$ we obtain

$$|f(\delta_\tau(t_1), x) - f(\delta_\tau(t_2), x)| < \frac{\varepsilon}{2}, \quad \forall x \in S.$$

Since γ is convergent, there exists $N = N(\delta)$, when $p, m \geq N$ one has $|\gamma_p - \gamma_m| < \delta^*$. Now, we can choose $\alpha \subset \alpha', \tau \subset \tau' = \{\tau'_n\}$ such that α, τ common with γ , then for any integers $p, m \geq N$, we can obtain

$$\begin{aligned} |f(\delta_{\tilde{\delta}(\tau_p, \tau_m^{-1})}(t), x) - f(t, x)| &\leq |f(\delta_{\tilde{\delta}(\tau_p, \tau_m^{-1})}(t), x) - f(\delta_{\tau_p}(t), x)| + |f(\delta_{\tau_p}(t), x) - f(t, x)| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \end{aligned}$$

For $\tilde{\delta}(\alpha'_n, (\gamma'_n)^{-1}) = \tau'_n$, we obtain

$$\tilde{\delta}(\tilde{\delta}(\alpha_p, \alpha_m^{-1}), \tilde{\delta}^{-1}(\gamma_p, \gamma_m^{-1})) = \tilde{\delta}(\tilde{\delta}(\alpha_p, \gamma_p^{-1}), \tilde{\delta}^{-1}(\alpha_m, \gamma_m^{-1})) = \tilde{\delta}(\tau_p, \tau_m^{-1}) \in E\left\{\frac{\varepsilon}{2}, f, S\right\}.$$

Hence, we get

$$\begin{aligned} |f(\delta_{\alpha_p}(t), x) - f(\delta_{\alpha_m}(t), x)| &\leq \sup_{(t,x) \in \mathbb{T}^* \times S} |f(\delta_{\alpha_p}(t), x) - f(\delta_{\alpha_m}(t), x)| \\ &\leq \sup_{(t,x) \in \mathbb{T}^* \times S} |f(\delta_{\tilde{\delta}(\alpha_p, \alpha_m^{-1})}(t), x) - f(t, x)| \\ &\leq \sup_{(t,x) \in \mathbb{T}^* \times S} |f(\delta_{\tilde{\delta}(\alpha_p, \alpha_m^{-1})}(t), x) - f(\delta_{\tilde{\delta}(\gamma_p, \gamma_m^{-1})}(t), x)| \\ &\quad + \sup_{(t,x) \in \mathbb{T}^* \times S} |f(\delta_{\tilde{\delta}(\gamma_p, \gamma_m^{-1})}(t), x) - f(t, x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, one can select sequences $\alpha^{(k)} = \{\alpha_n^{(k)}\}, k = 1, 2, \dots$, and $\alpha^{(k+1)} \subset \alpha^{(k)} \subset \alpha$ such that

$$|f(\delta_{\alpha_p^{(k)}}(t), x) - f(\delta_{\alpha_m^{(k)}}(t), x)| < \frac{1}{k}, k = 1, 2, \dots$$

for any integers m, p , and all $(t, x) \in \mathbb{T}^* \times S$. Moreover, for each sequence $\alpha^{(k)}, k = 1, 2, \dots$, we can choose a sequence $\beta = \{\beta_n\}, \beta_n = \alpha_n^{(n)}$, and then for any integers p, m with $p < m$, one has $\{f(\delta_{\beta_n}(t), x)\} \subset \{f(\delta_{\alpha_n}(t), x)\}$ and

$$|f(\delta_{\beta_p}(t), x) - f(\delta_{\beta_m}(t), x)| < \frac{1}{p}, \forall (t, x) \in \mathbb{T}^* \times S.$$

Therefore, the sequence $\{f(\delta_{\beta_n}(t), x)\}$ is convergent uniformly on $\mathbb{T}^* \times S$, i.e., $T_{\beta}^{\delta} f(t, x) = g(t, x)$ holds uniformly on $\mathbb{T}^* \times S$, where $\beta = \{\beta_n\} \subset \alpha$.

In the following, we will show that $g(t, x)$ is continuous on $\mathbb{T}^* \times D$. Assume it is not true, so there will be $(t_0, x_0) \in \mathbb{T}^* \times D$ such that $g(t, x)$ is discontinuous at (t_0, x_0) . Then there exist $\varepsilon_0 > 0$ and sequences $\{\delta_m^*\}, \{t_m\}, \{x_m\}$, where $\delta_m^* > 0, \delta_m^* \rightarrow 0$ as $m \rightarrow +\infty, |t_0 - t_m| + |x_0 - x_m| < \delta_m^*$ and

$$|g(t_0, x_0) - g(t_m, x_m)| \geq \varepsilon_0. \tag{3}$$

Let $X = \{x_m\} \cup \{x_0\}$. It is easy to observe that X is a compact subset of D . Hence, for all $m \in \mathbb{Z}^+$, there is a positive integer $N = N(\varepsilon_0, X)$, and when $n > N$ we have

$$|f(\delta_{\beta_n}(t_m), x_m) - g(t_m, x_m)| < \frac{\varepsilon_0}{3} \tag{4}$$

and

$$|f(\delta_{\beta_n}(t_0), x_0) - g(t_0, x_0)| < \frac{\varepsilon_0}{3}. \tag{5}$$

Furthermore, from the uniform continuity of $f(t, x)$ on $\mathbb{T}^* \times D$, there exists a sufficiently large m such that

$$|f(\delta_{\beta_n}(t_0), x_0) - f(\delta_{\beta_n}(t_m), x_m)| < \frac{\varepsilon_0}{3}. \tag{6}$$

It follows from (4)–(6) that

$$|g(t_0, x_0) - g(t_m, x_m)| < \varepsilon_0,$$

which contradicts (3). Therefore, $g(t, x)$ is continuous on $\mathbb{T}^* \times D$.

Now, we will show $E\{\varepsilon, g, S\}$ is relatively dense. In fact, for any compact set $S \subset D$ and given $\varepsilon > 0$, one can select $\tau \in E\{\varepsilon, f, S\}$ such that

$$|f(\delta_{\beta_n}(\delta_\tau(t)), x) - f(\delta_{\beta_n}(t), x)| < \varepsilon, \forall (t, x) \in \mathbb{T}^* \times S.$$

Letting $n \rightarrow +\infty$, we can obtain

$$|g(\delta_\tau(t), x) - g(t, x)| \leq \varepsilon, \forall (t, x) \in \mathbb{T}^* \times S,$$

which indicates that $E\{\varepsilon, g, S\}$ is relatively dense. Therefore, $g(t, x)$ is uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$. This completes the proof. \square

A sequentially compact criterion of δ -almost periodic functions is given below.

Theorem 3. Assume $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$ and for arbitrary sequence $\alpha' \subset \tilde{\Pi}$, there is $\alpha \subset \alpha'$ such that $T_\alpha^\delta f(t, x)$ exists uniformly on $\mathbb{T}^* \times S$. Then $f(t, x)$ is an uniformly δ -almost periodic function under the matched space $(\mathbb{T}, F, \Pi, \delta)$.

Proof. We will give the proof by contradiction. Assume it is not true. Then there exist $\varepsilon_0 > 0$ and $S \subset D$ such that for any sufficiently large $l > 0$, there is an interval with the length of l in which there is no ε_0 -shift numbers of $f(t, x)$, that is, there is no intersection between the whole interval and $E\{\varepsilon_0, f, S\}$.

First, we select a number $\alpha'_1 \in \tilde{\Pi}$ and an interval (a_1, b_1) with $b_1 - a_1 > 2|\alpha'_1|$ and $a_1, b_1 \in \tilde{\Pi}$ so that there is no ε_0 -shift numbers of $f(t, x)$ in this interval. Then, choosing $\alpha'_2 \in (\tilde{\delta}(\alpha'_1, a_1), \tilde{\delta}(\alpha'_1, b_1))$, it is easy to observe that $\tilde{\delta}(\alpha'_2, (\alpha'_1)^{-1}) \in (a_1, b_1)$, and we obtain $\tilde{\delta}(\alpha'_2, (\alpha'_1)^{-1}) \notin E\{\varepsilon_0, f, S\}$. Next, we can choose an interval (a_2, b_2) with $b_2 - a_2 > 2(|\alpha'_1| + |\alpha'_2|)$ and $a_2, b_2 \in \tilde{\Pi}$ such that there is no ε_0 -shift numbers of $f(t, x)$ in this interval. Then, selecting $\alpha'_3 \in (\tilde{\delta}(\alpha'_2, a_2), \tilde{\delta}(\alpha'_2, b_2)) \cap (\tilde{\delta}(\alpha'_1, a_2), \tilde{\delta}(\alpha'_1, b_2))$, we can easily observe that $\tilde{\delta}(\alpha'_3, (\alpha'_2)^{-1}), \tilde{\delta}(\alpha'_3, (\alpha'_1)^{-1}) \notin E\{\varepsilon_0, f, S\}$. Now, we can repeat this process to select $\alpha'_4, \alpha'_5, \dots$, such that $\tilde{\delta}(\alpha'_i, (\alpha'_j)^{-1}) \notin E\{\varepsilon_0, f, S\}, i > j$. Hence, for any $i \neq j, i, j = 1, 2, \dots$, without loss of generality, letting $i > j$, for $x \in S$ we can obtain

$$\sup_{(t,x) \in \mathbb{T}^* \times S} |f(\delta_{\alpha'_i}(t), x) - f(\delta_{\alpha'_j}(t), x)| = \sup_{(t,x) \in \mathbb{T}^* \times S} |f(\delta_{\tilde{\delta}(\alpha'_i, (\alpha'_j)^{-1})}(t), x) - f(t, x)| \geq \varepsilon_0.$$

Hence, there does not exist a subsequence with uniform convergence in $\{f(\delta_{\alpha'_n}(t), x)\}$ for $(t, x) \in \mathbb{T}^* \times S$, which is a contradiction. Therefore, $f(t, x)$ is uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$. This completes the proof. \square

From Theorems 2 and 3, the following equivalent concept of uniformly δ -almost periodic functions is immediate.

