mathematics

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On the Semilocal Convergence of the Multi–Point Variant of Jarratt Method: Unbounded Third Derivative Case

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Received: 10 May 2019; Accepted: 10 June 2019; Published: 13 June 2019

Abstract: In this paper, we study the semilocal convergence of the multi-point variant of Jarratt method under two different mild situations. The first one is the assumption that just a second-order Fréchet derivative is bounded instead of third-order. In addition, in the next one, the bound of the norm of the third order Fréchet derivative is assumed at initial iterate rather than supposing it on the domain of the nonlinear operator and it also satisfies the local ω-continuity condition in order to prove the convergence, existence-uniqueness followed by a priori error bound. During the study, it is noted that some norms and functions have to recalculate and its significance can be also seen in the numerical section.

Keywords: Banach space; semilocal convergence; ω-continuity condition; Jarratt method; error bound

MSC: 65J15; 65H10; 65G99; 47J25

1. Introduction

The problem of finding a solution of the nonlinear equation affects a large area of various fields. For instance, kinetic theory of gases, elasticity, applied mathematics and also engineering dynamic systems are mathematically modeled by difference or differential equations. Likewise, there are numerous problems in the field of medical, science, applied mathematics and engineering that can be reduced in the form of a nonlinear equation. Many of those problems cannot be solved directly through any of the methods. For this, we opt for numerical procedure and are able to find at least an approximate solution of the problem using various iterative methods. In this concern, Newton's method [1] is one of the best and most renowned quadratically convergent iterative methods in Banach spaces, which is frequently used by the authors as it is an efficient method and has a smooth execution. Now, consider a nonlinear equation having the form

$$L(m) = 0, \tag{1}$$

where L is a nonlinear operator defined as $L : B \subseteq \nabla_1 \to \nabla_2$, where B is a non-empty open convex domain of a Banach space ∇_1 with values in a Banach space ∇_2 which is usually known as the Newton–Kantorovich method that can be defined as

$$\begin{cases} m_0 \ \ given \ \ in \ \ B, \\ m_n = m_{n-1} - [L'(m_{n-1})]^{-1} L(m_{n-1}), \ n \in \ \mathbb{N}, \end{cases}$$

where $L'(m_{n-1})$ is the Fréchet derivative of L at m_{n-1}. The results on semilocal convergence have been originally studied by L.V. Kantorovich in [2]. In the early stages, he gave the method of recurrence relations and afterwards described the method of majorant principle. Subsequently, Rall in [3] and many researchers have studied the improvements of the results based on recurrence relations. A large number of researchers studied iterative methods of various order to solve the nonlinear equations extensively. The convergence of iterative methods generally relies on two types: semilocal and local convergence analysis. In the former type, the convergence of iterative methods depends upon the information available around the starting point, whereas, in the latter one, it depends on the information around the given solution.

In the literature, researchers have developed various higher order schemes in order to get better efficiency and also discussed their convergence. Various types of convergence analysis using different types of continuity conditions viz. Lipschitz continuity condition has been studied by Wang et al. in [4,5], Singh et al. in [6], and Jaiswal in [7], to name a few. Subsequently, many authors have studied the weaker continuity condition than Lipschitz namely Hölder by Hernández in [8], Parida and Gupta in [9,10], Wang and Kou in [11] are some of them. Usually, there are some nonlinear equations that neither satisfy Lipschitz nor Hölder continuity conditions; then, we need a generalized form of continuity condition such as ω-continuity, which has been studied by Ezquerro and Hernández in [12,13], Parida and Gupta in [14,15], Prashanth and Gupta in [16,17], Wang and Kou in [18–20], etc.

The algorithms having higher order of convergence plays an important role where the quick convergence is required like in the stiff system of equations. Thus, it is quite interesting to study higher order methods. In this article, we target our study on the semilocal convergence analysis using recurrence relations technique on the multi-point variant of Jarratt method when the third order Fréchet derivative becomes unbounded in the given domain.

2. The Method and Some Preliminary Results

Throughout the paper, we use the below mentioned notations:

B $\equiv$ non-empty open subset of ∇_1; $B_0 \subseteq B$ is a non-empty convex subset; $\nabla_1,\ \nabla_2 \equiv$ Banach spaces, $U(m,b) = \{n \in \nabla_1 : \|n-m\| < b\}$, $\overline{U(m,b)} = \{n \in \nabla_1 : \|n-m\| \leq b\}$.

Here, we consider the multi-point variant of the Jarratt method suggested in [21]

$$\begin{aligned} n_n &= m_n + \tfrac{2}{3}(p_n - m_n), \\ o_n &= m_n - \Upsilon_L(m_n)\wp_n L(m_n), \\ m_{n+1} &= o_n - \left[\tfrac{3}{2}L'(n_n)^{-1}\Upsilon_L(m_n) + \wp_n\left(I - \tfrac{3}{2}\Upsilon_L(m_n)\right)\right] L(o_n), \end{aligned} \tag{2}$$

where $\Upsilon_L(m_n) = [6L'(n_n) - 2L'(m_n)]^{-1}[3L'(n_n) + L'(m_n)]$, $\wp_n = [L'(m_n)]^{-1}$, $p_n = m_n - \wp_n L(m_n)$ and I is the identity operator. In the same article for deriving semilocal convergence results, the researchers have assumed the following hypotheses:

$(A1)\|\wp_0 L(m_0)\| \leq \kappa$,
$(A2)\|\wp_0\| \leq \lambda$,
$(A3)\|L''(m)\| \leq P, m \in B$,
$(A4)\|L'''(m)\| \leq Q, m \in B$,
$(A5)\|L'''(m) - L'''(n)\| \leq \omega(\|m-n\|),\ \forall\, m,n \in B$,

where $\omega : R_+ \to R_+$, is a continuous and non-decreasing function for $m > 0$ such that $\omega(m) \geq 0$ and satisfying $\omega(\epsilon z) \leq \phi(\epsilon)\omega(z), \epsilon \in [0,1]$ and $z \in [0,+\infty)$ with $\phi : [0,1] \to R_+$, is also continuous and non-decreasing. One can realize that, if $\omega(m) = Lm$, then this condition is reduced into Lipschitz and when $\omega(m) = Lm^q$, $q \in (0,1]$ to the Hölder. Furthermore, we found some nonlinear functions which are unbounded in a given domain but seem to be bounded on a particular point of the domain.

