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Back to Basics: Meaning of the Parameters of Fractional Order PID Controllers

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Abstract: The beauty of the proportional-integral-derivative (PID) algorithm for feedback control is its simplicity and efficiency. Those are the main reasons why PID controller is the most common form of feedback. PID combines the three natural ways of taking into account the error: the actual (proportional), the accumulated (integral), and the predicted (derivative) values; the three gains depend on the magnitude of the error, the time required to eliminate the accumulated error, and the prediction horizon of the error. This paper explores the new meaning of integral and derivative actions, and gains, derived by the consideration of non-integer integration and differentiation orders, i.e., for fractional order PID controllers. The integral term responds with selective memory to the error because of its non-integer order λ , and corresponds to the area of the projection of the error curve onto a plane (it is not the classical area under the error curve). Moreover, for a fractional proportional-integral (PI) controller scheme with automatic reset, both the velocity and the shape of reset can be modified with λ . For its part, the derivative action refers to the predicted future values of the error, but based on different prediction horizons (actually, linear and non-linear extrapolations) depending on the value of the differentiation order, μ . Likewise, in case of a proportional-derivative (PD) structure with a noise filter, the value of μ allows different filtering effects on the error signal to be attained. Similarities and differences between classical and fractional PIDs, as well as illustrative control examples, are given for a best understanding of new possibilities of control with the latter. Examples are given for illustration purposes.

Keywords: fractional; control; PID; parameter; meaning

1. Introduction

The proportional-integral-derivative (PID) controller is distinguished as the most common form of feedback. In process control today, more than 95% of the control loops are of PID type, but these controllers can be found in all areas where control is used.

Despite its straightforward structure, the popularity of PID controllers lies in the simplicity of the design procedures and in the effectiveness obtained to the system performance [1]. Those are the main reasons why PID controllers have survived many changes in technology, from mechanics and pneumatics to microprocessors via electronic tubes, transistors, integrated circuits, among others. Actually, practically all PID controllers made today are based on microprocessors, so this electronic element has had a dramatic influence on this kind of control providing PIDs additional advances features, such as gain scheduling, continuous adaptation, and automatic tuning [2].

From the control engineering point of view, improving system behavior is the major concern. To that end, the generalization of classical PID controllers to non-integer orders of integration and

differentiation was firstly proposed in [3]. Intuitively, with this extension of classical PIDs there are more tuning parameters and, consequently, more flexibilities in adjusting time and frequency responses of the control system. This also translates in more robustness in designs.

However, the first step when applying an existing or new controller is to understand exactly what their actions can do in closed-loop in order to take full advantage of the possible effects on the system response. In the case of integer order, the interpretation of the three actions of PIDs seems to be clear [4–6]:

- the proportional action is simply proportional to the current control error;
- the integral action is related to the past values of the control error, so represents the accumulated error, i.e., the area under the error curve;
- the derivative action predicts future values of the error or, in other words, corrects based on the rate of change of the deviation from the set-point.

Since the pioneering work of Podlubny, there is ample evidence that supports that fractional order PIDs (FOPIDs) have been extensively studied and applied in many fields. Undoubtedly, the lists that are reported below are quite far from aiming at completeness, failing to mention literal hundreds of other published texts related to FOPIDs; relevant and recent papers were selected for giving the reader an idea of the development volume on this topic that can be found in the specialized literature. Fundamentals of FOPIDs can be found in, e.g., [7–10]. In what concerns design methods, among the reported, the following can be highlighted: Ziegler–Nichols-type rules [11–14], optimal tuning [15–18], tuning for robustness purposes [19–21], auto-tuning [22,23], and tuning based on reducing the number of parameters [24]. Likewise, numerous advanced control schemes based on FOPID controllers were proposed, such as, Smith predictors structures [25–27], internal mode control [28–30], hybrid control [31], gain scheduling [32–34], gain and order scheduling [35,36], among others. Current reviews in the development of FOPIDs can be found in [37–42]; likewise, few current examples of application in industry are described in [43–50].

Despite so many variations and applications of the FOPID algorithm, as well as design and tuning methods, up to now a detailed description of the meaning of the parameters of such controllers cannot be found in the literature. Nevertheless, although few, studies on the geometric and physical interpretation of integrals and derivatives of arbitrary (not necessary integer) order have been published [51,52], but it still remains as an open problem.

These circumstances make the understanding of the meaning of the parameters of FOPIDs a priority. With so much in play, this paper explores the new meaning of integral and derivative actions, and gains, derived by the consideration of non-integer integration and differentiation orders. Similarities and differences between classical and fractional PIDs, as well as illustrative control examples, are given for a best understanding of new possibilities of control with the latter.

The remainder of this paper is organized as follows. Section 2 describes the control algorithm of classical PID and its generalization to non-integer order. Section 3 explains the meaning of the parameters of non-integer PIDs. Section 4 discusses similarities and differences between classical and fractional PIDs. Illustrative examples are given in Section 5. Main conclusions of this paper are drawn in Section 6.

2. Generalities

This section describes the control algorithm of classical PIDs and its generalization to non-integer order.

2.1. Classical PID Controller

The use of PID control consists of applying properly the combination of three types of corrective actions to the error signal, which represents how far or near is the desired output from the actual output. As widely known, these three control actions are proportional, integral and derivative.

The key aspect when tuning PID controllers is in deciding how to best combine those three terms to achieve the most efficient regulation of the process variable for the considered problem. As well known, the most obvious way is to use a simple weighted sum where each term is multiplied by a tuning constant or gain, and the results are then added together as follows:

$$u(t) = K_p \left(e(t) + \frac{1}{T_i} \int_0^t e(\tau) d\tau + T_d \frac{de(t)}{dt} \right), \quad (1)$$

where $u(t)$ is the control signal, $e(t)$ is the control error ($e = y_{sp} - y$, i.e., the difference between the desired set-point, y_{sp} , and the measured process variable, y), and K_p , T_i , and T_d are the controller parameters: proportional gain, integral time constant, and derivative time constant, respectively.