Definition 11. Assume $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$ and for arbitrary sequence $\alpha' \subset \tilde{\Pi}$, there is a subsequence $\alpha \subset \alpha'$ such that $T_\alpha^\delta f(t, x)$ exists uniformly on $\mathbb{T}^* \times S$. Then $f(t, x)$ is said to be an uniformly δ -almost periodic function under the matched space $(\mathbb{T}, F, \Pi, \delta)$.

Theorem 4. Assume that $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$ is an uniformly δ -almost periodic function and $\varphi : \mathbb{T} \rightarrow S$ is δ -almost periodic. Then $f(\cdot, \varphi(\cdot)) : \mathbb{T} \rightarrow \mathbb{E}^n$ is a δ -almost periodic function.

Proof. For any given sequence $\alpha' \subset \tilde{\Pi}$, there exists $\alpha \subset \alpha'$, $\psi(t)$, $g(t, x)$ such that $T_\alpha^\delta \varphi(t) = \psi(t)$ exists uniformly on \mathbb{T}^* and $T_\alpha^\delta f(t, x) = g(t, x)$ exists uniformly on $\mathbb{T}^* \times S$, where $\psi(t)$ is δ -almost periodic and $g(t, x)$ is uniformly δ -almost periodic. Therefore, $g(t, x)$ is uniformly continuous on $\mathbb{T}^* \times S$, so for any given $\varepsilon > 0$, there exists $\delta^* \left(\frac{\varepsilon}{2}\right) > 0$ such that for any $x_1, x_2 \in S$ and all $t \in \mathbb{T}^*$, when $|x_1 - x_2| < \delta^* \left(\frac{\varepsilon}{2}\right)$, we have

$$|g(t, x_1) - g(t, x_2)| < \frac{\varepsilon}{2}.$$

For a sufficiently large $N_0(\varepsilon) > 0$ so that $n \geq N_0(\varepsilon)$ we have

$$|f(\delta_{\alpha_n}(t), x) - g(t, x)| < \frac{\varepsilon}{2}, \quad \forall (t, x) \in \mathbb{T}^* \times S,$$

$$|\varphi(\delta_{\alpha_n}(t)) - \psi(t)| < \delta^* \left(\frac{\varepsilon}{2}\right), \quad \forall t \in \mathbb{T}^*,$$

where $\{\varphi(\delta_{\alpha_n}(t)) : t \in \mathbb{T}^*\} \subset S$, $\{\psi(t) : t \in \mathbb{T}^*\} \subset S$. Therefore, when $n \geq N_0(\varepsilon)$, we have

$$\begin{aligned} |T_\alpha^\delta f(t, \varphi(t)) - g(t, \psi(t))| &= |f(\delta_{\alpha_n}(t), \varphi(\delta_{\alpha_n}(t))) - g(t, \psi(t))| \leq |f(\delta_{\alpha_n}(t), \varphi(\delta_{\alpha_n}(t))) - g(t, \varphi(\delta_{\alpha_n}(t)))| \\ &\quad + |g(t, \varphi(\delta_{\alpha_n}(t))) - g(t, \psi(t))| < \varepsilon. \end{aligned}$$

Please note that

$$\lim_{n \rightarrow \infty} f(\delta_{\alpha_n}(t), \varphi(\delta_{\alpha_n}(t))) = T_\alpha^\delta (f(t, \varphi(t))),$$

thus, $T_\alpha^\delta f(t, \varphi(t)) = g(t, \psi(t))$ exists uniformly on $\mathbb{T}^* \times S$. Thus, $f(t, \varphi(t))$ is δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$. The proof is complete. \square

Definition 12. Let $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$. Then $H_\delta(f) = \{g(t, x) : \mathbb{T} \rightarrow \mathbb{E}^n \mid \text{there is } \alpha \in \tilde{\Pi} \text{ such that } T_\alpha^\delta f(t, x) = g(t, x) \text{ exists uniformly on } \mathbb{T}^* \times S\}$ is said to be the δ -hull of $f(t, x)$ under the matched space $(\mathbb{T}, F, \Pi, \delta)$.

Theorem 5. $H_\delta(f)$ is a compact set iff $f(t, x)$ is an uniformly δ -almost periodic function.

Proof. If $H_\delta(f)$ is compact, for any given $\alpha' = \{\alpha'_n\} \subseteq \tilde{\Pi}$, there must exist subsequence $\{f(\delta_{\alpha_{n_k}}(t), x)\}_{k \in \mathbb{N}}$ of the sequence $\{f(\delta_{\alpha'_n}(t), x)\}_{n \in \mathbb{N}}$ such that

$$f(\delta_{\alpha_{n_k}}(t), x) \rightarrow g(t, x) \quad (k \rightarrow \infty), \forall (t, x) \in \mathbb{T}^* \times S.$$

Noting that $\alpha = \{\alpha_{n_k}\}$, obviously, $\alpha \subset \alpha'$, so $T_\alpha^\delta f(t, x)$ exists uniformly on $\mathbb{T}^* \times S$.

Conversely, if $f(t, x)$ is uniformly δ -almost periodic and $\{g_n(t, x)\} \subset H_\delta(f)$, then we can choose $\alpha' = \{\alpha'_n\}$ such that

$$\|f(\delta_{\alpha'_n}(t), x) - g_n(t, x)\| < \frac{1}{n}, \quad \forall (t, x) \in \mathbb{T}^* \times S.$$

One can choose $\alpha \subset \alpha'$ so that $T_\alpha^\delta f(t, x)$ exists uniformly. Let $\beta \subset \gamma = \{n\}$ such that β and α are common subsequences, then

$$f(\delta_{\alpha_n}(t), x) - g_{\beta_n}(t, x) \rightarrow 0 \quad (n \rightarrow \infty), \quad \forall (t, x) \in \mathbb{T}^* \times S,$$

so that

$$\lim_{n \rightarrow \infty} g_{\beta_n}(t, x) = T_\alpha^\delta f(t, x) \in H_\delta(f), \forall (t, x) \in \mathbb{T}^* \times S,$$

and thus, $H_\delta(f)$ is compact. The proof is complete. \square

Theorem 6. Assume $f(t, x)$ is uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$. Then for any $g \in H_\delta(f)$ we have $H_\delta(f) = H_\delta(g)$.

Proof. For any $h(t, x) \in H_\delta(g)$ there exists $\alpha' \subseteq \tilde{\Pi}$ such that $T_{\alpha'}^\delta g(t, x) = h(t, x)$. Since $f(t, x)$ is uniformly δ -almost periodic, from the sequence $\{\alpha'_n\} \subseteq \tilde{\Pi}$ one can extract a sequence $\{\alpha_n\}$ such that $T_{\alpha_n}^\delta f(t, x) = \lim_{n \rightarrow \infty} f(\delta_{\alpha_n}(t), x)$ exists uniformly on $\mathbb{T}^* \times S$.

For $g \in H_\delta(f)$, there is $\alpha^{(1)} \subseteq \tilde{\Pi}$ such that

$$\lim_{n \rightarrow +\infty} f(\delta_{\alpha_n^{(1)}}(t), x) = g(t, x), \quad \forall (t, x) \in \mathbb{T}^* \times S,$$

so we have

$$\lim_{n \rightarrow +\infty} f(\delta_{\tilde{\delta}(\alpha_n^{(1)}, \alpha_n)}(t), x) = g(\delta_{\alpha_n}(t), x), \quad \forall (t, x) \in \mathbb{T}^* \times S,$$

and then we can take $\beta = \{\beta_n\} = \{\tilde{\delta}(\alpha_n^{(1)}, \alpha_n)\}$ such that

$$|f(\delta_{\beta_n}(t), x) - g(\delta_{\alpha_n}(t), x)| < \frac{1}{n}, \quad \forall (t, x) \in \mathbb{T}^* \times S.$$

It follows that $T_\beta^\delta f(t, x) = T_\alpha^\delta g(t, x) = T_{\alpha'}^\delta g(t, x) = h(t, x)$. Hence $h \in H_\delta(f)$. Thus, $H_\delta(g) \subseteq H_\delta(f)$.

On the other hand, for any $g \in H_\delta(f)$, there exists α such that $T_\alpha^\delta f(t, x) = g(t, x)$, then

$$|f(\delta_{\alpha_n}(t), x) - g(t, x)| \rightarrow 0, \quad n \rightarrow \infty, \forall (t, x) \in \mathbb{T}^* \times S,$$

so making the change of variable $\delta_{\alpha_n}(t) = s$, one has

$$|f(s, x) - g(\delta_{\alpha_n^{-1}}(s), x)| \rightarrow 0, \quad \forall (s, x) \in \mathbb{T}^* \times S,$$

that is $T_{\alpha_n^{-1}}^\delta g(t, x) = f(t, x)$. Thus, $f(t, x) \in H_\delta(g)$ and so by what was shown above, $H_\delta(f) \subseteq H_\delta(g)$.

According to the above, we have $H_\delta(f) = H_\delta(g)$. The proof is complete. \square

From Definition 12 and Theorem 6, the result below is obvious.

Theorem 7. Assume $f(t, x)$ is an uniformly δ -almost periodic function. Then for any $g \in H_\delta(f)$, $g(t, x)$ is an uniformly δ -almost periodic function under the matched space $(\mathbb{T}, F, \Pi, \delta)$.

Theorem 8. Assume that $f(t, x)$ is uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$. Then for all $\varepsilon > 0$, there exist constants $\eta > 0$ and $\alpha \in \tilde{\Pi}$ such that $[\alpha, \tilde{\delta}(\alpha, \eta)]_{\Pi^*} \subset [a, \tilde{\delta}(a, L)]_{\Pi^*}$ and $[\alpha, \tilde{\delta}(\alpha, \eta)]_{\Pi^*} \subset E(\varepsilon, f, S)$, where $L(\varepsilon, S) > 0$ and $a \in \tilde{\Pi}$.