For a motivational example, consider a function h on $(-2,2)$. We can verify the above fact by considering the following example [22]

$$h(m) = \begin{cases} m^3 ln(m^2) - 6m^2 - 3m + 8 & m \in (-2,0) \cup (0,2), \\ 0, & m = 0. \end{cases} \tag{3}$$

Clearly, we can see this fact that $h'''(m)$ is unbounded in $(-2,2)$. Hence, for avoiding the unboundedness of the function, we replace the condition $(A4)$ by the milder condition since the given example is bounded at $m = 1$. Thus, here we can assume that the norm of the third order Fréchet derivative is bounded on the initial iterate as:

$(B1)\|L'''(m_0)\| \leq \overline{A},\ m_0 \in B_0,$
where m_0 be an initial approximation. Moreover, we also assume
$(B2)\|L'''(m) - L'''(n)\| \leq \omega(\|m-n\|)\ \forall\ m,n \in B(m_0,\varepsilon),$
where $\varepsilon > 0$. For now, we choose $\varepsilon = \frac{\kappa}{\tilde{\tau}_0}$, where $\tilde{\tau}_0$ will be defined later and the rationality of this choice of such ε will be proved. Moreover, some authors have considered partial convergence conditions. The following nonlinear integral equation of mixed Hammerstein type [23]

$$m(s) = 1 + \int_0^1 G(s,t)\left(\frac{1}{2}m(t)^{\frac{5}{2}} + \frac{7}{16}m(t)^3\right)dt, s \in [0,1], \tag{4}$$

where $m \in [0,1], t \in [0,1], G(s,t)$ is the Green function defined by

$$G(s,t) = \begin{cases} (1-s)t & t \leq s, \\ s(1-t) & s \leq t, \end{cases}$$

is an example that justified this idea which will be proved later in the numerical application section. In this study, on using recurrence relations, we first discuss the semilocal convergence of the above-mentioned algorithm by just assuming that the second-order Fréchet derivative is bounded. In addition, next, we restrict the domain of the nonlinear operator and consider the bound of the norm of the third-order Fréchet derivative on an initial iterate only rather than supposing it on the given domain of the nonlinear operator.

We start with a nonlinear operator $L : B \subseteq \nabla_1 \to \nabla_2$ and let the Hypotheses $(A1)$–$(A3)$ be fulfilled. Consider the following auxiliary scalar functions out of which Δ and Λ function are taken from the reference [21] and Γ and Θ have been recalculated:

$$\begin{aligned} \Gamma(\theta) &= 1 + \tfrac{1}{2}\tfrac{\theta}{1-\theta} + \left[1 + \tfrac{\theta}{1-\frac{2}{3}\theta}\left(1 + \tfrac{1}{2}\tfrac{\theta}{1-\theta}\right)\right] \\ &\quad \times \tfrac{\theta}{2}\left[\tfrac{1}{1-\theta} + \left(1 + \tfrac{1}{2}\tfrac{\theta}{1-\theta}\right)^2\right], \end{aligned} \tag{5}$$

$$\Delta(\theta) = \frac{1}{1 - \theta\Gamma(\theta)}, \tag{6}$$

$$\begin{aligned} \Theta(\theta) = \ & \Bigg[\tfrac{\theta}{1-\frac{2}{3}\theta}\left(1 + \tfrac{1}{2}\tfrac{\theta}{1-\theta}\right) + \theta\left[1 + \tfrac{\theta}{1-\frac{2}{3}\theta}\left(1 + \tfrac{1}{2}\tfrac{\theta}{1-\theta}\right)\right] \\ & + \tfrac{\theta^2}{2(1-\theta)}\left[1 + \tfrac{\theta}{1-\frac{2}{3}\theta}\left(1 + \tfrac{1}{2}\tfrac{\theta}{1-\theta}\right)\right] \\ & + \tfrac{\theta}{2}\left[1 + \tfrac{\theta}{1-\frac{2}{3}\theta}\left(1 + \tfrac{1}{2}\tfrac{\theta}{1-\theta}\right)\right]^2 \Lambda(\theta)\Bigg]\Lambda(\theta), \end{aligned} \tag{7}$$

where

$$\Lambda(\theta) = \frac{\theta}{2}\left[\frac{1}{1-\theta} + \left(1+\frac{1}{2}\frac{\theta}{1-\theta}\right)^2\right]. \tag{8}$$

Next, we study some of the properties of the above-stated functions. Let $k(\theta) = \Gamma(\theta)\theta - 1$. Since $k(0) = -1 < 0$ and $k(\frac{1}{2}) \approx 1.379 > 0$, then the function $k(t)$ has at least one real root in $(0, \frac{1}{2})$. Suppose γ is the smallest positive root, then clearly $\gamma < \frac{1}{2}$. Now, we begin with the following lemmas that will be used later in the main theorem(s).

Lemma 1. *Let the functions Γ, Δ and Θ be given in Equations (5)–(7), respectively, and γ be the smallest positive real root of $\Gamma(\theta)\theta - 1$. Then,*

(a) $\Gamma(\theta)$ *and* $\Delta(\theta)$ *are increasing and* $\Gamma(\theta) > 1, \Delta(\theta) > 1$ *for* $\theta \in (0, \gamma)$,

(b) *for* $\theta \in (0, \gamma)$, $\Theta(\theta)$ *is an increasing function.*

Proof. The proof is straightforward from the expressions of Γ, Δ and Θ given in Relations (5)–(7), respectively. □

Define $\kappa_0 = \kappa, \lambda_0 = \lambda, \tau_0 = P\lambda\kappa$ and $\zeta_0 = \Delta(\tau_0)\Theta(\tau_0)$. Furthermore, we designate the following sequences as:

$$\kappa_{n+1} = \zeta_n \kappa_n, \tag{9}$$

$$\lambda_{n+1} = \Delta(\tau_n)\lambda_n, \tag{10}$$

$$\tau_{n+1} = P\lambda_{n+1}\kappa_{n+1} = \Delta(\tau_n)\zeta_n\tau_n, \tag{11}$$

$$\zeta_{n+1} = \Delta(\tau_{n+1})\Theta(\tau_{n+1}), \tag{12}$$

where $n \geq 0$. Some important properties of the immediate sequences are given by the following lemma.