Control law (1) guarantees that the present (due to the proportional action), the past (by means of the integral action) and the future of the error (by the derivative action) are taken into account, as shown in Figure 1. Two main observations can be made to (1): (1) the controller needs only compute the current error between the measured process variable and the desired set-point to calculate how much and how fast that difference has been changing over time, and (2) the relative contributions of each term then can be then adjusted by choosing appropriate values of the controller parameters.

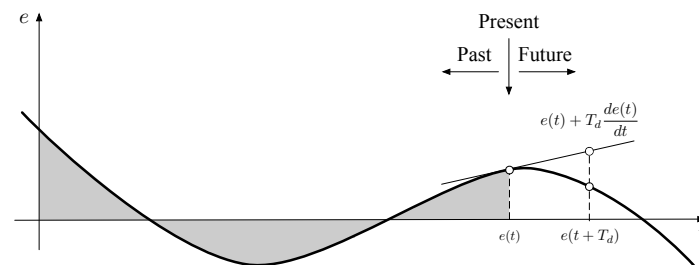


Figure 1. Evolution of the error (classical proportional-integral-derivative (PID) controllers).

2.2. Fractional Order PID Controller

The generalization to non-integer orders of (1) leads to the typical algorithm of a FOPID controller, i.e.,

$$u(t) = K_p \left(e(t) + \frac{1}{T_i} \mathcal{D}^{-\lambda} e(t) + T_d \mathcal{D}^{\mu} e(t) \right), \quad (2)$$

where $\lambda, \mu \in \mathbb{R}^+$ are the non-integer orders of the integral and derivative terms, respectively, and $\mathcal{D}$ is the fractional operator defined as Riemann–Liouville as [10,53]

$$\mathcal{D}^{-n} f(t) = \frac{1}{\Gamma(n)} \int_0^t f(y) (t - y)^{n-1} dy, \quad (3)$$

(n is a general non-integer order, and $\Gamma(-)$, the gamma function).

Similarly to the classical version, control law (2) combines the three natural ways of taking into account the error: current, accumulated, and predicted error. However, in contrast to the integer counterpart, fractional operators are non local, which results in new meaning of the integral and derivative actions, and gains. In particular, the fractional integration of the error is not already the area under its curve; it can be viewed as the area of the projection of the curve onto a plane. Hence, what the integration order λ is doing is a selection of the history of the error or, what is the same, the integral term responds with selective memory to the past values of the error. For its part, the action of a controller with proportional and fractional derivative action may be interpreted as if the control is made proportional to the predicted process output, where the prediction is definitively different from the classical case: it is made by extrapolating the error by a straight line that is not tangent to the error curve at the current value of the error, or by a curve (i.e., linear and non-linear extrapolations). More details of these and other possibilities will be explained in Section 3.

Figure 2 is an attempt to illustrate these effects. It should be remarked that, in this figure, the error was considered as the function $e(t) = t + 0.5 \sin(t)$ (taken from [51]). To show the integral and derivative effects for non-integer orders, it must be said that:

- For integral action, left-side Riemann-Liouville fractional integral of the error was considered as:

$${}_0I_t^\lambda e(t) = \int_0^t e(\tau) g_t(\tau) d\tau, \quad (4)$$

where

$$g_t(\tau) = \frac{1}{\Gamma(\lambda + 1)} \left(t^\lambda - (t - \tau)^\lambda \right). \quad (5)$$

Then, taking the axes τ , g_t and e , function $g_t(\tau)$ was plotted in the plane (τ, g_t) for $0 \leq \tau \leq t$. Along the obtained curve, a “fence” was plotted varying $e(\tau)$, so the top edge of the “fence” is a 3D line $(\tau, g_t(\tau), e(\tau))$ for $0 \leq \tau \leq t$. The area of the projection of this “fence” onto the plane (τ, e) corresponds to the value of the integer order integral of the error (i.e., the classical area under the curve), whereas that of the projection onto the plane (g_t, e) corresponds to the value of fractional order integral (4). All this reasoning was adapted from [51].

- For derivative action, straight lines that pass through the point $e(t)$ and whose slope matches the value of the differentiation of order μ of the error curve at that point were plotted.

The following values of the orders were taken for simulations: $\lambda = 0.75$, $\mu_1 = 1.4$, $\mu_2 = 1$, and $\mu_3 = 0.4$.

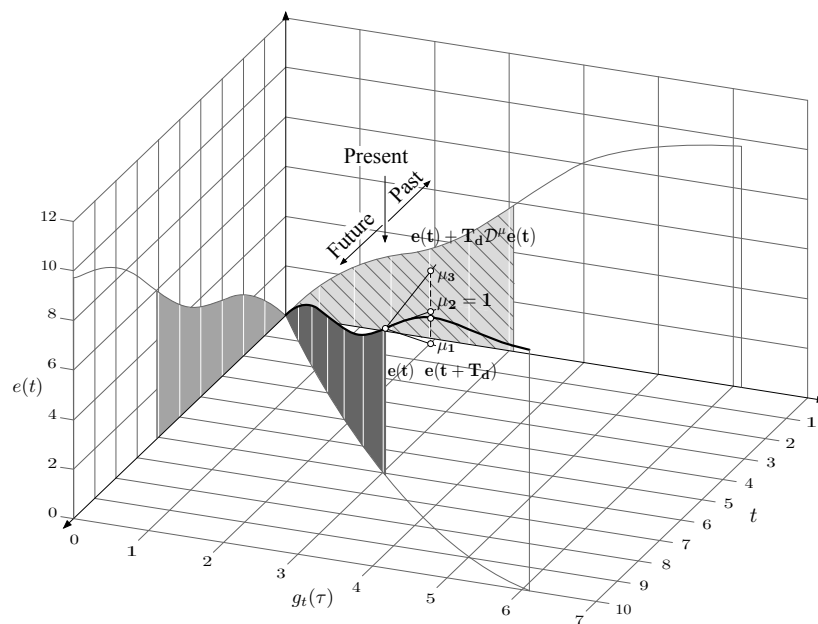


Figure 2. Evolution of the error (fractional PID controllers).