Proof. From the uniform continuity of $f(t, x)$ on $\mathbb{T}^* \times S$, we have that for any given $\varepsilon > 0$, there exists $\delta^*(\varepsilon_1, S) > 0$, so when $|t_1 - t_2| < \delta^*(\varepsilon_1, S)$ we obtain

$$|f(t_1, x) - f(t_2, x)| < \varepsilon_1, \quad \forall x \in S,$$

where $\varepsilon_1 = \frac{\varepsilon}{2}$.

Now, let $l(\varepsilon_1, S)$ be the inclusion length of $E(\varepsilon_1, f, S)$. We choose $\eta = \delta^*\left(\frac{\varepsilon}{2}, S\right) = \delta^*(\varepsilon_1, S)$ and $L = \tilde{\delta}(l(\varepsilon_1, S), \eta)$. For an arbitrary $a \in \tilde{\Pi}$, considering an interval $[a, \tilde{\delta}(a, L)]_{\Pi^*}$ and selecting

$$\tau \in E(f, \varepsilon_1, S) \cap [\tilde{\delta}(a, \eta_*), \tilde{\delta}(\eta_*^{-1}, \tilde{\delta}(a, l(\varepsilon_1, S)))]_{\Pi^*},$$

where $\eta_* > e_{\Pi^*}$ satisfying $\tilde{\delta}(\eta_*, \eta_*) \leq \frac{\eta}{2}$ (in fact, we have $\eta_* - \eta_*^{-1} \leq \eta_* < \tilde{\delta}(\eta_*, \eta_*) \leq \frac{\eta}{2}$), so we obtain

$$[\tilde{\delta}(\tau, \eta_*^{-1}), \tilde{\delta}(\tau, \eta_*)]_{\Pi^*} \subset [a, \tilde{\delta}(a, L)]_{\Pi^*}.$$

Thus, for any $\xi \in [\tilde{\delta}(\tau, \eta_*^{-1}), \tilde{\delta}(\tau, \eta_*)]_{\Pi^*}$, we can obtain

$$|\xi - \tau| \leq |\tilde{\delta}(\tau, \eta_*^{-1}) - \tilde{\delta}(\tau, \eta_*)| \leq 2\tilde{\delta}(\eta_*, \eta_*) \leq \eta.$$

Hence, for all $(t, x) \in \mathbb{T}^* \times S$,

$$|f(\delta_\xi(t), x) - f(t, x)| \leq |f(\delta_\xi(t), x) - f(\delta_\tau(t), x)| + |f(\delta_\tau(t), x) - f(t, x)| \leq \varepsilon.$$

Now, by taking $\alpha = \tilde{\delta}(\tau, \eta_*^{-1})$, we have $[\alpha, \tilde{\delta}(\alpha, \eta)]_{\Pi^*} \subset E(\varepsilon, f, S)$. This completes the proof. \square

In the following theorem, for $\eta \in \Pi^*$, we will use the notation $\tilde{\delta}_\eta = \tilde{\delta}(\eta, e_{\Pi^*}) = \eta$, $\tilde{\delta}_{2\eta} = \tilde{\delta}(\eta, \tilde{\delta}_\eta)$, $\tilde{\delta}_{3\eta} = \tilde{\delta}(\eta, \tilde{\delta}_{2\eta}), \dots, \tilde{\delta}_{m\eta} := \tilde{\delta}(\eta, \tilde{\delta}_{(m-1)\eta})$ for simplicity.

Theorem 9. Assume that f, g are uniformly δ -almost periodic functions under the matched space $(\mathbb{T}, F, \Pi, \delta)$. Then $E(f, \varepsilon, S) \cap E(g, \varepsilon, S)$ is relatively dense in Π^* and nonempty for any $\varepsilon > 0$.

Proof. Because f, g are uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$, it follows from Theorem 1 that f, g are uniformly continuous on $\mathbb{T}^* \times S$. For any given $\varepsilon > 0$, we choose $\delta_i^* = \delta_i^*\left(\frac{\varepsilon}{2}, S\right)$ ($i = 1, 2$); and select $l_1 = l_1\left(\frac{\varepsilon}{2}, S\right), l_2 = l_2\left(\frac{\varepsilon}{2}, S\right)$ as the inclusion lengths of $E(f, \frac{\varepsilon}{2}, S), E(g, \frac{\varepsilon}{2}, S)$, respectively.

From Theorem 5, one can choose

$$\eta = \eta(\varepsilon, S) = \min(\delta_1^*, \delta_2^*) \in \tilde{\Pi}, L_i = \tilde{\delta}(l_i, \eta) \ (i = 1, 2), L = \max(L_1, L_2).$$

Hence, one can select $\frac{\varepsilon}{2}$ -shift numbers of $f(t, x)$ and $g(t, x)$: $\tau_1 = \tilde{\delta}_{m\eta}$ and $\tau_2 = \tilde{\delta}_{n\eta}$, respectively, and $\tau_1, \tau_2 \in [a, \tilde{\delta}(a, L)]_{\Pi^*}, m, n \in \mathbb{Z}$. It follows from $\tau_2^{-1} \in [\tilde{\delta}(a^{-1}, L^{-1}), a^{-1}]_{\Pi^*}$ that

$$\tilde{\delta}(\tilde{\delta}(a^{-1}, L^{-1}), a) \leq \tilde{\delta}(\tau_1, \tilde{\delta}(a^{-1}, L^{-1})) \leq \tilde{\delta}(\tau_1, \tau_2^{-1}) \leq \tilde{\delta}(\tau_1, a^{-1}) \leq \tilde{\delta}(a^{-1}, \tilde{\delta}(a, L)),$$

so we obtain $A(\tilde{\delta}(\tau_1, \tau_2^{-1})) \leq L$. Let $m - n = s$, then s can only be chosen from a finite number set $\{s_1, s_2, \dots, s_p\}$. Without loss of generality, we assume $m - n = s_j, j = 1, 2, \dots, p$, and the $\frac{\varepsilon}{2}$ -shift numbers of $f(t)$ and $g(t)$ are denoted by τ_1^j, τ_2^j , respectively. It is obvious that $\tilde{\delta}(\tau_1^j, (\tau_2^j)^{-1}) = \tilde{\delta}_{s_j\eta}, j = 1, 2, \dots, p$, and one can select $T = \max_j \{A(\tau_1^j), A(\tau_2^j)\}$.

For any $a \in \Pi^*$, one can choose $\frac{\varepsilon}{2}$ -shift numbers τ_1, τ_2 of f and g from the interval $[\tilde{\delta}(a, T), \tilde{\delta}(\tilde{\delta}(a, T), L)]$, and then there is some integer s_j such that

$$\tilde{\delta}(\tau_1, \tau_2^{-1}) = \tilde{\delta}_{s_j, \eta} = \tilde{\delta}(\tau_1^j, (\tau_2^j)^{-1}).$$

Let

$$\tau(\varepsilon, S) = \tilde{\delta}(\tau_1, (\tau_1^j)^{-1}) = \tilde{\delta}(\tau_2, (\tau_2^j)^{-1}),$$

then $\tau(\varepsilon, S) \in [a, \tilde{\delta}(a, \tilde{\delta}(L, \tilde{\delta}(T, T)))]_{\Pi^*}$. Hence, for any $(t, x) \in \mathbb{T}^* \times S$, we obtain

$$\begin{aligned} |f(\delta_\tau(t), x) - f(t, x)| &\leq |f(\delta_{\tilde{\delta}(\tau_1, (\tau_1^j)^{-1})}(t), x) - f(\delta_{(\tau_1^j)^{-1}}(t), x)| \\ &\quad + |f(\delta_{(\tau_1^j)^{-1}}(t), x) - f(t, x)| < \varepsilon \end{aligned}$$

and

$$\begin{aligned} |g(\delta_\tau(t), x) - g(t, x)| &\leq |g(\delta_{\tilde{\delta}(\tau_2, (\tau_2^j)^{-1})}(t), x) - g(\delta_{(\tau_2^j)^{-1}}(t), x)| \\ &\quad + |g(\delta_{(\tau_2^j)^{-1}}(t), x) - g(t, x)| < \varepsilon. \end{aligned}$$

Hence, there is at least a $\tau = \tau(\varepsilon, S)$ on any interval $[a, \tilde{\delta}(a, \tilde{\delta}(L, \tilde{\delta}(T, T)))]_{\Pi^*}$ with the length $\tilde{\delta}(L, \tilde{\delta}(T, T))$ such that $\tau \in E(f, \varepsilon, S) \cap E(g, \varepsilon, S)$. The proof is complete. \square

By Definition 9, the following theorem is immediate.

Theorem 10. Assume that $f(t, x)$ is uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$. Then for arbitrary $\alpha \in \mathbb{R}, b \in \tilde{\Pi}$, the functions $\alpha f(t, x), f(\delta_b(t), x)$ are uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$.

Theorem 11. Assume that f, g are uniformly δ -almost periodic functions under the matched space $(\mathbb{T}, F, \Pi, \delta)$. Then fg is uniformly δ -almost periodic.

Proof. According to Theorem 9, for any given $\varepsilon > 0, E(f, \frac{\varepsilon}{2}, S) \cap E(g, \frac{\varepsilon}{2}, S)$ is nonempty relatively dense.

Let $\sup_{(t,x) \in \mathbb{T}^* \times S} |f(t, x)| = M_1, \sup_{(t,x) \in \mathbb{T}^* \times S} |g(t, x)| = M_2$ and choose $\tau \in E(f, \frac{\varepsilon}{2}, S) \cap E(g, \frac{\varepsilon}{2}, S)$. Then for all $(t, x) \in \mathbb{T}^* \times S$ we can obtain

$$\begin{aligned} |f(\delta_\tau(t), x)g(\delta_\tau(t), x) - f(t, x)g(t, x)| &\leq |g(\delta_\tau(t), x)||f(\delta_\tau(t), x) - f(t, x)| + |f(t, x)||g(\delta_\tau(t), x) - g(t, x)| \\ &\leq (M_1 + M_2)\varepsilon \equiv \varepsilon_1. \end{aligned}$$

Therefore, $E(fg, \varepsilon_1, S)$ is a relatively dense set and $\tau \in E(fg, \varepsilon_1, S)$, i.e., fg is uniformly δ -almost periodic. This completes the proof. \square

Theorem 12. Assume that f, g are uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$. Then $f + g$ is uniformly δ -almost periodic. If g, f are uniformly δ -almost periodic and $\inf_{t \in \mathbb{T}} |g(t, x)| > 0$. Then $\frac{f(t, x)}{g(t, x)}$ is uniformly δ -almost periodic.