Lemma 2. *If $\tau_0 < \gamma$ and $\Delta(\tau_0)\zeta_0 < 1$, where γ is the smallest positive root of $\Gamma(\theta)\theta - 1 = 0$, then we have*

(a) $\Delta(\tau_n) > 1$ *and* $\zeta_n < 1$ *for* $n \geq 0$,

(b) *the sequences* $\{\kappa_n\}, \{\tau_n\}$ *and* $\{\zeta_n\}$ *are decreasing,*

(c) $\Gamma(\tau_n)\tau_n < 1$ *and* $\Delta(\tau_n)\zeta_n < 1$ *for* $n \geq 0$.

Proof. The proof can be done readily using mathematical induction. □

Lemma 3. *Let the functions Γ, Δ and Θ be given in the Relations (5)–(7), respectively. Assume that $\alpha \in (0, 1)$, then $\Gamma(\alpha\theta) < \Gamma(\theta)$, $\Delta(\alpha\theta) < \Delta(\theta)$ and $\Theta(\alpha\theta) < \alpha^2\Theta(\theta)$, for $\theta \in (0, \gamma)$.*

Proof. For $\alpha \in (0, 1)$, $\theta \in (0, \gamma)$ and by using the Equations (5)–(7), this lemma can be proved. □

3. Recurrence Relations for the Method

Here, we characterized some norms which are already derived in the reference [21] for the Method (2) and some are recalculated here.

For $n = 0$, the existence of $\wp_0$ implies the existence of p_0, n_0 and further, we have

$$\|p_0 - m_0\| \leq \kappa_0, \; \|n_0 - m_0\| \leq \frac{2}{3}\kappa_0, \tag{13}$$

i.e., p_0 and $n_0 \in U(m_0, \rho\kappa)$, where $\rho = \frac{\Gamma(\tau_0)}{1-\zeta_0}$. Let $R(m_0) = \wp_0[L'(n_0) - L'(m_0)]$ also; since $\tau_0 < 1$, we have

$$\|R(m_0)\| \leq \frac{2}{3}\tau_0\,, \quad \left\| \left[I + \frac{3}{2}R(m_0) \right]^{-1} \right\| \leq \frac{1}{1-\tau_0}. \tag{14}$$

Moreover,

$$\begin{aligned} \|\Upsilon_L(m_0)\| &= \left\| I - \tfrac{3}{4}\left[I + \tfrac{3}{2}R(m_0) \right]^{-1} R(m_0) \right\| \\ &\leq 1 + \left\| \tfrac{3}{4}\left[I + \tfrac{3}{2}R(m_0) \right]^{-1} \right\| \|R(m_0)\| \\ &\leq 1 + \tfrac{1}{2}\tfrac{\tau_0}{1-\tau_0}. \end{aligned} \tag{15}$$

From the second sub-step of the considered scheme, it is obvious that

$$\|o_0 - m_0\| \leq \left[1 + \frac{1}{2}\frac{\tau_0}{1-\tau_0} \right] \kappa_0. \tag{16}$$

It is similar to obtain

$$\|o_0 - p_0\| \leq \left[\frac{1}{2}\frac{\tau_0}{1-\tau_0} \right] \kappa_0. \tag{17}$$

Using the Banach Lemma, we realize that $L'(n_0)^{-1}$ exists and can be bounded as

$$\|L'(n_0)^{-1}\| \leq \frac{\lambda_0}{1 - \frac{2}{3}\tau_0}. \tag{18}$$

From Taylor's formula, we have

$$\begin{aligned} L(o_0) &= L(m_0) + L'(m_0)(o_0 - m_0) \\ &+ \int_0^1 [L'(m_0 + \theta(o_0 - m_0)) - L'(m_0)] d\theta (o_0 - m_0). \end{aligned} \tag{19}$$

From the above relation, it follows that

$$\|L(o_0)\| \leq \Lambda(\tau_0)\frac{\kappa}{\lambda}. \tag{20}$$

Though in the considered reference [21] the norm $\|m_1 - o_0\|$ has already been calculated, here we are recalculating it in a more precise way such that the recalculated norm becomes finer than the given in the reference [21] and its significance can be seen in the numerical section. The motivation for recalculating this norm has been also discussed later. From the last sub-step of the Equation (2),

$$\begin{aligned} m_1 - o_0 &= - \left[\frac{3}{2}L'(n_0)^{-1}\Upsilon_L(m_0) + \wp_n \left(I - \frac{3}{2}\Upsilon_L(m_0) \right) \right] L(o_0) \\ &= - \left[\wp_0 + \frac{3}{2}[L'(n_0)^{-1} + L'(n_0)^{-1}] \right] \Upsilon_L(m_0) L(o_0). \end{aligned}$$

On taking the norm, we have

$$\begin{aligned} \|m_1 - o_0\| \leq \tfrac{\kappa_0}{2} & \left[1 + \tfrac{\tau_0}{1-\frac{2}{3}\tau_0} \left(1 + \tfrac{1}{2}\tfrac{\tau_0}{1-\tau_0} \right) \right] \\ & \times \left[\tfrac{\tau_0}{1-\tau_0} + \tau_0 \left(1 + \tfrac{1}{2}\tfrac{\tau_0}{1-\tau_0} \right)^2 \right], \end{aligned} \tag{21}$$

and thus we obtain

$$\|m_1 - m_0\| \leq \|m_1 - o_0\| + \|o_0 - m_0\| \leq \Gamma(\tau_0)\kappa_0. \tag{22}$$

Hence, $m_1 \in U(m_0, \rho\kappa)$. Now, since the assumption $\zeta_0 < \frac{1}{\Delta(\tau_0)} < 1$, notice that $\tau_0 < \gamma$ hence $\Gamma(\tau_0) < \Gamma(\gamma)$ and it can be written as

$$\|I - \wp_0 L'(m_1)\| \leq \tau_0 \Lambda(\tau_0) < 1. \tag{23}$$

Thus, $\wp_1 = [L'(m_1)]^{-1}$ exists and, by virtue of Banach lemma, it may be written as

$$\|\wp_1\| \quad \leq \quad \frac{\lambda_0}{1 - \tau_0\Gamma(\tau_0)} = \lambda_1.$$