3. Going Into Detail About Parameters

This section contains the explanation of the meaning of the parameters of FOPID controllers, in comparison with those classical of integer order. It should be remarked that, despite the fact the proportional action is not affected by a fractional order, it is also included for completeness of information.

3.1. Proportional Action

The proportional action increases the deviation between the set-point value y_{sp} and the system output y (i.e., the error) with the proportional gain K_p . The main drawback of using a pure proportional control action is that it produces a steady-state error, which motivates that it can be also considered as:

$$u(t) = K_p e(t) + u_b, \quad (6)$$

where u_b is a bias or reset [4]. Indeed, when $e = 0$, the control signal reduces to $u(t) = u_b$. The parameter u_b usually takes the value $(u_{max} + u_{min})/2$, where u_{max} and u_{min} are the maximum and minimum limits of the actuator, respectively. However, sometimes u_b can be adjusted manually to a value that ensures that the steady-state error is zero at a given set-point.

Likewise, the proportional gain can be specified in terms of its inverse proportional band, P_b , which represents the percentage of change in the error signal necessary to cause a full-scale change in the proportional action. As can be observed in Figure 3, the tendency of y to oscillate increases as P_b decreases. The large oscillations occurring with a small P_b are due to the fact that the power is reduced very quickly when the system output enters the proportional band, meaning a balanced state cannot be established immediately.

Indeed, confusing “proportional band” with “proportional gain” leads to a decreased proportional action when the control engineer wants more, and vice-versa.

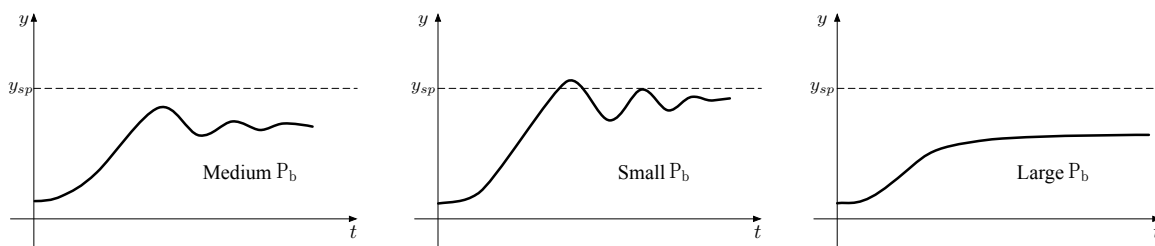


Figure 3. System response for different P_b .

3.2. Integral Action

The main function of the integral action is to guarantee that the steady-state error is zero, i.e., the output of the controlled system is equal to the desired set-point in steady-state. The following simple explanation proves this affirmation. Let us consider a system controlled by a fractional order proportional-integral (PI) controller in steady-state where both the control signal (u_{ss}) and the error (e_{ss}) are constant. If this is the case, the control signal will be given by [3]

$$u_{ss} = K_p \left(e_{ss} + \frac{e_{ss}}{T_i} \frac{t^\lambda}{\Gamma(\lambda + 1)} \right). \quad (7)$$

While $e_{ss} \neq 0$, this clearly contradicts the hypothesis that the control signal u_{ss} is constant. Thus, similarly to the integer case, a controller with integral action of fractional order will always give zero error in steady-state.

Integral action was known originally as a device that automatically resets the bias u_b of a proportional controller. Figure 4 shows the scheme of the extension to non-integer orders of a PI controller with automatic reset. As can be observed, the first order filter in the feedback loop is replaced by its fractional version of order λ . From the block diagram, the control signal can be obtained as

$$u = K_p e + I, \quad (8)$$

with

$$\begin{aligned}
 I &= U(s) \frac{1}{1 + T_i s^\lambda} \\
 \Rightarrow I + T_i \frac{d^\lambda I}{dt} &= e, \\
 \Rightarrow T_i \mathcal{D}^\lambda I &= K_p e \\
 \Rightarrow I &= K_p \frac{1}{T_i} \mathcal{D}^{-\lambda} e(t).
 \end{aligned} \tag{9}$$

Substituting (9) in (8), the control signal is given by

$$u(t) = K_p \left(e(t) + \frac{1}{T_i} \mathcal{D}^{-\lambda} e(t) \right), \tag{10}$$

thus the control scheme of the figure is equivalent to a fractional order PI controller. Figure 5 shows unit step responses of the fractional system in the feedback loop in Figure 4. Unlike the integer order case (see Figure 5a), both the velocity and the shape of resetting can be controlled by the integration order, λ , as shown in Figure 5b. It can be observed that it is even possible to obtain underdamped responses when $\lambda > 1$.

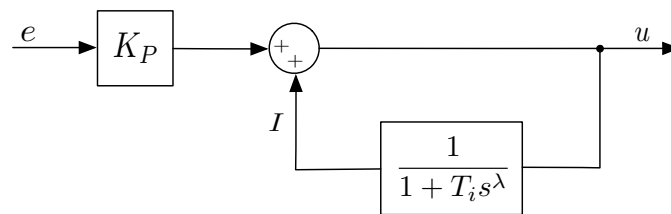


Figure 4. Fractional proportional-integral (PI) controller scheme (classical implementation with automatic reset).

It is important to remark that function `fode_sol()` was taken from [10] to obtain the step responses in MATLAB.