Proof. According to Theorem 9, for any $\varepsilon > 0$, $E(f, \frac{\varepsilon}{2}, S) \cap E(g, \frac{\varepsilon}{2}, S)$ is a nonempty relatively dense set. Clearly, if $\tau \in E(f, \frac{\varepsilon}{2}, S) \cap E(g, \frac{\varepsilon}{2}, S)$, then $\tau \in E(f + g, \varepsilon, S)$. Hence

$$(E(f, \frac{\varepsilon}{2}, S) \cap E(g, \frac{\varepsilon}{2}, S)) \subset E(f + g, \varepsilon, S).$$

Thus, $E(f + g, \varepsilon, S)$ is a relatively dense set, i.e., $f + g$ is uniformly δ -almost periodic.

Now, let $\inf_{(t,x) \in \mathbb{T}^* \times S} |g(t, x)| = N$ and select $\tau \in E(g, \varepsilon, S)$, then for all $(t, x) \in \mathbb{T}^* \times S$ we can obtain

$$\left| \frac{1}{g(\delta_\tau(t), x)} - \frac{1}{g(t, x)} \right| = \left| \frac{g(\delta_\tau(t), x) - g(t, x)}{g(\delta_\tau(t), x)g(t, x)} \right| < \frac{\varepsilon}{N^2} \equiv \varepsilon_2,$$

i.e., $\tau \in E(\frac{1}{g}, \varepsilon_2, S)$. Hence, $\frac{1}{g}$ is uniformly δ -almost periodic. Meanwhile, it follows from Theorem 11 that $\frac{f}{g}$ is uniformly δ -almost periodic. The proof is complete. \square

In the following, a convergence theorem of δ -almost periodic function sequences is established.

Theorem 13. If $f_n \in C(\mathbb{T} \times D, \mathbb{E}^n)$, $n = 1, 2, \dots$ are uniformly δ -almost periodic and the sequence $f_n(t, x) \rightarrow f(t, x)$ uniformly on $\mathbb{T}^* \times S$ as $n \rightarrow \infty$. Then $f(t, x)$ is uniformly δ -almost periodic.

Proof. For any given $\varepsilon > 0$, there is a sufficiently large \tilde{n}_0 such that

$$|f(t, x) - f_{\tilde{n}_0}(t, x)| < \frac{\varepsilon}{3}, \forall (t, x) \in \mathbb{T}^* \times S.$$

Selecting $\tau \in E\{f_{\tilde{n}_0}, \frac{\varepsilon}{3}, S\}$, then we have

$$\begin{aligned} |f(\delta_\tau(t), x) - f(t, x)| &\leq |f(\delta_\tau(t), x) - f_{\tilde{n}_0}(\delta_\tau(t), x)| + |f_{\tilde{n}_0}(\delta_\tau(t), x) - f_{\tilde{n}_0}(t, x)| \\ &\quad + |f_{\tilde{n}_0}(t, x) - f(t, x)| < \varepsilon, \forall (t, x) \in \mathbb{T}^* \times S, \end{aligned}$$

so $\tau \in E(f, \varepsilon, S)$. Therefore, $E(f, \varepsilon, S)$ is relatively dense, i.e., $f(t, x)$ is uniformly δ -almost periodic. This completes the proof. \square

Theorem 14. Assume that $f(t, x)$ is uniformly Δ_1^δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$, and denote

$$\mathcal{F}(t, x) = \int_{t_0}^t f(s, x) \Delta s, \quad t_0 \in \mathbb{T}^*.$$

Then $\mathcal{F}(t, x)$ is uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$ iff $\mathcal{F}(t, x)$ is a bounded function on $\mathbb{T}^* \times S$.

Proof. Note if we assume that $\mathcal{F}(t, x)$ is uniformly δ -almost periodic, one can easily observe that $\mathcal{F}(t, x)$ is a bounded function on $\mathbb{T}^* \times S$.

Without loss of generality, if $\mathcal{F}(t, x)$ is bounded, let $\mathcal{F}(t, x)$ be a real-valued function and

$$\mathcal{G} := \sup_{(t,x) \in \mathbb{T}^* \times S} \mathcal{F}(t, x) > \tilde{\mathcal{G}} := \inf_{(t,x) \in \mathbb{T}^* \times S} \mathcal{F}(t, x).$$

For any given $\varepsilon > 0$, there are t_1 and t_2 such that

$$\mathcal{F}(t_1, x) < \tilde{g} + \frac{\varepsilon}{6}, \quad \mathcal{F}(t_2, x) > \mathcal{G} - \frac{\varepsilon}{6}, \quad \forall x \in S.$$

Set $l = l(\varepsilon_1, S)$ an inclusion length of $E(f, \varepsilon_1, S)$, and $\varepsilon_1 = \frac{\varepsilon}{6d}, d = |t_1 - t_2|$. For any $\alpha \in \Pi^*$, we can choose $\tau \in E(f, \varepsilon_1, S)$ such that $\delta(\tau, t_1) \in [\delta(\alpha, t_0), \delta(\tilde{\delta}(\alpha, l), t_0)]_{\mathbb{T}}$. We introduce the notations $s_i = \delta(\tau, t_i), (i = 1, 2), L = \tilde{\delta}(l, \tilde{d}),$ where $\tilde{d} \in \Pi^*$ and $\tilde{d} > d$, then $s_1, s_2 \in [\delta(\alpha, t_0), \delta(\tilde{\delta}(\alpha, L), t_0)]_{\mathbb{T}}$. Hence, for all $x \in S$, we obtain

$$\begin{aligned} \mathcal{F}(s_2, x) - \mathcal{F}(s_1, x) &= \mathcal{F}(t_2, x) - \mathcal{F}(t_1, x) - \int_{t_1}^{t_2} f(t, x) \Delta t + \int_{\delta_\tau(t_1)}^{\delta_\tau(t_2)} f(t, x) \Delta t \\ &= \mathcal{F}(t_2, x) - \mathcal{F}(t_1, x) + \int_{t_1}^{t_2} [f(\delta_\tau(t), x) \delta_\tau^\Delta(t) - f(t, x)] \Delta t \\ &> \mathcal{G} - \tilde{g} - \frac{\varepsilon}{3} - \varepsilon_1 d = \mathcal{G} - \tilde{g} - \frac{\varepsilon}{2}, \end{aligned}$$

which yields

$$(\mathcal{F}(s_1, x) - \tilde{g}) + (\mathcal{G} - \mathcal{F}(s_2, x)) < \frac{\varepsilon}{2}.$$

Because for any interval with length L , we have

$$\mathcal{F}(s_1, x) - \tilde{g} \geq 0, \quad \mathcal{G} - \mathcal{F}(s_2, x) \geq 0,$$

so there exist s_1, s_2 such that

$$\mathcal{F}(s_1, x) < \tilde{g} + \frac{\varepsilon}{2}, \quad \mathcal{F}(s_2, x) > \mathcal{G} - \frac{\varepsilon}{2}.$$

Next, note that $L \neq e_{\Pi^*}$, and we have $\inf |\delta_L(t) - t| > q > 0$ for all $t \in \mathbb{T}^*$, where q is some positive constant. Let $\varepsilon_2 = \frac{\varepsilon}{2q}$. We claim that if $\tau \in E(f, \varepsilon_2, S)$, then $\tau \in E(F, \varepsilon, S)$. In fact, for all $(t, x) \in \mathbb{T}^* \times S$, we can choose $s_1, s_2 \in [t, \delta_L(t)]_{\mathbb{T}}$ such that

$$\mathcal{F}(s_1, x) < \tilde{g} + \frac{\varepsilon}{2}, \quad \mathcal{F}(s_2, x) > \mathcal{G} - \frac{\varepsilon}{2}.$$

Thus, for $\tau \in E(f, \varepsilon_2, S)$, we can get

$$\begin{aligned} \mathcal{F}(\delta_\tau(t), x) - \mathcal{F}(t, x) &= \mathcal{F}(\delta_\tau(s_1), x) - \mathcal{F}(s_1, x) \\ &\quad + \int_t^{s_1} f(t, x) \Delta t - \int_{\delta_\tau(t)}^{\delta_\tau(s_1)} f(t, x) \Delta t \\ &> \tilde{g} - (\tilde{g} + \frac{\varepsilon}{2}) - \int_t^{s_1} [f(\delta_\tau(t), x) \delta_\tau^\Delta(t) - f(t, x)] \Delta t \\ &> -\frac{\varepsilon}{2} - \varepsilon_2 q = -\varepsilon \end{aligned}$$

since $|t - s_1| = |\delta(\tau, t_1) - t| \leq |\delta_L(t) - t|$, and

$$\begin{aligned} \mathcal{F}(\delta_\tau(t), x) - \mathcal{F}(t, x) &= \mathcal{F}(\delta_\tau(s_2), x) - \mathcal{F}(s_2, x) + \int_t^{s_2} f(t, x)\Delta t - \int_{\delta_\tau(t)}^{\delta_\tau(s_2)} f(t, x)\Delta t \\ &< \mathcal{G} - \left(\mathcal{G} - \frac{\varepsilon}{2}\right) + \varepsilon_2 q = \varepsilon. \end{aligned}$$

Therefore, for $\tau \in E(f, \varepsilon_2, S)$, we can obtain $\tau \in E(\mathcal{F}, \varepsilon, S)$, i.e., $\mathcal{F}(t, x)$ is uniformly δ -almost periodic. The proof is complete. \square

Theorem 15. Assume that $f(t, x)$ is uniformly δ -almost periodic, $\mathcal{F}(\cdot)$ is uniformly continuous on the range of $f(t, x)$. Then $\mathcal{F} \circ f$ is uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$.