Again by Taylor's expansion along o_n, we can write

$$\begin{aligned} L(m_{n+1}) \quad &= L(o_n) + L'(p_n)(m_{n+1} - o_n) \\ &+ \textstyle\int_0^1 [L'(o_n + \theta(m_{n+1} - o_n)) - L'(p_n)] d\theta (m_{n+1} - o_n), \end{aligned} \tag{24}$$

and

$$L'(p_n) = L'(m_n) + \int_0^1 L''(m_n + \theta(p_n - m_n)) d\theta (p_n - m_n). \tag{25}$$

On using the above relation and, for $n = 0$, Equation (24) assumes the form

$$\begin{aligned} L(m_1) \quad = \quad & L(o_0) + L'(m_0)(m_1 - o_0) \\ & + \int_0^1 L''(m_0 + \theta(p_0 - m_0)) d\theta (p_0 - m_0)(m_1 - o_0) \\ & + \int_0^1 [L'(o_0 + \theta(m_1 - o_0)) - L'(p_0)] d\theta (m_1 - o_0). \end{aligned}$$

Using the last sub-step of the Scheme given in the Equation (2), the above expression can be rewritten as

$$\begin{aligned} L(m_1) \quad = \quad & \frac{3}{2}[L'(n_0) - L'(m_0)] L'(n_0)^{-1} \Upsilon_L(m_0) L(o_0) \\ & + \int_0^1 L''(m_n + \theta(p_n - m_n)) d\theta (p_n - m_n)(m_{n+1} - o_n) \\ & + \int_0^1 [L'(o_n + \theta(m_{n+1} - o_n)) - L'(p_n)] d\theta (m_{n+1} - o_n). \end{aligned}$$

In addition, thus,

$$\|L(m_1)\| \leq \Theta(\tau_0) \frac{\kappa}{\lambda}. \tag{26}$$

Hence,

$$\|p_1 - m_1\| \leq \Delta(\tau_0)\Theta(\tau_0)\kappa_0 = \kappa_1.$$

In addition, because $\Gamma(\tau_0) > 1$ and by triangle inequality, we find

$$\|p_1 - m_0\| \leq \rho\kappa,$$

and

$$\|n_1 - m_0\| \leq \|m_1 - m_0\| + \left\| \frac{2}{3}(p_1 - m_1) \right\| \leq (\Gamma(\tau_0) + \zeta_0)\kappa_0 < \rho\kappa,$$

which implies $p_1, n_1 \in U(m_0, \rho\kappa)$. Furthermore, we have

$$P\|\wp_1\|\|\wp_1 L(m_1)\| \leq \Delta^2(\tau_0)\Theta(\tau_0)\tau_0 = \tau_1. \tag{27}$$

Moreover, we can state the following lemmas.

Lemma 4. *Under the hypotheses of Lemma 2, let $\sigma = \Delta(\tau_0)\zeta_0$ and $\varsigma = \frac{1}{\Delta(\tau_0)}$, we have*

$$\begin{aligned}
\zeta_i &\leq \varsigma\sigma^{3^n}, && (28)\\
\prod_{i=0}^{n}\zeta_i &\leq \varsigma^{n+1}\sigma^{\frac{3^{n+1}-1}{2}}, && (29)\\
\kappa_n &\leq \kappa\varsigma^n\sigma^{\frac{3^n-1}{2}}, && (30)\\
\sum_{i=n}^{n+m}\kappa_i &\leq \kappa\varsigma^n\sigma^{\frac{3^n-1}{2}}\left(\frac{1-\varsigma^{m+1}\sigma^{\frac{3^n(3^m+1)}{2}}}{1-\varsigma\sigma^{3^n}}\right), && (31)
\end{aligned}$$

where $n \geq 0$ and $m \geq 1$.

Proof. In order to prove this lemma, first, we need to derive

$$\zeta_n \leq \varsigma\sigma^{3^n}.$$

We will prove it by executing the induction. By Lemma 3 and since $\tau_1 = \sigma\tau_0$, hence for $n = 1$,

$$\zeta_1 = \Delta(\sigma\tau_0)\Theta(\sigma\tau_0) < \sigma^2\zeta_0 < \varsigma\sigma^{3^1}.$$

Let it be true for $n = k$, then

$$\zeta_k \leq \varsigma\sigma^{3^k}, k \geq 1.$$

Now, we will prove it for $n = k + 1$. Thus,

$$\zeta_{k+1} < \Delta(\sigma\tau_k)\Theta(\sigma\tau_k) < \varsigma\sigma^{3^{k+1}}.$$

Therefore, $\zeta_n \leq \varsigma\sigma^{3^n}$ is true for $n \geq 0$. Making use of this inequality, we have

$$\prod_{i=0}^{k}\zeta_i \leq \prod_{i=0}^{k}\varsigma\sigma^{3^i} = \varsigma^{k+1}\prod_{i=0}^{k}\sigma^{3^i} = \varsigma^{k+1}\sigma^{\frac{3^{k+1}-1}{2}}, k \geq 0.$$

By making use of the above-derived inequality in the Relation (9), we have

$$\kappa_n = \zeta_{n-1}\kappa_{n-1} = \zeta_{n-1}\zeta_{n-2}\kappa_{n-2} = \cdots = \kappa_0\prod_{i=0}^{n-1}\zeta_i \leq \kappa\varsigma^n\sigma^{\frac{3^n-1}{2}}, n \geq 0.$$

With the evidence that $0 < \varsigma < 1$ and $0 < \sigma < 1$, we can say that $\kappa_n \to 0$ as $n \to \infty$. Let us denote

$$\varpi = \sum_{i=k}^{k+m}\varsigma^i\sigma^{\frac{3^i}{2}}, k \geq 0, m \geq 1.$$