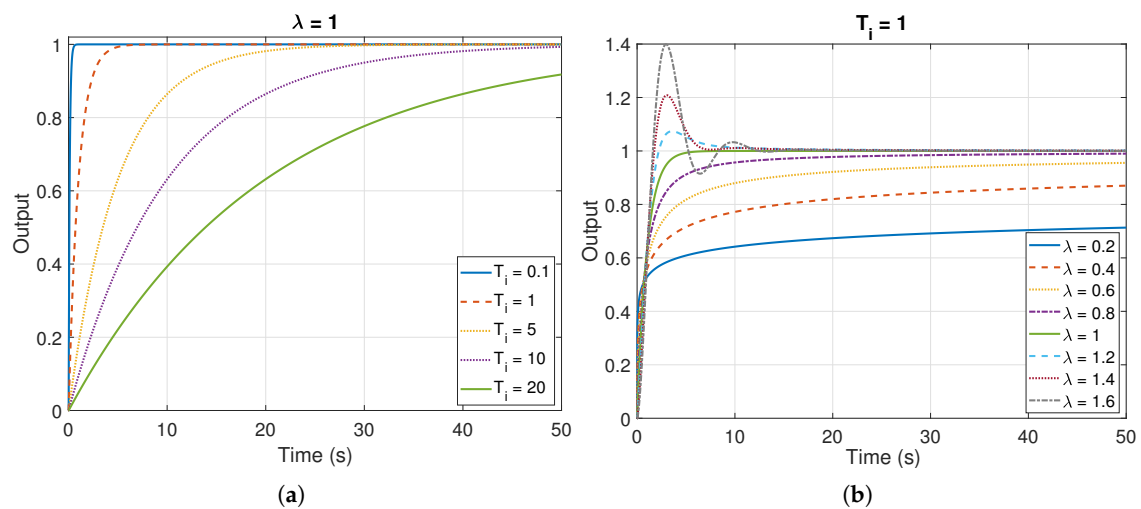


Figure 5. Effects of the automatic reset of the integral action when changing: (a) integral time constant T_i (integer case) (b) integration order λ (fractional case).

3.3. Derivative Action

The objective of the derivative action is to improve the system stability in closed-loop. Classical derivative controllers give responses to changing error signals but do not, however, respond to constant error signals, since with a constant error the rate of change of error with time is zero. Because of this, the derivative term is combined with, at least, the proportional term. By contrast, fractional derivative controllers do respond to constant error signals; the differentiation of fractional order of a constant is different from zero.

The derivative action of a proportional-derivative (PD) controller can be viewed as a crude prediction of the error in future, where the prediction is made by extrapolating the error by the tangent to the error curve at time t , being T_d the prediction horizon (see Figure 1). (Actually, the derivative action uses linear extrapolation, not prediction.) For the fractional case, the basic structure of the controller is

$$u(t) = K_p (e(t) + T_d \mathcal{D}^\mu e(t)). \quad (11)$$

Analogously to the classical case, an approximation of $e(t + T_d)$ may be

$$e(t + T_d) \approx e(t) + T_d \mathcal{D}^\mu e(t). \quad (12)$$

The control signal is then proportional to an estimation of the error at time T_d ahead over a straight line that, in general, is not tangent to the error curve, and whose slope matches the value of the differentiation of μ -th order of the error curve at the point $e(t)$ [52], as shown in Figure 2. Likewise, another way of viewing the prediction is from fractional Taylor series for fractional derivatives. In this case, the prediction horizon is done over a curve (i.e., non-linear extrapolation). More details about prediction can be found in Appendix A.

Therefore, different prediction horizons (in fact, linear and non-linear extrapolations) for the error can be obtained choosing accordingly the value of μ .

The classical implementation of a fractional derivative action is illustrated in Figure 6. From this figure, the following relation can be obtained

$$U(s) = \left(1 - \frac{1}{1 + T_d s^\mu}\right) E(s) = \frac{T_d s^\mu}{1 + T_d s^\mu} E(s), \quad (13)$$

which corresponds to a derivative action of fractional order with noise filter. It is well known that the derivative part of the PID controller requires low-pass filtering to limit the high-frequency gain.

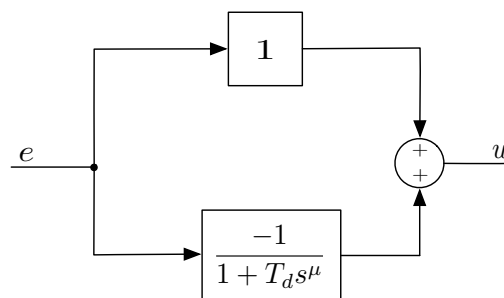


Figure 6. Fractional order derivative action scheme (classical implementation).

Let us focus only on the noise filter of (13), i.e.,

$$G_f(s) = \frac{1}{1 + T_d s^\mu}. \quad (14)$$

Notice that: (1) it has the property $G_f(0) = 1$ to ensure that the process output equals the set-point in steady-state; (2) as usual, it has a low-pass character; and (3) the order of the filter is μ .

In the literature, the classical filters to reduce noise effects in PID controllers are up to second order, usually given as follows [54]:

$$G_{f1}(s) = \frac{1}{1 + T_f s}, \quad (15)$$

$$G_{f2}(s) = \frac{1}{1 + T_f s + T_f^2/2s^2}, \quad (16)$$

where T_f is the filter time constant. Note that filter (16) has complex poles with damping ratio $\delta = 0.707$.

Figure 7 shows Bode plots of the filters, of both integer and fractional order, for $T_d = T_f = 0.1$. As can be seen in Figure 7b, changes in the order μ allow to have different frequency responses ranged between those of the two integer filters (Figure 7a), and consequently, different filtering effects on the error signal can be attained. Another issue to take into account is that most design methods for classical PIDs do not consider noise. Due to this fact, the filter time constant is often suggested to be chosen as a fraction of the derivative time, i.e., $T_f = T_d/N$. However, this solution has severe drawbacks, as reported in [55]. On the contrary, this is not necessary for the fractional case because there is an only parameter to be tuned, μ , and it is included in the controller design.