Proof. Please note that \mathcal{F} is uniformly continuous on the range of $f(t, x)$, because $f(t, x)$ is uniformly δ -almost periodic, then there is a sequence $\alpha = \{\alpha_n\} \subseteq \tilde{\Pi}$ such that

$$T_\alpha^\delta(\mathcal{F} \circ f) = T_\alpha^\delta(\mathcal{F}(f(t, x))) = \lim_{n \rightarrow +\infty} \mathcal{F}(f(\delta_{\alpha_n}(t), x)) = \mathcal{F}\left(\lim_{n \rightarrow +\infty} f(\delta_{\alpha_n}(t), x)\right) = \mathcal{F}(T_\alpha^\delta f)$$

holds uniformly on $\mathbb{T}^* \times S$. Therefore, $\mathcal{F} \circ f$ is an uniformly δ -almost periodic function. \square

In the following, a sufficient and necessary criterion for δ -almost periodic functions is obtained.

Theorem 16. A function $f(t, x)$ is uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$ iff there are common subsequences α, β of $\alpha', \beta' \subseteq \tilde{\Pi}$, respectively, such that

$$T_{\delta(\alpha, \beta)}^\delta f(t, x) = T_\alpha^\delta T_\beta^\delta f(t, x). \tag{7}$$

Proof. If $f(t, x)$ is uniformly δ -almost periodic, for any two sequences $\alpha', \beta' \subseteq \tilde{\Pi}$, there exists subsequence $\beta'' \subset \beta'$ such that

$$T_{\beta''}^\delta f(t, x) = g(t, x)$$

holds uniformly on $\mathbb{T}^* \times S$ and $g(t, x)$ is uniformly δ -almost periodic.

Take $\alpha'' \subset \alpha'$ and α'', β'' are the common subsequences of α', β' , respectively, then there exists $\alpha''' \subset \alpha''$ such that

$$T_{\alpha'''}^\delta g(t, x) = h(t, x)$$

holds uniformly on $\mathbb{T}^* \times S$.

Similarly, take $\beta''' \subset \beta''$, and β''', α''' are the common subsequences of β'', α'' , respectively, then there exist common subsequence $\alpha \subset \alpha''', \beta \subset \beta'''$ such that

$$T_{\delta(\alpha, \beta)}^\delta f(t, x) = k(t, x)$$

holds uniformly on $\mathbb{T}^* \times S$. According to the above, it is easy to observe that

$$T_\beta^\delta f(t, x) = g(t, x), \quad T_\alpha^\delta g(t, x) = h(t, x)$$

hold uniformly on $\mathbb{T}^* \times S$. Thus, for all $\varepsilon > 0$, if n is sufficiently large, then for any $(t, x) \in \mathbb{T}^* \times S$, we have

$$|f(\delta_{\delta(\alpha_n, \beta_n)}^\delta(t), x) - k(t, x)| < \frac{\varepsilon}{3},$$

$$|g(t, x) - f(\delta_{\beta_n}(t), x)| < \frac{\varepsilon}{3},$$

$$|h(t, x) - g(\delta_{\alpha_n}(t), x)| < \frac{\varepsilon}{3}.$$

Therefore

$$|h(t, x) - k(t, x)| \leq |h(t, x) - g(\delta_{\alpha_n}(t), x)| + |g(\delta_{\alpha_n}(t), x) - f(\delta_{\tilde{\delta}(\alpha_n, \beta_n)}(t), x)| + |f(\delta_{\tilde{\delta}(\alpha_n, \beta_n)}(t), x) - k(t, x)| < \varepsilon$$

holds for all $(t, x) \in \mathbb{T}^* \times S$. Since $\varepsilon > 0$ is arbitrary, we have $h(t, x) \equiv k(t, x)$, that is, $T_{\tilde{\delta}(\alpha, \beta)}^\delta f(t, x) = T_\alpha^\delta T_\beta^\delta f(t, x)$ holds uniformly on $\mathbb{T}^* \times S$.

On the other hand, assume (7) holds, then for any sequence $\gamma' \subseteq \tilde{\Pi}$, there is a subsequence $\gamma \subset \gamma'$ such that $T_\gamma^\delta f(t, x)$ exists uniformly on $\mathbb{T}^* \times S$.

Next, we will prove that $f(t, x)$ is uniformly δ -almost periodic.

Assume this is not true, i.e., $T_\gamma^\delta f(t, x)$ is not convergent uniformly on $\mathbb{T}^* \times S$. Then there exist $\varepsilon_0 > 0$ and $t_0 \in \mathbb{T}$, subsequences $\alpha' \subset \gamma, \beta' \subset \gamma, s' \subseteq \tilde{\Pi}$ and $\alpha' = \{\alpha'_n\}, \beta' = \{\beta'_n\}, s = \{s'_n\}$ such that

$$|f(\delta_{\tilde{\delta}(s'_n, \alpha'_n)}(t_0), x) - f(\delta_{\tilde{\delta}(s'_n, \beta'_n)}(t_0), x)| \geq \varepsilon_0 > 0. \tag{8}$$

According to (7), there exist common subsequences $\alpha'' \subset \alpha', s'' \subset s'$ such that for all $(t, x) \in \mathbb{T}^* \times S$, we have

$$T_{\tilde{\delta}(s'', \alpha'')}^\delta f(t, x) = T_{s''}^\delta T_{\alpha''}^\delta f(t, x). \tag{9}$$

Taking $\beta'' \subset \beta'$ and β'', α'', s'' are common subsequences of β', α', s' , respectively, such that for all $(t, x) \in \mathbb{T}^* \times S$, we have

$$T_{\tilde{\delta}(s, \beta)}^\delta f(t, x) = T_s^\delta T_\beta^\delta f(t, x). \tag{10}$$

Similarly, taking $\alpha \subset \alpha''$ satisfying α, β, s are common subsequences of α'', β'', s'' , respectively, according to (9), for all $(t, x) \in \mathbb{T}^* \times S$, we have

$$T_{\tilde{\delta}(s, \alpha)}^\delta f(t, x) = T_s^\delta T_\alpha^\delta f(t, x). \tag{11}$$

Since $T_\alpha^\delta f(t, x) = T_\beta^\delta f(t, x) = T_\gamma^\delta f(t, x)$, from (10) and (11), for all $(t, x) \in \mathbb{T}^* \times S$, we have

$$T_{\tilde{\delta}(s, \beta)}^\delta f(t, x) = T_{\tilde{\delta}(s, \alpha)}^\delta f(t, x),$$

that is, for all $(t, x) \in \mathbb{T}^* \times S$, we have

$$\lim_{n \rightarrow +\infty} f(\delta_{\tilde{\delta}(s_n, \beta_n)}(t), x) = \lim_{n \rightarrow +\infty} f(\delta_{\tilde{\delta}(s_n, \alpha_n)}(t), x).$$

Taking $t = t_0$, this contradicts (8). Therefore, $f(t, x)$ is uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$. The proof is complete. \square

Definition 13. Assume each element of matrix-valued function $\mathcal{M}(t, x) = (k_{ij}(t, x))_{n \times m}$, where $k_{ij}(t, x) \in C(\mathbb{T} \times \mathbb{E}, \mathbb{E})(1, 2, \dots, n; j = 1, 2, \dots, m)$ is uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$, then $\mathcal{M}(t, x)$ is said to be uniformly δ -almost periodic.

Now, we adopt the matrix norm $|\mathcal{M}(t, x)| = \sqrt{\sum_{ij} k_{ij}^2(t, x)}$, and then the Definition turns into:

Definition 14. A matrix function $M(t, x)$ is uniformly δ -almost periodic iff for any $\varepsilon > 0$, the shift set

$$E(\mathcal{M}, \varepsilon, S) = \{\tau \in \Pi : |\mathcal{M}(\delta_\tau(t), x) - \mathcal{M}(t, x)| < \varepsilon, \forall (t, x) \in \mathbb{T}^* \times S\}$$

is relatively dense in Π .

Theorem 17. Definitions 13 and 14 are equivalent.

Proof. First, assume that $\mathcal{M}(t, x)$ is uniformly δ -almost periodic. It follows from Definition 13 that each element $k_{ij}(t, x)$ is uniformly δ -almost periodic. Hence, for any given $\varepsilon > 0$, there is a nonempty relatively dense set $\mathcal{A} = \bigcap_{ij} E(k_{ij}(t, x), \frac{\varepsilon}{\sqrt{mn}}, S)$ such that $\tau \in \mathcal{A}$ implies

$$|\mathcal{M}(\delta_\tau(t), x) - \mathcal{M}(t, x)| = \left[\sum_{ij} |k_{ij}(\delta_\tau(t), x) - k_{ij}(t, x)|^2 \right]^{1/2} < \varepsilon.$$

Conversely, assume that for any given $\varepsilon > 0$, $E(\mathcal{M}, \varepsilon, S)$ is relatively dense. Then for any $i = 1, 2, \dots, n; j = 1, 2, \dots, m$ and $\tau \in E(\mathcal{M}, \varepsilon, S)$, one can obtain

$$|k_{ij}(\delta_\tau(t), x) - k_{ij}(t, x)| < |\mathcal{M}(\delta_\tau(t), x) - \mathcal{M}(t, x)| < \varepsilon, \forall (t, x) \in \mathbb{T}^* \times S,$$

so each element $k_{ij}(t, x)$ is uniformly δ -almost periodic, i.e., $\mathcal{M}(t, x)$ is uniformly δ -almost periodic. The proof is complete. \square

Definition 15. Let $\mathcal{M}(t, x)$ be a continuous matrix function. \mathcal{M} is said to be δ -normal if for any sequence $\alpha' \subseteq \tilde{\Pi}$, there is a subsequence $\alpha \subset \alpha'$ such that $T_\alpha^\delta \mathcal{M}(t, x)$ exists uniformly on $\mathbb{T}^* \times S$.

Theorem 18. Let $\mathcal{M}(t, x)$ be a continuous matrix function. \mathcal{M} is δ -normal iff $\mathcal{M}(t, x)$ is uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$.

Proof. First, it is easy to observe that if $\mathcal{M}(t, x)$ is δ -normal, then each element $k_{ij}(t, x)$ satisfies Definition 11, which implies that $\mathcal{M}(t, x)$ is uniformly δ -almost periodic.