The above equation may also be rewritten in the following form

$$\begin{aligned} \varpi &\leq \varsigma^k \sigma^{\frac{3^k}{2}} + \varsigma\sigma^{3^k} \sum_{i=k}^{k+m-1} \varsigma^i \sigma^{\frac{3^i}{2}} \\ &= \varsigma^k \sigma^{\frac{3^k}{2}} + \varsigma\sigma^{3^k} \left(\varpi - \varsigma^{k+m} \sigma^{\frac{3^{k+m}}{2}} \right), \end{aligned}$$

and then it becomes

$$\varpi < \varsigma^k \sigma^{\frac{3^k}{2}} \left(\frac{1 - \varsigma^{m+1} \sigma^{\frac{3^k(3^m+1)}{2}}}{1 - \varsigma\sigma^{3^k}} \right).$$

Moreover,

$$\sum_{i=k}^{k+m} \kappa_i \leq \sum_{i=k}^{k+m} \kappa \varsigma^i \sigma^{\frac{3^i-1}{2}} \leq \kappa \varsigma^k \sigma^{\frac{3^k-1}{2}} \left(\frac{1 - \varsigma^{m+1} \sigma^{\frac{3^k(3^m+1)}{2}}}{1 - \varsigma\sigma^{3^k}} \right).$$

□

Lemma 5. *Let the hypotheses of Lemma 2 and the conditions $(A1)$–$(A3)$ hold; then, the following conditions are true for all $n \geq 0$:*

$$\begin{aligned} &(i) \wp_n = [L'(m_n)]^{-1} \text{exists and } \|\wp_n\| \leq \lambda_n, \\ &(ii) \|\wp_n L(m_n)\| \leq \kappa_n, \\ &(iii) P\|\wp_n\| \|\wp_n L(m_n)\| \leq \tau_n, \\ &(iv) \|p_n - m_n\| \leq \kappa_n, \\ &(v) \|m_{n+1} - m_n\| \leq \Gamma(\tau_n)\kappa_n, \\ &(vi) \|m_{n+1} - m_0\| \leq \rho\kappa, \text{where } \rho = \tfrac{\Gamma(\tau_0)}{1-\zeta_0}. \end{aligned} \tag{32}$$

Proof. By using the mathematical induction of Lemma 4, we can prove $(i) - (v)$ for $n \geq 0$. Now, for $n \geq 1$, by making use of Relation (31) and the above results, we get

$$\|m_{n+1} - m_0\| \leq \sum_{i=0}^{n} \|m_{i+1} - m_i\| < \rho\kappa.$$

□

Lastly, the following lemma can be proved in a similar way of the article by Wang and Kou [22].

Lemma 6. *Let $\rho = \frac{\Gamma(\tau_0)}{1-\zeta_0}$ and $\Delta(\tau_0)\zeta_0 < 1$ and $\tau_0 < \gamma$, where γ is the smallest positive root of $\Gamma(\theta)\theta - 1 = 0$; then, $\rho < \frac{1}{\tau_0}$.*

4. Semilocal Convergence When L''' Condition Is Omitted

In the ensuing section, our objective is to prove the convergence of the Algorithm mentioned in the Equation(2) by assuming the Hypotheses $(A1)$–$(A3)$ only. Furthermore, we will find a ball with center m_0 and of radius $\rho\kappa$ in which the solution exists and will be unique as well together with which we will define its error bound.

Theorem 1. *Suppose $L : B \subseteq \nabla_1 \to \nabla_2$ is a continuously second-order Fréchet differentiable on B. Suppose the hypotheses (A1)–(A3) are true and $m_0 \in B$. Assume that $\tau_0 = P\lambda\kappa$ and $\zeta_0 = \Delta(\tau_0)\Theta(\tau_0)$ satisfy $\tau_0 < \gamma$ and $\Delta(\tau_0)\zeta_0 < 1$, where γ is the smallest root of $\Gamma(\theta)\theta - 1 = 0$ and Γ, Δ and Θ are defined by Equations (5)–(7), respectively. In addition, suppose $\overline{U(m_0, \rho\kappa)} \subseteq B$, where $\rho = \frac{\Gamma(\tau_0)}{1-\zeta_0}$. Then, initiating* *with m_0, the iterative sequence $\{m_n\}$ creating from the Scheme given in the Equation (2) converges to a zero m^* of $L(m) = 0$ with $m_n, m^* \in \overline{U(m_0, \rho\kappa)}$ and m^* is an exclusive zero of $L(m) = 0$ in $U(m_0, \frac{2}{P\lambda} - \rho\kappa) \cap B$. Furthermore, its error bound is given by*

$$\|m_n - m^*\| \leq \Gamma(\tau_0)\kappa\varsigma^n \sigma^{\frac{3^n-1}{2}} \left(\frac{1}{1 - \varsigma\sigma^{3^n}} \right), \tag{33}$$

where $\sigma = \Delta(\tau_0)\zeta_0$ and $\varsigma = \frac{1}{\Delta(\tau_0)}$.

Proof. Clearly, the sequence $\{m_n\}$ is well established in $\overline{U(m_0, \rho\kappa)}$. Now,

$$\begin{aligned} \|m_{k+l} - m_k\| &\leq \textstyle\sum_{i=k}^{k+l-1} \|m_{i+1} - m_i\| \\ &\leq \Gamma(\tau_0)\kappa\varsigma^k \sigma^{\frac{3^k-1}{2}} \left(\frac{1 - \varsigma^l \sigma^{\frac{3^k(3^{l-1}+1)}{2}}}{1 - \varsigma\sigma^{3^k}} \right), \end{aligned} \tag{34}$$

which shows that $\{m_k\}$ is a Cauchy sequence. Hence, there exists m^* satisfying

$$\lim_{k\to\infty} m_k = m^*.$$

Letting $k = 0, l \to \infty$ in Equation (34), we obtain

$$\|m^* - m_0\| \leq \rho\kappa,$$

which implies that $m^* \in \overline{U(m_0, \rho\kappa)}$. Next, we will show that m^* is a zero of $L(m) = 0$. Because

$$\|\wp_0\| \, \|L(m_n)\| \leq \|\wp_n\| \, \|L(m_n)\|,$$

and in the above inequality by tending $n \to \infty$ and using the continuity of L in B, we find that $L(m^*) = 0$. Finally, for unicity of m^* in $U(m_0, \frac{2}{P\lambda} - \rho\kappa) \cap B$, let m^{**} be another solution of $L(m)$ in $U(m_0, \frac{2}{P\lambda} - \rho\kappa) \cap B$. Using Taylor's theorem, we get

$$0 = L(m^{**}) - L(m^*) = \int_0^1 L'((1-t\theta)m^* + \theta m^{**})d\theta(m^{**} - m^*).$$