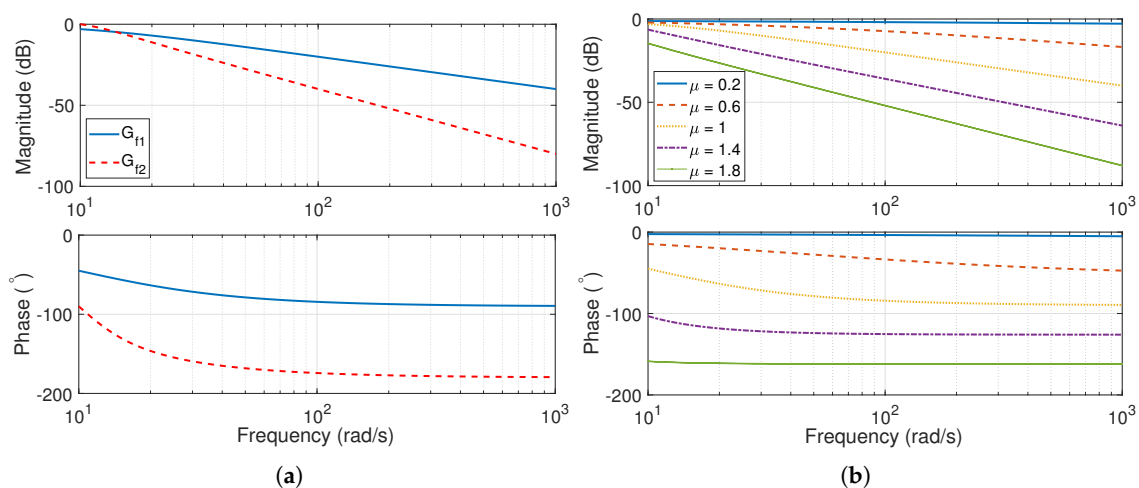


Figure 7. Frequency response of the noise filter: (a) first and second order low-pass filters (integer case) (b) fractional filter for different values of μ (fractional case). For simulations, the following values were taken: $T_d = T_f = 0.1$.

4. Classical Versus Fractional PIDs

This section discusses the similarities and differences between classical and fractional PIDs. In particular, only integral and derivatives parts are compared—the proportional part is similar for both kinds of controllers. The properties described below are summarized in Table 1.

Table 1. Similarities and differences between classical and fractional proportional-integral-derivatives (PID) controllers.

Action	Domain	Effect on	Integer	Fractional
I	Time	Steady-state error	Elimination	
		u with respect to the sign of e	If $e > 0$, u grows linearly with time If $e < 0$, u decreases linearly with time	The same but the growing or the decrease is not linear with time
		Automatic reset	The velocity of reset can be changed	Both the velocity and the shape of reset can be changed
	Frequency	Frequency response	The magnitude curve decreases with a slope of 20 dB/dec Decrease of $\pi/2$ rad in the phase curve	The magnitude curve decreases with a slope of 20λ dB/dec Decrease of $(\pi\lambda)/2$ rad in the phase curve
D	Time	Constant errors	Does not response, so the derivative term needs to be combined with, at least, the proportional term	Does response, so the derivative term can be used individually
		Prediction horizon	Time T_d ahead over the tangent to the error curve at $e(t)$	Time T_d ahead over a curve (non-linear extrapolation) or over a straight line (linear extrapolation) that passes through the point $e(t)$ and whose slope matches the value of the fractional differentiation of order μ of the error curve at that point
	Frequency	Frequency response	The magnitude curve grows with a slope of 20 dB/dec Increment of $\pi/2$ rad in the phase curve	The magnitude curve grows with a slope of 20λ dB/dec Increment of $(\pi\lambda)/2$ rad in the phase curve
			Low-pass filters up to second order	Low-pass filter of order μ
		Filtering	Usually needs two parameters to be tuned, i.e., T_f (filter time constant) and N (ratio between T_d and T_f)	Parameter μ allows to have different frequency responses, ranged between those of the two integer filters, and consequently, have different filtering effects on the error signal.

4.1. Integral Part

As is well known, the main effects of the integral action are those that make the system response slower, decrease its relative stability, and eliminate the steady-state error for inputs for which previously had a finite error. These effects can be observed in the different domains of analysis as follows. In the time-domain, the effects on the transient response cause a decrease of the rising time and an increase of the settling time and the overshoot. In the complex plane, the integral action causes a displacement of the root locus of the system towards the right half-plane. Finally, in the frequency-domain, an increase of -20 dB/dec in the slopes of the magnitude curves and a decrement of $\pi/2$ in the phase plots can be observed.

In the case of a fractional integration order $\lambda \in (0, 1)$, the selection of the value of λ translates into a certain weighting of the effects mentioned above. In the time-domain, for example, due to the fact that the integral action only responds to errors other than zero by increasing the control action, for positive errors, and decreasing it in case of negative, if the error is constant, the control action can be ramped up with different slopes or velocities, as can be observed in Figure 8a. For a square error signal (Figure 8b), it can be observed that there is a set of effects of the control action over the error that range from the pure proportional action ($\lambda = 0$) to the classical integral action ($\lambda = 1$). For intermediate values of λ , the control action grows when the error is constant, which results in a removal of the steady-state error, whereas decreases when the error goes to zero, which reduces the instability of the system. In the complex plane, it can be seen that the selection of a value of parameter λ moves the root locus of the system towards the right half-plane. In the frequency-domain, the fact that λ can vary between 0 and 1 introduces the possibility of adding a constant increment to the slope of the magnitude curve between 0 and -20 dB/dec (actually, a value of -20λ dB/dec), as well as a constant lag to the phase curve between 0 and $-\pi/2$ rad (specifically, a value of $-(\pi\lambda)/2$ rad).