Conversely, assume that $\mathcal{M}(t, x)$ is uniformly δ -almost periodic. According to Definition 13, for any sequence $\alpha' \subseteq \tilde{\Pi}$, there is subsequence $\alpha_1 \subset \alpha'$ such that $T_{\alpha_1}^\delta k_{11}(t, x)$ exists uniformly on $\mathbb{T}^* \times S$. Hence, there exists $\alpha_2 \subset \alpha_1$, such that $T_{\alpha_2}^\delta k_{12}(t, x)$ exists uniformly on $\mathbb{T}^* \times S$; by repeating this process mn times, then one can obtain a series of subsequences fulfilling:

$$\alpha = \{\alpha_k\} = \alpha_{mn} \subset \alpha_{mn-1} \subset \dots \subset \alpha_2 \subset \alpha_1 \subset \alpha'$$

so that

$$T_\alpha^\delta k_{ij}(t, x), \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m$$

exist uniformly on $\mathbb{T}^* \times S$. Hence, there is subsequence $\alpha \subset \alpha'$ such that $T_\alpha^\delta \mathcal{M}(t, x)$ exists uniformly on $\mathbb{T}^* \times S$, i.e., $\mathcal{M}(t, x)$ is δ -normal. The proof is complete. \square

4. Almost Periodic Dynamic Equations under Matched Spaces

In this section, under matched spaces, we introduce some new concepts related to dynamic equations and establish some basic results of almost periodic dynamic equations with shift operators.

Consider the nonlinear dynamic system as follows:

$$x^\Delta = f(t, x), \tag{12}$$

where $f \in C(\mathbb{T} \times \mathbb{E}^n, \mathbb{E}^n)$, and let $\Omega = \{x(t) : x(t) \text{ is a bounded solution to (12)}\}$.

Definition 16. Assume that $\Omega \neq \emptyset$ and $\lambda = \inf_{x \in \Omega} \|x\|$ exists. Then λ is said to be the least-value of solutions for (12). Suppose there exists $\varphi(t) \in \Omega$ such that $\|\varphi\| = \lambda$, and then we say $\varphi(t)$ is a minimum norm solution for (12), where $\|\cdot\| = \sup_{t \in \mathbb{T}} |\cdot|$.

Similar to the proof of Theorem 5.1 in [29], the following lemma is immediate.

Lemma 1. Assume that $0 \in S$ and $f \in C(\mathbb{T} \times S, \mathbb{E}^n)$ is bounded on $\mathbb{T}^* \times S$. If a bounded function $\varphi : \mathbb{T} \rightarrow S$ is a solution of (12), then there is a minimum norm solution for (12).

Lemma 2. Assume that $f(t, x)$ is uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$, $S = \overline{\{\varphi(t) : t \geq t_0\}}$ and (12) has a bounded solution $\varphi(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Then (12) has an δ -almost periodic solution $\psi(t)$ fulfilling $\{\psi(t), t \in \mathbb{T}\} \subset S$.

Proof. First, we choose $\alpha' = \{\alpha'_k\} \subset \tilde{\Pi}$ such that $\lim_{k \rightarrow +\infty} \alpha'_k = +\infty$ and

$$T_{\alpha'}^\delta f(t, x) = \lim_{k \rightarrow \infty} f(\delta_{\alpha'_k}(t), x) = f(t, x)$$

holds uniformly on $\mathbb{T}^* \times S$. Let $\varphi_k(t) = \varphi(\delta_{\alpha'_k}(t))$.

Next, for $\forall a \in \mathbb{T}^*$, we will prove that for sufficiently large k , $\{\varphi_k\}$ is defined on $(a, \infty)_{\mathbb{T}}$ and is a solution to $x^\Delta = f(t, x)$. Because $\varphi(t)$ is a solution to (12), then we have $T_{\alpha'}^\delta(\varphi^\Delta(t)) = T_{\alpha'}^\delta(f(t, x))$, i.e.,

$$\lim_{k \rightarrow \infty} (\varphi(\delta_{\alpha'_k}(t)))^\Delta = (\lim_{k \rightarrow \infty} (\varphi(\delta_{\alpha'_k}(t))))^\Delta = \lim_{k \rightarrow \infty} (f(\delta_{\alpha'_k}(t), x) = f(t, x).$$

Obviously, $\{\varphi_k(t)\}$ is uniformly bounded and equicontinuous on $(a, \infty)_{\mathbb{T}}$. Let α be a sequence which goes to $+\infty$, by Corollary 3.4 from [30], there exist $\alpha = \{\alpha'_{k_n}\} \subset \alpha'$ such that $T_\alpha^\delta \varphi(t) = \lim_{n \rightarrow \infty} \varphi(\delta_{\alpha'_{k_n}}(t)) = \psi(t)$ holds uniformly on \mathbb{T}^* . Therefore, we get $\psi(t) \in S, \forall t \in \mathbb{T}^*$. Moreover, it follows from $T_\alpha^\delta f(t, x) = \lim_{n \rightarrow \infty} f(\delta_{\alpha'_{k_n}}(t), x) = f(t, x)$ that $\psi(t)$ is an δ -almost periodic solution for (12). This completes the proof. \square

Lemma 3. Assume that $f \in C(\mathbb{T} \times \mathbb{E}^n, \mathbb{E}^n)$ is uniformly δ -almost periodic under the matched space $(\mathbb{T}, F, \Pi, \delta)$. Assume (12) has a minimum norm solution. Then for any $g \in H_\delta(f)$, the equation

$$x^\Delta = g(t, x) \tag{13}$$

has the same least-value of solutions with (12).

Proof. First, let $\varphi(t)$ be the minimum norm solution for (12) and λ be the least-value. Because $g \in H_\delta(f)$, there is a sequence $\alpha' \in \tilde{\Pi}$ such that $T_{\alpha'}^\delta f(t, x) = g(t, x)$ holds uniformly on $\mathbb{T}^* \times S$. According to Corollary

3.4 from [30], there exists $\alpha \subset \alpha'$ such that $T_\alpha^\delta \varphi(t) = \psi(t)$ holds uniformly on \mathbb{T}^* . Therefore, $\psi(t)$ is a solution to (13). Moreover, it follows from $|\varphi(t)| \leq \lambda$ that $|\psi(t)| \leq \lambda$. Hence, $\lambda' = \|\psi\| \leq \lambda$. In view of $\varphi(t) = T_{\alpha^{-1}}^\delta \psi(t)$ and $|\psi(t)| \leq \lambda'$, one can obtain $|\varphi(t)| \leq \lambda'$. Therefore, $\lambda = \|\varphi\| \leq \lambda'$, i.e., $\lambda = \lambda'$. The proof is complete. \square

According to the proof of Lemma 3, the following lemma is immediate.

Lemma 4. Assume that $\varphi(t)$ is a minimum norm solution for (12) and there is a sequence $\alpha' \subseteq \tilde{\Pi}$ such that $T_{\alpha'}^\delta f(t, x) = g(t, x)$ exists uniformly on $\mathbb{T}^* \times S$. Furthermore, if there is a subsequence $\alpha \subset \alpha'$ such that $T_\alpha^\delta \varphi(t) = \psi(t)$ holds uniformly on \mathbb{T}^* , then $\psi(t)$ is a minimum norm solution for (13).

Lemma 5. Assume that $f \in C(\mathbb{T} \times \mathbb{E}^n, \mathbb{E}^n)$ is uniformly δ -almost periodic and for each $g \in H_\delta(f)$, (13) has a unique minimum norm solution. Then these minimum norm solutions are δ -almost periodic.

Proof. For any given $g \in H_\delta(f)$, there is a unique minimum norm solution $\psi(t)$ for (13). Because $g(t, x)$ is uniformly δ -almost periodic, for any sequences $\alpha', \beta' \subseteq \tilde{\Pi}$, there are common subsequences $\alpha \subset \alpha', \beta \subset \beta'$ such that

$$T_{\tilde{\delta}(\alpha, \beta)}^\delta g(t, x) = T_\alpha^\delta T_\beta^\delta g(t, x)$$

holds uniformly on $\mathbb{T}^* \times S$ and $T_\alpha^\delta T_\beta^\delta \psi(t), T_{\tilde{\delta}(\alpha, \beta)}^\delta \psi(t)$ exist uniformly on \mathbb{T}^* . According to Lemmas 3 and 4, one can obtain $T_\alpha^\delta T_\beta^\delta \psi(t)$ and $T_{\tilde{\delta}(\alpha, \beta)}^\delta \psi(t)$ are minimum norm solutions for the following equation:

$$x^\Delta = T_{\tilde{\delta}(\alpha, \beta)}^\delta g(t, x).$$

Thus, one has $T_\alpha^\delta T_\beta^\delta \psi(t) = T_{\tilde{\delta}(\alpha, \beta)}^\delta \psi(t)$ because of the uniqueness of the minimum norm solution. Therefore, $\psi(t)$ is a δ -almost periodic solution. The proof is complete. \square

Consider the linear δ -almost periodic dynamic equation

$$x^\Delta = A(t)x + f(t) \tag{14}$$

and its associated homogeneous equation

$$x^\Delta = A(t)x, \tag{15}$$

where $A : \mathbb{T} \rightarrow \mathbb{E}^{n \times n}$ and $f : \mathbb{T} \rightarrow \mathbb{E}^n$ are δ -almost periodic.

Definition 17. If $B \in H_\delta(A)$, the dynamic equation

$$y^\Delta = B(t)y \tag{16}$$

is said to be a homogeneous hull equation of (14).

Definition 18. If $B \in H_\delta(A)$ and $g \in H_\delta(f)$, the dynamic equation

$$y^\Delta = B(t)y + g(t) \tag{17}$$

is said to be a hull equation of (14).

Next, we define an exponential dichotomy on time scales.