In addition,

$$\begin{aligned} & \|\wp_0\| \left\| \int_0^1 [L'((1-\theta)m^* + \theta m^{**}) - L'(m_0)]d\theta \right\| \\ \leq \; & P\lambda \int_0^1 [(1-\theta)\|m^* - m_0\| + \theta\|m^{**} - m_0\|]d\theta \\ \leq \; & \frac{P\lambda}{2} \left[\rho\kappa + \frac{2}{P\lambda} - \rho\kappa \right] = 1, \end{aligned}$$

which implies $\int_0^1 L'((1-\theta)m^* + \theta m^{**})d\theta$ is invertible and hence $m^{**} = m^*$. □

5. Semilocal Convergence When L''' Is Bounded on Initial Iterate

In the current section, we establish the existence and uniqueness theorem of the solution based on the weaker conditions $(A1)$–$(A3)$, $(B1)$ and $(B2)$. Define the sequences as

$$\tilde{\kappa}_{n+1} = \tilde{\zeta}_n \tilde{\kappa}_n, \tag{35}$$

$$\tilde{\lambda}_{n+1} = \Delta(\tilde{\tau}_n)\tilde{\lambda}_n, \tag{36}$$

$$\tilde{\tau}_{n+1} = P\tilde{\lambda}_{n+1}\tilde{\kappa}_{n+1} = \Delta(\tilde{\tau}_n)\tilde{\zeta}_n\tilde{\tau}_n, \tag{37}$$

$$\tilde{\mu}_{n+1} = Q\tilde{\lambda}_{n+1}\tilde{\kappa^2}_{n+1} = \Delta(\tilde{\tau}_n)\tilde{\zeta^2}_n\tilde{\mu}_n, \tag{38}$$

$$\tilde{\nu}_{n+1} = \tilde{\lambda}_{n+1}\tilde{\kappa^2}_{n+1}\omega(\tilde{\kappa}_{n+1}) \leq \Delta(\tilde{\tau}_n)\phi(\tilde{\zeta}_n)\tilde{\zeta}_n^2\tilde{\nu}_n, \tag{39}$$

$$\tilde{\zeta}_{n+1} = \Delta(\tilde{\tau}_{n+1})\Theta'(\tilde{\tau}_{n+1}, \tilde{\mu}_{n+1}, \tilde{\nu}_{n+1}), \tag{40}$$

where $n \geq 0$ and $Q = \overline{A} + \omega\left(\frac{\kappa}{\tilde{\tau}_0}\right)$. Here, we assign $\tilde{\kappa}_0 = \kappa, \tilde{\lambda}_0 = \lambda, \tilde{\tau}_0 = P\lambda\kappa, \tilde{\mu}_0 = Q\lambda\kappa^2$, $\tilde{\nu}_0 = \lambda\kappa^2\omega(\kappa)$ and $\tilde{\zeta}_0 = \Delta(\tilde{\tau}_0)\Theta'(\tilde{\tau}_0, \tilde{\mu}_0, \tilde{\nu}_0)$. From Lemma (5), it is known that

$$\|m_n - m_0\| < \rho\kappa < \frac{\kappa}{\tilde{\tau}_0}.$$

Therefore, $m_n \in U(m_0, \frac{\kappa}{\tilde{\tau}_0})$. Similarly, for $t \in [0,1]$ and $n \geq 1$ and using Lemma (6), we get

$$\begin{aligned}
\|m_n + st(p_n - m_n) - m_0\| &\leq \|m_n - m_0\| + \|p_n - m_n\| \\
&\leq \sum_{i=0}^{n-1} \|m_{i+1} - m_i\| + \tilde{\kappa}_n \\
&\leq \Gamma(\tilde{\tau}_0)\sum_{i=0}^{n} \tilde{\kappa}_i \leq \rho\kappa < \frac{\kappa}{\tilde{\tau}_0}.
\end{aligned}$$

Therefore, $\{m_n + st(p_n - m_n)\} \in U(m_0, \frac{\kappa}{\tilde{\tau}_0})$. This shows that the choice for $\varepsilon = \frac{\kappa}{\tilde{\tau}_0}$ is relevant. Assume that there exists a root $\tilde{\tau}_0 \in (0, \gamma)$ of the equation

$$m = \left[\overline{A} + \omega\left(\frac{\kappa}{m}\right)\right]\lambda\kappa^2.$$

It is obvious that $\tilde{\mu}_0 = Q\lambda\kappa^2$, where $Q = \overline{A} + \omega\left(\frac{\kappa}{\tilde{\tau}_0}\right)$. Notice that here we don't define $\tilde{\tau}_0$ as the root of the following equation:

$$m = \left[\overline{A} + \omega\left(\frac{\Gamma(m)\kappa}{1 - \Delta(m)\Theta'(m, \tilde{\mu}_0, \tilde{\nu}_0)}\right)\right]\lambda\kappa^2.$$

It would be remembered that, for all $m \in U(m_0, \frac{\kappa}{\tilde{\tau}_0})$, we have

$$\begin{aligned}
\|L'''(m)\| = \ & \|L'''(m_0)\| + \|L'''(m) - L'''(m_0)\| \\
& \leq \overline{A} + \omega(\|m - m_0\|) \\
& \leq \overline{A} + \omega\left(\tfrac{\kappa}{\tilde{\tau}_0}\right) = Q.
\end{aligned}$$