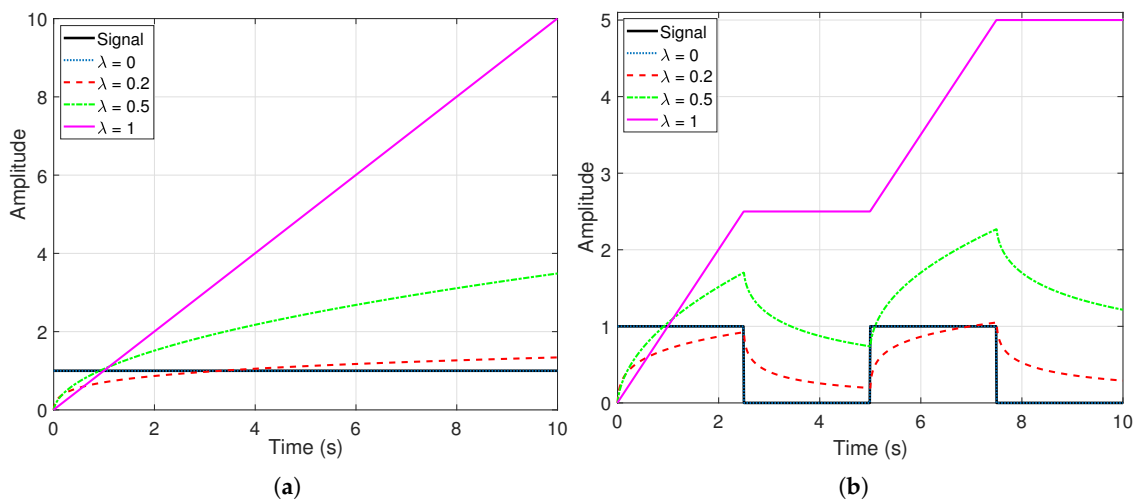


Figure 8. Effects of the integral action according to integration order λ on different error signals: (a) constant (b) square.

4.2. Derivative Part

The derivative action increases the system stability and tends to accentuate the noise effects at high frequencies. In the time-domain, this is manifested, mainly, by a decrease in both the overshoot and the settling time. In the complex plane, it produces a displacement of the system root locus towards the left half-plane. In the frequency-domain, a constant phase lead of $\pi/2$ and an increase of 20 dB/dec in the slopes of the magnitude curves.

Following a reasoning parallel to that made for the integral action, it is easy to understand that all these effects are weighted by choosing an appropriate value of the order of the derivative, $\mu \in (0, 1)$ (see Figure 9).

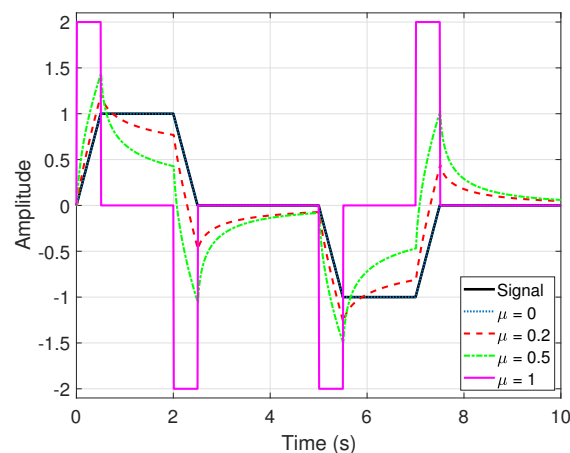


Figure 9. Effects of the derivative action according to differentiation order μ on a trapezoidal signal.

5. Illustrative Examples

This section provides two examples to illustrate the properties of both integral and derivative actions, as well the effects of their fractional orders.

Example 1. Let us consider a system given by the following transfer function [4]

$$G(s) = \frac{1}{(s+1)^3} \quad (17)$$

controlled by a fractional order PI of the form:

$$C(s) = K_p \left(1 + \frac{1}{T_i s^\lambda} \right) \quad (18)$$

To illustrate the properties of the integral action and the effect of its fractional order, Figure 10 shows unit step responses in closed-loop of system (17) controlled by fractional PI controller (18). The proportional gain is constant, $K_p = 1$, whereas the integral time T_i and λ are changed individually in Figure 10a,b, respectively. More precisely, responses for the classical case ($\lambda = 1$) are plotted in Figure 10a. As expected, the steady-state error is removed when T_i takes finite values; the case $T_i = \infty$ corresponds to pure proportional control, where the steady-state error is 50 percent. Likewise, the smaller the values of T_i , the faster the response, but the more oscillatory [4]. The effects of changing the integration order, λ , can be seen in Figure 10b (T_i is fixed to 2). In this case, the smaller the values of λ , the slower the response. Unlike T_i , oscillations and system response are not affected by parameter λ .

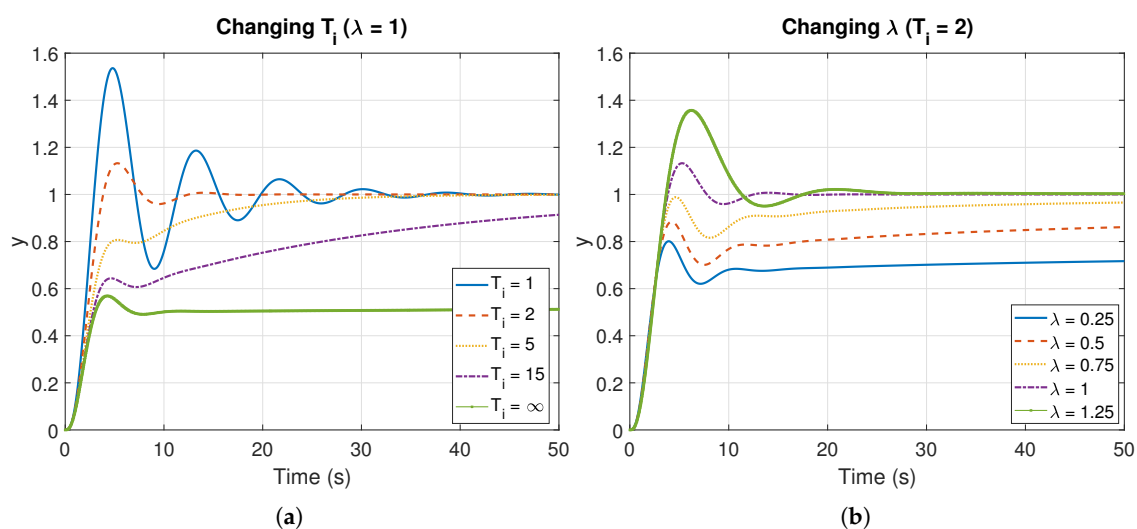


Figure 10. Step responses in closed-loop of system (17) controlled by fractional PI controller (18) when changing: (a) integral time constant T_i ($K_p = 1$, $\lambda = 1$) (b) integration order λ ($K_p = 1$, $T_i = 2$).