Definition 19. *The linear system*

$$x^\Delta(t) = A(t)x(t) \tag{18}$$

is called admitting an exponential dichotomy on \mathbb{T} if there are positive constants $K \geq 1, \alpha$, projection P and the fundamental solution matrix $X(t)$ of (18), fulfilling

$$\begin{cases} |X(t)PX^{-1}(s)| \leq Ke_{\ominus\alpha}(t, \rho(s)), s, t \in \mathbb{T}, t \geq s, \\ |X(t)(I - P)X^{-1}(s)| \leq Ke_{\ominus\alpha}(s, \rho(t)), s, t \in \mathbb{T}, t \leq s. \end{cases} \tag{19}$$

Based on Definition 19, a Favard’s theorem for homogeneous linear dynamic equation can be established as follows.

Lemma 6. *Assume that $A : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ is δ -almost periodic and $x(t)$ is a δ -almost periodic solution of the homogeneous linear dynamic equation*

$$x^\Delta = A(t)x.$$

Then $\inf_{t \in \mathbb{T}} |x(t)| > 0$ or $x(t) \equiv 0$.

Proof. If $\inf_{t \in \mathbb{T}} |x(t)| = 0$, there exists $\{t_n\} \subset \mathbb{T}$ such that $|x(t_n)| \rightarrow 0$ as $n \rightarrow \infty$, and from $x \in H_\delta(x)$, then there exists $\alpha' \subset \tilde{\mathbb{T}}$ such that $\lim_{n \rightarrow +\infty} x(\delta_{\alpha'_n}(t)) = x(t)$ for all $t \in \mathbb{T}^*$, and this implies that for any $\varepsilon > 0$, there exists $N > 0$ so that $n > N$ implies $|x(\delta_{\alpha'_n}(t_n)) - x(t_n)| < \frac{\varepsilon}{2}$. Furthermore, since $x(t)$ is δ -almost periodic on \mathbb{T} , it is uniformly continuous on \mathbb{T} , we can take $t_0 \in \mathbb{T}$ with $|t_0 - t_n| < \delta^*$ implies $|x(\delta_{\alpha'_n}(t_0)) - x(\delta_{\alpha'_n}(t_n))| < \frac{\varepsilon}{2}$. Therefore, for sufficiently large $n \in \mathbb{N}$, we have $|x(\delta_{\alpha'_n}(t_0)) - x(t_n)| \leq |x(\delta_{\alpha'_n}(t_0)) - x(\delta_{\alpha'_n}(t_n))| + |x(\delta_{\alpha'_n}(t_n)) - x(t_n)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$. Hence, we can easily see that $|x(\delta_{\alpha'_n}(t_0))| \rightarrow 0$ as $n \rightarrow +\infty$. Since $A(t)$ is δ -almost periodic on \mathbb{T} , there exists sequence $\alpha \subset \alpha'$ such that

$$\begin{aligned} T_\alpha^\delta A(t) &= B(t), & T_\alpha^\delta x(t) &= y(t), \\ T_{\alpha^{-1}}^\delta B(t) &= A(t), & T_{\alpha^{-1}}^\delta y(t) &= x(t) \end{aligned}$$

hold uniformly on \mathbb{T}^* , since $x^\Delta = A(t)x(t)$ is equivalent to

$$(x(t))^\Delta = A(t)x(t),$$

so we have

$$\begin{aligned} y^\Delta(t) &= (T_\alpha^\delta x(t))^\Delta = \left(\lim_{n \rightarrow \infty} x(\delta_{\alpha_n}(t))\right)^\Delta = \lim_{n \rightarrow \infty} (x(\delta_{\alpha_n}(t)))^\Delta \\ &= \lim_{n \rightarrow \infty} A(\delta_{\alpha_n}(t))x(\delta_{\alpha_n}(t)) = \lim_{n \rightarrow \infty} A(\delta_{\alpha_n}(t)) \cdot \lim_{n \rightarrow \infty} x(\delta_{\alpha_n}(t)), \end{aligned}$$

that is, $y(t)$ is a solution to the following equation:

$$y^\Delta = B(t)y$$

satisfying the initial condition:

$$y(t_0) = T_\alpha^\delta x(t_0) = \lim_{n \rightarrow +\infty} x(\delta_{\alpha_n}(t_0)) = 0.$$

Hence, according to Theorem 2.77 from [25], $y(t) = y(t_0)e_B(t, t_0) \equiv 0$, therefore, $x(t) = T_{\alpha^{-1}}^\delta y(t) \equiv 0$. The proof is complete. \square

Lemma 7. Assume that (15) has a δ -almost periodic solution $x(t)$ and $\inf_{t \in \mathbb{T}} |x(t)| > 0$. If a solution of (14) on $[t_0, \infty)_{\mathbb{T}}$ is bounded, then (14) has a solution that is δ -almost periodic.

Proof. According to Lemmas 1 and 2, there is a minimum norm solution for (14) on \mathbb{T}^* and for each pair of $T_\alpha^\delta A(t) = B(t)$ and $T_\alpha^\delta f(t) = g(t)$, (17) has a minimum norm solution.

Now, we claim that the minimum norm solution for (17) is unique.

In fact, for a fixed pair of $B(t)$ and $g(t)$, we consider (17). Let (17) have two different minimum norm solutions $x_1(t)$ and $x_2(t)$, and their least-values are equivalent to λ . Since $\frac{1}{2}[x_1(t) - x_2(t)]$ is a bounded non-trivial solution for (16), by Lemma 7, there is a real number $\rho > 0$ such that

$$\inf_{t \in \mathbb{T}} \frac{1}{2} |x_1(t) - x_2(t)| \geq \rho > 0.$$

Now by the parallelogram law, we have

$$\left| \frac{1}{2}(x_1(t) + x_2(t)) \right|^2 + \left| \frac{1}{2}(x_1(t) - x_2(t)) \right|^2 = \frac{1}{2}(|x_1(t)|^2 + |x_2(t)|^2) \leq \lambda^2,$$

and noting that

$$\begin{aligned} \left(\frac{1}{2}(x_1(t) + x_2(t)) \right)^\Delta &= \frac{1}{2}x_1^\Delta + \frac{1}{2}x_2^\Delta \\ &= \frac{1}{2}(B(t)x_1 + g(t)) + \frac{1}{2}(B(t)x_2 + g(t)) \\ &= B(t)\left(\frac{1}{2}(x_1 + x_2)\right) + g(t), \end{aligned}$$

so $\frac{1}{2}(x_1(t) + x_2(t))$ is a solution to (17), and thus

$$\left| \frac{1}{2}(x_1(t) + x_2(t)) \right| < \sqrt{\lambda^2 - \rho^2} < \lambda.$$

This is a contradiction. The proof is complete. \square

Lemma 8. Assume that each bounded solution of a homogeneous hull equation of (14) is δ -almost periodic. Then all bounded solutions of (14) are δ -almost periodic.

Proof. It follows from Lemma 6 that each non-trivial bounded solution for the hull equations of (14) fulfills $\inf_{t \in \mathbb{T}} |x(t)| > 0$. According to Lemma 7, we have that if (14) has bounded solutions on \mathbb{T} , then (14) has a δ -almost periodic solution $\psi(t)$. Assume $\varphi(t)$ is an arbitrary bounded solution of (14). Then $\eta(t) = \psi(t) - \varphi(t)$ is a δ -almost periodic solution of its associated homogeneous Equation (15). Therefore, $\varphi(t)$ is a δ -almost periodic solution. This completes the proof. \square

Lemma 9. Assume that a homogeneous hull equation of (14) has the unique bounded solution $x(t) \equiv 0$. Then (14) has an unique δ -almost periodic solution.

Proof. If $\psi(t), \varphi(t)$ are two bounded solutions to (14), then $x(t) = \varphi(t) - \psi(t)$ is a solution of a homogeneous hull equation of (14). It follows from $x(t) \equiv 0$ that $\varphi(t) \equiv \psi(t)$. According to Lemma 8, (14) has an unique δ -almost periodic solution. This completes the proof. \square

Using a proof similar to that in Lemmas 7.4-7.5 in [31], one can easily prove Lemmas 10 and 11 (we omit their proof).

Lemma 10. Assume that X is a Δ -differentiable invertible matrix and P is a projection such that XPX^{-1} is bounded on \mathbb{T} . Then there is a differentiable matrix S such that $XPX^{-1} = SPS^{-1}$ for all $t \in \mathbb{T}$ and S, S^{-1} are bounded on \mathbb{T} . Moreover, there exists an S of the form $S = XQ^{-1}$ and Q, P are commutable.

Lemma 11. Assume that (15) admits an exponential dichotomy and C non-singular. Then $X(t)C$ admits an exponential dichotomy with the same projection P iff $CP = PC$, where $X(t)$ is a fundamental solution matrix for (15).

Lemma 12. Assume that $A : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ is δ -almost periodic under the matched spaces $(\mathbb{T}, F, \Pi, \delta)$ and (15) admits an exponential dichotomy. Then for each $B \in H_\delta(A)$, there exist the same projection P and constants K, α such that (16) admits an exponential dichotomy.

Proof. By Lemma 10, let X be a fundamental solution matrix satisfying (19). Set Q and S the matrices given by Lemma 10. Let $T_\alpha^\delta A = B$ uniformly on \mathbb{T}^* . For any given $t_0 \in \mathbb{T}^*$, let $X_n(t) = X(\delta_{\alpha_n}(t))Q^{-1}(\delta_{\alpha_n}(t_0))$, and from (15), we obtain

$$(X(t))^\Delta = A(t)X(t), \tag{20}$$

and replacing t with $\delta_{\alpha_n}(t)$ in (20), we can obtain

$$(X(\delta_{\alpha_n}(t)))^\Delta = A(\delta_{\alpha_n}(t))X(\delta_{\alpha_n}(t)),$$

i.e., $X_n^\Delta(t) = A(\delta_{\alpha_n}(t))X_n(t)$, so $X_n(t)$ is a fundamental solution matrix to $x^\Delta = A(\delta_{\alpha_n}(t))x$, by Lemma 11, there exist the same projection P and the same constants such that it has an exponential dichotomy and Q^{-1}, P are commutable. Moreover, one can choose subsequences such that $X_n(t_0)$ and $X_n^{-1}(t_0)$ convergent because they are $S(\delta_{\alpha_n}(t_0))$ and $S^{-1}(\delta_{\alpha_n}(t_0))$, respectively, and bounded. Keeping the same symbols let $X_n(t_0) \rightarrow Y_0$, and then $X_n^{-1}(t_0) \rightarrow Z_0$ where $Z_0 = Y_0^{-1}$. Now for a suitable subsequence $X_n(t) \rightarrow Y$ as $n \rightarrow \infty$, where Y denotes a solution of $y^\Delta = By$ uniformly on \mathbb{T} , one has $Y(t_0) = Y_0$ is non-singular. Finally, since X_n satisfies (19) for all n , then Y also satisfies (19). The proof is complete. \square

Lemma 13. Assume that (15) admits an exponential dichotomy. Then (15) has an unique bounded solution $x(t) \equiv 0$.