Here, we include two auxiliary scalar functions taken from the reference [21]

$$\begin{aligned}
\Theta'(\theta, \eta, \xi) \ &= \Bigg[\tfrac{5}{6}\eta + \tfrac{(3\theta+\eta)(6\theta+2\eta)}{27-18\theta} + \tfrac{(2\theta+\eta)(3\theta+\eta)}{6-4\theta} \\
&+ \tfrac{(2+2\theta+\eta)(3\theta+\eta)\theta}{(12-8\theta)(1-\theta)}\Bigg]\tilde{\Lambda}(\theta, \eta, \xi) \\
&+ \tfrac{1}{2}\tfrac{\theta^2}{1-\theta}\left[\tfrac{9}{6-4\theta}\left(1 + \tfrac{1}{2}\tfrac{\theta}{1-\theta}\right) + \tfrac{3\theta}{4(1-\theta)} + \tfrac{1}{2}\right]\tilde{\Lambda}(\theta, \eta, \xi) \\
&+ \tfrac{\theta}{2}\left[\tfrac{9}{6-4\theta}\left(1 + \tfrac{1}{2}\tfrac{\theta}{1-\theta}\right) + \tfrac{3\theta}{4(1-\theta)} + \tfrac{1}{2}\right]^2\tilde{\Lambda}(\theta, \eta, \xi)^2,
\end{aligned} \tag{41}$$

where

$$\tilde{\Lambda}(\theta,\eta,\xi) = \frac{1}{8}\frac{\theta^3}{(1-\theta)^2} + \frac{1}{12}\frac{\theta\eta}{1-\theta} + \left(D_1 + \frac{1}{3}D_2\right)\xi. \quad (42)$$

$D_1 = \int_0^1 \int_0^1 \phi(s\theta)\theta(1-\theta)dsd\theta$ and $D_2 = \int_0^1 \int_0^1 \phi\left(\frac{2}{3}s\theta\right)\theta dsd\theta$.

Using the property of the induction and from the conditions $(A1)$–$(A3)$, $(B1)$ and $(B2)$, the following relations are true for all $n \geq 0$:

$$\begin{aligned}
&(i)\wp_n = [L'(m_n)]^{-1} \text{exists and } \|\wp_n\| \leq \tilde{\lambda}_n,\\
&(ii)\|\wp_n L(m_n)\| \leq \tilde{\kappa}_n,\\
&(iii)P\|\wp_n\|\|\wp_n L(m_n)\| \leq \tilde{\tau}_n,\\
&(iv)Q\|\wp_n\|\|\wp_n L(m_n)\| \leq \tilde{\mu}_n,\\
&(v)\|\wp_n\|\|\wp_n L(m_n)\|^2\omega(\|\wp_n L(m_n)\|) \leq \tilde{\nu}_n,\\
&(vi)\|m_{n+1} - m_n\| \leq \Gamma(\tilde{\tau}_n)\tilde{\kappa}_n,\\
&(vii)\|m_{n+1} - m_0\| \leq \tilde{\rho}\kappa, \ \text{where } \tilde{\rho} = \frac{\Gamma(\tilde{\tau}_0)}{1-\tilde{\zeta}_0}.
\end{aligned} \quad (43)$$

The second theorem of this article is based on the weaker assumptions, which is stated as:

Theorem 2. *Suppose $L : B \subseteq \nabla_1 \to \nabla_2$ is a continuously third-order Fréchet differentiable on a non-empty open convex subset $B_0 \subseteq B$. Suppose the hypotheses $(A1)$–$(A3)$, $(B1)$ and $(B2)$ are true and $m_0 \in B_0$. Assume that $\tilde{\tau}_0 = P\lambda\kappa, \tilde{\mu}_0 = Q\lambda\kappa^2, \tilde{\nu}_0 = \lambda\kappa^2\omega(\kappa)$ and $\tilde{\zeta}_0 = \Delta(\tilde{\tau}_0)\Theta'(\tilde{\tau}_0, \tilde{\mu}_0, \tilde{\nu}_0)$ satisfy $\tilde{\tau}_0 < \gamma$ and $\Delta(\tilde{\tau}_0)\tilde{\zeta}_0 < 1$, where γ is the smallest root of $\Gamma(\theta)\theta - 1 = 0$ and Γ, Δ and Θ' are defined by Equations (5), (6) and (41). In addition, suppose $\overline{U(m_0, \tilde{\rho}\kappa)} \subseteq B_0$, where $\tilde{\rho} = \frac{\Gamma(\tilde{\tau}_0)}{1-\tilde{\zeta}_0}$. Then, initiating with m_0, the iterative sequence $\{m_n\}$ created from the Scheme given in the Equation (2) converges to a zero m^* of $L(m) = 0$ with $m_n, m^* \in \overline{U(m_0, \tilde{\rho}\kappa)}$ and m^* is an exclusive zero of $L(m) = 0$ in $U(m_0, \frac{2}{P\lambda} - \tilde{\rho}\kappa) \cap B$. Furthermore, its error bound is given by*

$$\|m_n - m^*\| \leq \Gamma(\tilde{\tau}_0)\kappa\tilde{\xi}^n\tilde{\sigma}^{\frac{5^n-1}{4}}\left(\frac{1}{1-\tilde{\xi}\tilde{\sigma}^{5^n}}\right), \quad (44)$$

where $\tilde{\sigma} = \Delta(\tilde{\tau}_0)\tilde{\zeta}_0$ and $\tilde{\xi} = \frac{1}{\Delta(\tilde{\tau}_0)}$.

Proof. Analogous to the proof of Theorem 1. □

6. Numerical Example

Example 1. *Consider nonlinear integral equation from the reference [23] already mentioned in the introduction is given as*

$$m(s) = 1 + \int_0^1 G(s,t)\left(\frac{1}{2}m(t)^{\frac{5}{2}} + \frac{7}{16}m(t)^3\right)dt, s \in [0,1], \quad (45)$$

where $m \in [0,1], t \in [0,1]$ and G is the Green's function defined by

$$G(s,t) = \begin{cases} (1-s)t & t \leq s, \\ s(1-t) & s \leq t. \end{cases}$$

Proof. Solving Equation (45) is equivalent to find the solution for $L(m) = 0$, where $L : B \subseteq C[0,1] \to C[0,1]$:

$$[L(m)](s) = m(s) - 1 - \int_0^1 G(s,t)\left(\frac{1}{2}m(t)^{\frac{5}{2}} + \frac{7}{16}m(t)^3\right) dt, s \in [0,1].$$