Example 2. Now, consider the double integrator with unit gain, i.e.,

$$G(s) = \frac{1}{s^2} \quad (19)$$

controlled by a fractional order PD of the form:

$$C(s) = K_p (1 + T_d s^\mu). \quad (20)$$

The properties of the derivative action and its non-integer order are illustrated in Figure 11, which shows unit step responses in closed-loop of system (19) controlled by fractional PD controller (20). Similarly to the previous example, the proportional gain is constant, $K_p = 1$, whereas the derivative time T_d and μ are changed in an individual way in Figure 11a,b, respectively. In particular, responses for the classical case ($\mu = 1$) are plotted in Figure 11a. As expected, damping increases when T_d increases. In other words, the higher the values of T_d , the faster the response. The effects of changing μ can be observed in Figure 11b (T_d is set to 2). In this case, the smaller the values of μ , the slower the damping. Unlike T_d , μ only affects the overshoot.

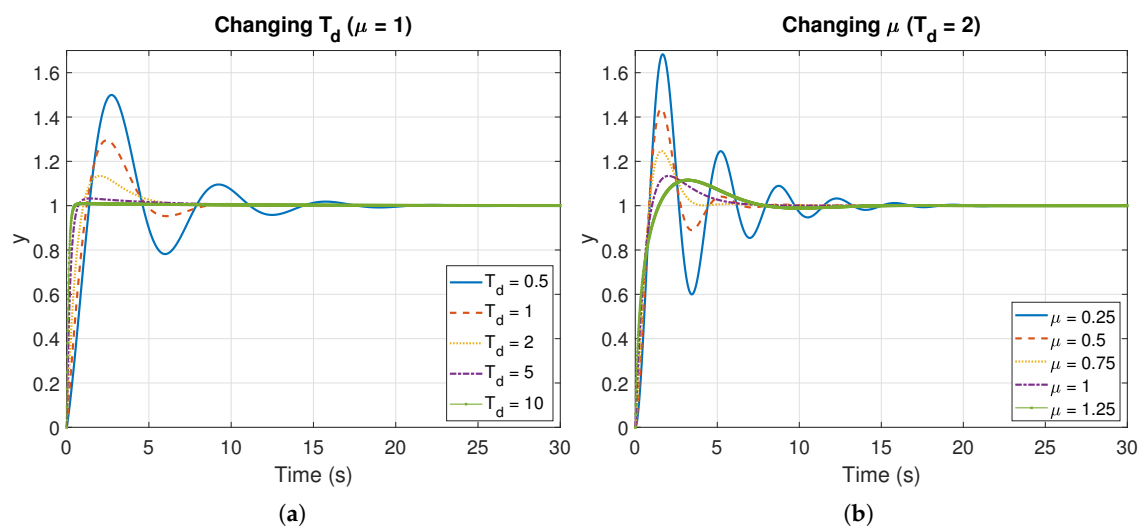


Figure 11. Step responses in closed-loop of system (19) controlled by fractional PD controller (20) when changing: (a) derivative time constant T_i ($K_p = 1$, $\mu = 1$) (b) integration order μ ($K_p = 1$, $T_d = 2$).

It must be said that the function `fode_sol()` was also used to obtain the step responses of both examples.

6. Conclusions

Although ample studied and successfully applied to many kinds of control problems, the extension of classical proportional-integral-derivative (PID) controllers to non-integer orders presents a main weakness: basis, in terms of understanding of the effects of their parameters on the system response, is sometimes omitted, and even unknown. This can be explained, in part, because the geometric and physical interpretation of integrals and derivatives of arbitrary (not necessary integer) order still remains as an open problem. This paper has focused on the new meaning of integral and derivative actions, and gains, derived by the consideration of non-integer integration and differentiation orders, i.e., for fractional order PID (FOPID) controllers. Similarities and differences between classical and fractional PIDs, as well as illustrative examples were given for a best understanding of the possibilities of control with the latter.

When the integral term is concerned, it was shown that it responds with selective memory because of the non-integer order λ . That was also viewed as the area of the projection of the error curve onto a plane, which is definitively different from the classical area under the error curve. Moreover, taking into account a fractional proportional-integral (PI) controller scheme with automatic reset, unlike the integer order case, it was also shown that both the velocity and the shape of resetting can be controlled by the integration order, λ .

With respect to derivative action, it was illustrated that different prediction horizons (in fact, linear and non-linear extrapolations) for the error can be obtained for a fractional proportional-derivative (PD) choosing accordingly the value of the differentiation order, μ . Likewise, in case of an structure with noise filter, the value of μ allows different filtering effects on the error signal to be attained.

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Appendix A. Analysis of Prediction Horizon

This appendix gives information about the approximation of the prediction of the error at time T_d ahead, i.e., the approximation of $e(t + T_d)$ to understand the meaning of a fractional order derivative action.