Proof. Assume $X(t)$ is the fundamental solution matrix for (15). For any sequence $\alpha \subset \tilde{\Pi}$, we use the notations $A_n = A(\delta_{\alpha_n}(t)), X_n(t) = X(\delta_{\alpha_n}(t))$. For the homogeneous equation (15) has an exponential dichotomy, and then it follows that there is a constant M such that $\|X_n(t)\| \leq M$ and $\|X_n^\Delta(t)\| = \|A_n(t)X_n(t)\| \leq \bar{A}M$, where $\bar{A} = \sup_{t \in \mathbb{T}} \|A(t)\|$. Therefore, according to Corollary 3.4 from [30], there is $\{\alpha_{n_k}\} := \alpha' \subset \alpha$ such that $\{X_{n_k}\}$ converges uniformly on \mathbb{T}^* and $\lim_{n \rightarrow +\infty} X(\delta_{\alpha_n}(t))$ exists uniformly on \mathbb{T}^* . Hence, one can get that $X(t)$ is δ -almost periodic. Moreover, because the homogeneous equation (15)

has an exponential dichotomy, then it follows that $\inf_{t \in \mathbb{T}} x(t) = 0$. From Lemma 6, we have $x(t) \equiv 0$. This completes the proof. \square

Lemma 14. Assume that (15) admits an exponential dichotomy. Then all hull equations of (15) have an unique bounded solution $x(t) \equiv 0$.

Proof. According to Lemma 12, all hull equations of (15) have an exponential dichotomy, and it follows from Lemma 13 that all hull equations (15) have an unique bounded solution $x(t) \equiv 0$. This completes the proof. \square

Theorem 19. Let $A : \mathbb{T} \rightarrow \mathbb{E}^{n \times n}$ and $f : \mathbb{T} \rightarrow \mathbb{E}^n$ be δ -almost periodic. Moreover, assume (15) admits an exponential dichotomy, then (14) has an unique δ -almost periodic solution

$$x(t) = \int_{-\infty}^t X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_t^{+\infty} X(t)(I - P)X^{-1}(\sigma(s))f(s)\Delta s, \tag{21}$$

where $X(t)$ is a fundamental solution matrix for (15).

Proof. Now, we show that $x(t)$ is a bounded solution of (14). In fact,

$$\begin{aligned} & x^\Delta(t) - A(t)x(t) \\ &= X^\Delta(t) \int_{-\infty}^t PX^{-1}(\sigma(s))f(s)\Delta s + X(\sigma(t))PX^{-1}(\sigma(t))f(t) \\ &\quad - X^\Delta(t) \int_t^{+\infty} (I - P)X^{-1}(\sigma(s))f(s)\Delta s + X(\sigma(t))(I - P)X^{-1}(\sigma(t))f(t) \\ &\quad - A(t)X(t) \int_{-\infty}^t PX^{-1}(\sigma(s))f(s)\Delta s + A(t)X(t) \int_t^{+\infty} (I - P)X^{-1}(\sigma(s))f(s)\Delta s \\ &= X(\sigma(t))(P + I - P)X^{-1}(\sigma(t))f(t) = f(t) \end{aligned}$$

and

$$\begin{aligned} \|x\| &= \sup_{t \in \mathbb{T}} \left| \int_{-\infty}^t X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_t^{+\infty} X(t)(I - P)X^{-1}(\sigma(s))f(s)\Delta s \right| \\ &\leq \sup_{t \in \mathbb{T}} \left(\left| \int_{-\infty}^t e_{\ominus\alpha}(t,s)\Delta s \right| + \left| \int_t^{+\infty} e_{\ominus\alpha}(\sigma(s),t)\Delta s \right| \right) K\|f\| \leq \frac{2K}{\alpha}\|f\|, \end{aligned}$$

where $\|\cdot\| = \sup_{t \in \mathbb{T}} |\cdot|$.

Next, we will show that $x(t)$ is δ -almost periodic. By Lemma 14, all hull equations of (15) have an unique bounded solution $\bar{x}(t) \equiv 0$. Therefore, for given α', β' , there exist common subsequences $\alpha \subset \alpha', \beta \subset \beta'$ so that $T_{\delta(\alpha,\beta)}^\delta A = T_\alpha^\delta T_\beta^\delta A, T_{\delta(\alpha,\beta)}^\delta f = T_\alpha^\delta T_\beta^\delta f, y = T_{\delta(\alpha,\beta)}^\delta x$, and $z = T_\alpha^\delta T_\beta^\delta x$ exists uniformly on \mathbb{T}^* . However, $y - z = T_{\delta(\alpha,\beta)}^\delta x - T_\alpha^\delta T_\beta^\delta x \equiv 0$ since $y - z$ is the bounded solution to all hull equations of (15). Hence, $T_{\delta(\alpha,\beta)}^\delta x = T_\alpha^\delta T_\beta^\delta x$, from Theorem 16, $x \in AP^\delta(\mathbb{T})$. This completes the proof. \square

Now, we will provide an application to demonstrate our obtained results. In the sense of matched spaces, consider the following δ -almost periodic dynamic equation with variable delays:

$$x^\Delta(t) = A(t)x(t) + \sum_{i=1}^n f(t, x(\delta(\tau_i(t), t))), \tag{22}$$

where $A : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ is δ -almost periodic, $\tau_i : \mathbb{T}^* \rightarrow \Pi^*$ is δ -almost periodic for every $i = 1, 2, \dots, n$, $f \in C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ is uniformly δ -almost periodic.

Theorem 20. We give the following assumptions:

- (H₁) There exist positive constants K and α such that $x^\Delta(t) = A(t)x(t)$ admits an exponential dichotomy.
- (H₂) There exists $M < \frac{\alpha}{2Kn}$ such that for all $t \in \mathbb{T}$ and $x, y \in \mathbb{R}^n$,

$$|f(t, x) - f(t, y)| \leq M|x - y|.$$

Then there is an unique δ -almost periodic solution for (22).

Proof. For any $\varphi \in AP^\delta(\mathbb{T})$, consider the following equation

$$x^\Delta(t) = A(t)x(t) + \sum_{i=1}^n f(t, \varphi(\delta(\tau_i(t), t))). \tag{23}$$

It follows from Theorem 19 that (23) has a unique solution $T\varphi \in AP^\delta(\mathbb{T})$ as follows:

$$\begin{aligned} T\varphi(t) &= \int_{-\infty}^t X(t)PX^{-1}(\sigma(s)) \sum_{i=1}^n f(s, \varphi(\delta(\tau_i(s), s)))\Delta s \\ &\quad - \int_t^{+\infty} X(t)(I - P)X^{-1}(\sigma(s)) \sum_{i=1}^n f(s, \varphi(\delta(\tau_i(s), s)))\Delta s. \end{aligned}$$

Consider a mapping $T : AP^\delta(\mathbb{T}) \rightarrow AP^\delta(\mathbb{T})$ given by $(T\varphi)(t) = x_\varphi(t)$, $\forall x \in AP^\delta(\mathbb{T})$. From (H₁), we can obtain

$$\begin{aligned} |X(t)PX^{-1}(s)| &\leq Ke_{\ominus\alpha}(t, \rho(s)), s, t \in \mathbb{T}, t \geq s, \\ |X(t)(I - P)X^{-1}(s)| &\leq Ke_{\ominus\alpha}(s, \rho(t)), s, t \in \mathbb{T}, t \leq s. \end{aligned}$$

For any $\varphi, \psi \in AP^\delta(\mathbb{T})$, one can obtain

$$\begin{aligned} \|T\varphi - T\psi\| &\leq \left\| \int_{-\infty}^t X(t)PX^{-1}(\sigma(s)) \sum_{i=1}^n |f(s, \varphi(\delta(\tau_i(s), s))) - f(s, \psi(\delta(\tau_i(s), s)))|\Delta s \right. \\ &\quad \left. - \int_t^{+\infty} X(t)(I - P)X^{-1}(\sigma(s)) \sum_{i=1}^n |f(s, \varphi(\delta(\tau_i(s), s))) - f(s, \psi(\delta(\tau_i(s), s)))|\Delta s \right\| \\ &\leq \left[\left| \int_{-\infty}^t Ke_{\ominus\alpha}(t, s)\Delta s \right| + \left| \int_t^{+\infty} Ke_{\ominus\alpha}(\sigma(s), t)\Delta s \right| \right] \sum_{i=1}^n M\|\varphi - \psi\| \\ &\leq \frac{2}{\alpha}KnM\|\varphi - \psi\|, \end{aligned}$$

and from (H₂), T is a contractive mapping. Therefore, we obtain the desired results. \square

5. Conclusions

In this paper, by employing the algebraic structure of matched spaces for time scales, some basic results of the shift closedness including non-translational shift closedness of time scales are established. This progress combines a larger scope of time scales without translation invariance. Based on matched

spaces of time scales, we studied δ -almost periodic functions and some fundamental theorems are established which can be used to study δ -almost periodic dynamic equations. Moreover, some sufficient conditions to guarantee the existence of δ -almost periodic solutions are established for a new type of delay dynamic equation whose delay function range is from a period set of the time scale and may be separated from the time scale \mathbb{T} . The results in this paper develop a new theory of almost periodic dynamic equations which include almost periodic q -difference equations and other almost periodic dynamic equations on irregular hybrid domains.

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