The Fréchet derivatives of L are given by

$$L'(m)n(s) = n(s) - \int_0^1 G(s,t)\left(\frac{5}{4}m(t)^{\frac{3}{2}} + \frac{21}{16}m(t)^2\right) n(t)dt, n \in B,$$

$$L''(m)no(s) = -\int_0^1 G(s,t)\left(\frac{15}{8}m(t)^{\frac{1}{2}} + \frac{21}{8}m(t)\right) n(t)o(t)dt, \;\; n, o \in B.$$

Using the max-norm and taking into account that a solution m^* of Equation (45) in $C[0,1]$ must satisfy

$$\|m^*\| - \frac{1}{16}\|m^*\|^{\frac{5}{2}} - \frac{7}{128}\|m^*\|^3 - 1 \leq 0,$$

i.e., $\|m^*\| \leq s_1 = 1.18771$ and $\|m^*\| \geq s_2 = 2.54173$, where s_1 and s_2 are the positive roots of the real equation $t - \frac{t^{\frac{5}{2}}}{16} - \frac{7}{128}t^3 - 1 = 0$. Consequently, if we look for a solution m^* such that $\|m^*\| \leq s_1$, we can consider $U(0,s) \subseteq C[0,1]$, where $s \in (s_1, s_2)$, as a non-empty open convex domain. We choose, for example, $s = 2$ and therefore $B = U(0,2)$. If $m_0 = 1$, then

$$\|\wp_0\| = \frac{128}{87} = \lambda, \|\wp_0 L(m_0)\| \leq \frac{15}{87} = \kappa, \|L''(m)\| \leq \frac{15\sqrt{2}}{64} + \frac{21}{32} = P.$$

Thus, $\tau_0 \approx 0.2505$. Hence, $\tau_0\Gamma(\tau_0) = 0.4068 < 1$ and $\Delta(\tau_0)\zeta_0 = 0.790 < 1$ (It is noticeable that, if we choose the function $\Gamma(m)$ from the reference [21], then we get $\Delta(\tau_0)\zeta_0 = 1.280 > 1$ which violates one of the assumed hypotheses considered in Theorem 1 and hence this motivates us to recalculate the function $\Gamma(m)$). In addition, $U(m_0, \rho\kappa) = U(1, 0.5270) \subseteq U(0,2) = B$. Thus, the conditions of Theorem 1 of Section 4 are satisfied and the nonlinear Equation (45) has the solution m^* in the region $\{u \in C[0,1] : \|u - 1\| \leq 0.5270\}$, which is unique in $\{u \in C[0,1] : \|u - 1\| < 0.8492\} \cap B$. Hence, we can deduce that the existence ball of solution based on our result is superior to that of Wang and Kou in [23], but our uniqueness ball is inferior. □

Example 2. *Now, consider another example discussed in [22] and also mentioned in the introduction, is given by*

$$h(m) = \begin{cases} m^3 ln(m^2) - 6m^2 - 3m + 8, & m \in (-2,0) \cup (0,2), \\ 0, & m = 0. \end{cases} \tag{46}$$

Proof. Taking $U(0,2) = B$. Let $m_0 = 1$ be an initial approximation. The derivatives of h are given by

$$\begin{aligned} h'(m) &= 3m^2 ln(m^2) + 2m^2 - 12m - 3, \\ h''(m) &= 6m ln(m^2) + 10m - 12, \\ h'''(m) &= 6ln(m^2) + 22. \end{aligned}$$

Clearly, h''' is unbounded in B and does not satisfy the condition $(A4)$ but satisfies assumption $(B1)$, and we have

$$\|\wp_0\| = \frac{1}{13} = \lambda, \|\wp_0 h(m_0)\| = \frac{1}{13} = \kappa, \|h''(m)\| \leq 12ln(4) + 32 = P.$$

$\|h'''(m_0)\| = 22$, $\|h'''(m) - h'''(n)\| \leq \frac{12}{1-\frac{13}{32+12log(4)}}|m-n|$, for all $m, n \in U\left(1, \frac{13}{32+12ln(4)}\right)$. Here, $\omega(z) = \frac{12}{1-\frac{13}{32+12log(4)}}z$ and $\phi(\epsilon) = 1$. Here, $\tau_0 \approx 0.2878$ and since $\tau_0\Gamma(\tau_0) = 0.51440 < 1$, $\Delta(\tau_0)\zeta_0 = 0.01742 < 1$. Thus, the assumptions of Theorem 2 of Section 5 are satisfied. In addition, thus, the solution lies in the ball $m \in U(1, 0.13867)$, which is unique in $U(1, 0.39592) \cap B$. Table 1 shows the comparison of error bounds for the considered Algorithm mentioned in the Equation 2 but with two different values of function $\Gamma(m)$ (One is given in the reference [21] and the other is recalculated here). This table also confirms that the value of the recalculated function is prominent.

Table 1. Comparison of the error bounds for Method 2.

n	With Recalculated $\Gamma(m)$	With $\Gamma(m)$ Calculated in [21]
1	0.00085294	0.0019139
2	1.4091×10^{-13}	2.7117×10^{-11}
3	3.1182×10^{-61}	2.0135×10^{-48}
4	2.9759×10^{-298}	5.9108×10^{-232}

□

7. Conclusions

In this contribution, we have analyzed the semilocal convergence of a well defined multi-point variant of the Jarratt method in Banach spaces. This iterative method can be used to solve various kinds of nonlinear equations that satisfy the assumed set of hypotheses. The analysis of this method has been examined using recurrence relations by relaxing the assumptions in two different approaches. In the first approach, we have softened the classical convergence conditions to the prove convergence, existence and uniqueness results together with a priori error bounds. In another way, we have assumed the norm of the third order Fréchet derivative on an initial iterate, so that it never gets unbounded on the given domain and, in addition, it satisfies the local ω-continuity condition as well. Two numerical applications are mentioned that sustain our theoretical consideration.

Author Contributions: For research All the authors have similar contribution.

Funding: This paper is supported by two project funds: the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant No: 16KJB110002.) and the National Science Foundation for Young Scientists of China (Grant No: 11701048).

Acknowledgments: The authors are grateful to the reviewers for their significant criticism which made the paper more elegant and readable.

Conflicts of Interest: The authors declare no conflict of interest.

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