Given a continuous function y , the fractional Euler method establishes that [56]

$$y(t_{i+1}) \approx y(t_i) + \mathcal{D}^\mu y(t_i) \frac{h^\mu}{\Gamma(\mu + 1)} ((i + 1)^\mu - i^\mu). \quad (\text{A1})$$

Let assume that the error $e(t)$ is a continuous function, and $T_d = kh$, $k \in \mathbb{Z}^+$, where h is a fixed step. Thus, $e(t + T_d)$ can be approximated by:

$$e(t + T_d) \approx e(t) + \mathcal{D}^\mu e(t) \frac{h^\mu}{\Gamma(\mu + 1)} (T_d^\mu - t^\mu), \quad (\text{A2})$$

which is referred to as approximation #1. Likewise, analogously with the integer case, the prediction may be also expressed as

$$e(t + T_d) \approx e(t) + T_d \mathcal{D}^\mu e(t), \quad (\text{A3})$$

It is referred to as approximation #2.

Hence, taking into account approximation #1, the prediction in future of the error is done from a non-linear extrapolation due to the fact that the term on the right in (A2) is non-linear. In contrast, the prediction with (A3) is carried out over a straight line (linear extrapolation) that passes for point t , and whose slope matches the value of the μ -th order differentiation of the error. Thus, the control signal is proportional to an estimation of the control error at time T_d ahead, where that estimation is obtained by non-linear or linear extrapolation. Both predictions are illustrated in Figure A1 for different values of μ and T_d , namely, $\mu = \{0.4, 0.8, 1, 1.25\}$ and $T_d = \{1, 2, 1.5\}$, and considering the error as $e(t) = t + 0.5 \sin(t)$ and $e(t) = 0.01t^2(1 - 2 \cos(t))$. As can be observed, although there are big differences between the approximations above, the prediction function assumed to the derivative term is guaranteed at time T_d ahead, especially when approximation given by (A2) is taken: it approaches, to a large extent, the actual curve of the error.

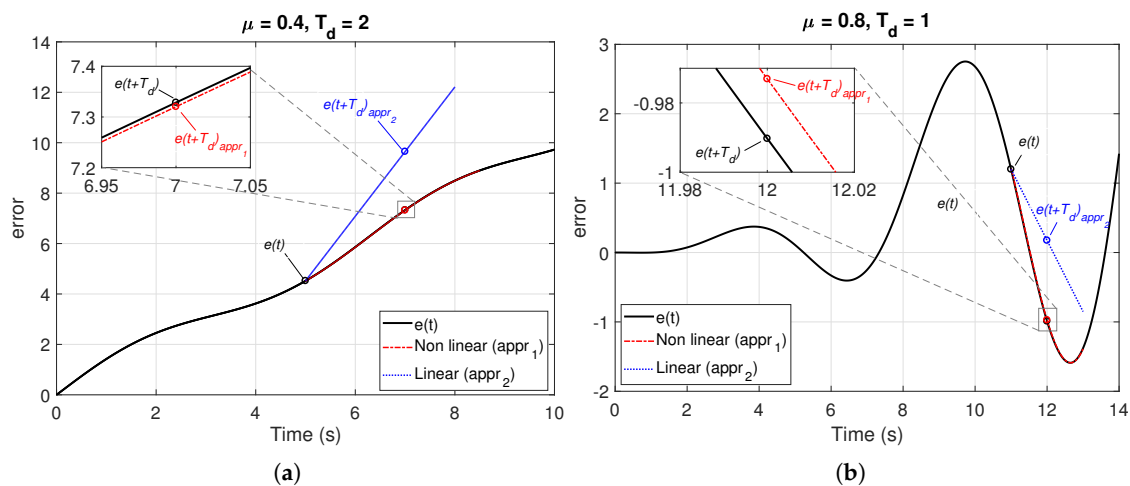


Figure A1. Cont.

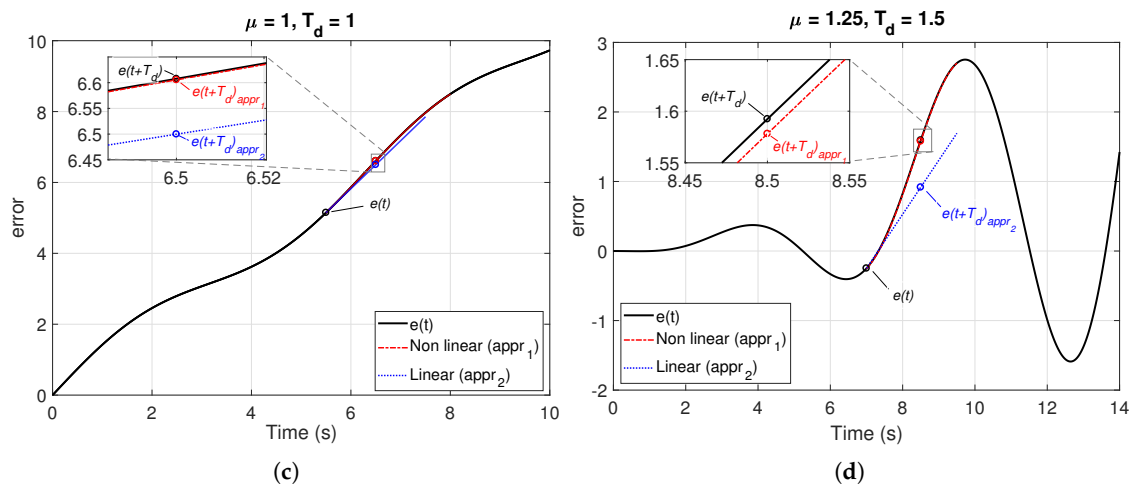


Figure A1. Prediction horizons of derivative part depending on the order μ : (a) $\mu = 0.4, T_d = 2$ (b) $\mu = 0.8, T_d = 1$ (c) $\mu = 1, T_d = 1$ (d) $\mu = 1.25, T_d = 1.5$. Notice that cases (a) and (c) correspond to the error as the function $e(t) = t + 0.5 \sin(t)$, whereas the error is given by $e(t) = 0.01t^2(1 - 2 \cos(t))$ in cases (b) and (d).

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