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Basic Fuzzy Event Space and Probability Distribution of Probability Fuzzy Space

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Abstract: In this paper, the problems of basic fuzzy event space and of probability fuzzy space are studied. Firstly, the concepts of basic fuzzy event, fuzzy event and basic fuzzy event space are defined, related properties are investigated, and some results that will be used in the next study of probability fuzzy space are obtained. Then, the definitions of the probability function for fuzzy events and probability fuzzy space are given, some properties of the defined probability function are obtained. In addition, some models of probability distribution of probability fuzzy space based on a known probability space are proposed, and some examples are given to show the usability of the proposed models of probability distribution.

Keywords: basic fuzzy event; basic fuzzy event space; fuzzy event; probability function of fuzzy events; probability fuzzy space

1. Introduction

It is known that the probability and randomness theory is the theoretical basis for studying and dealing with the problem of the uncertainty of whether or not an event happened. The fuzzy set theory is the theoretical basis for studying and dealing with the problem of the uncertainty of the boundary of an event concept (for example, see [1,2]). The theory from combination of random theory and fuzzy set theory can be used to study and deal with the problem with these two mixed uncertainty attributes.

On the combination of random theory and fuzzy set theory, many scholars have engaged in, or are engaging in the work in this area. For example, in [3], Zadeh defined the probability of a fuzzy set as the expectation of its membership function; in [4], Khalili studied the independence between fuzzy events; in [5], Smets introduced the concept of the conditional probability of a fuzzy event; in [6], Baldwin, Lawry and Martin discussed the problem of the conditional probability of fuzzy subsets of a continuous domain; in [7], Buoncristiani investigated probability on ϵ-fuzzy sets; in [8], Lee and Li determined an order of fuzzy numbers (about fuzzy numbers, we can see [9–12]) based on the concept of probability measure of fuzzy events due to Zadeh; in [13], Heilpern defined the expected value of fuzzy variables, and investigated its properties; in [14], Gil gave a discussion on treating fuzziness as a kind of randomness in studying statistical management of fuzzy elements in random experiments; in [15], Flaminio and Godo proposed a logic for reasoning about the probability of fuzzy events; in [16], Kato, Izuka et al. proposed a new fuzzy probability distribution function containing fuzzy numbers as its parameters; in [17], Xia provided a fuzzy probability system, which has a more original theoretical starting point, and appears to deal with such uncertainty as has subjectivity and fuzziness; in [18], Talašová and Pavlačka defined a fuzzy probability space that enables an adequate mathematical modeling of expertly set uncertain probabilities of states of the world; in [19], Biacino extended the definition of belief function to fuzzy events starting from a basic assignment of probability on some fuzzy focal events and using a suitable notion of inclusion for fuzzy subsets; in [20,21], Kßißç and

Leblebicioğlu studied the problems of fuzzy discrete event systems; in [22], Kahraman and Kaya made an investment analysis by using the concept of probability of a fuzzy event.

Recently, there has still been a lot of work on the theory and application of the combination of randomness and fuzziness. For example, in 2012, Liu and Dziong formalized the notion of codiagnosability for decentralized diagnosis of fuzzy discrete-event systems, in which the observability of fuzzy events is defined to be fuzzy instead of crisp in [23]; in 2014, Purba, Lu, Zhang and Pedrycz developed a fuzzy reliability algorithm to effectively generate basic event failure probabilities without reliance on quantitative historical failure data through qualitative data processing in [24]; in 2015, Purba and Tjahyani et al. proposed a fuzzy probability based fault tree analysis to propagate and quantify epistemic uncertainty raised in basic event reliability evaluations to complement conventional fault tree analysis which can only evaluate aleatory uncertainty in [25]; in [26], Zhao and Hu took fuzzy probability and interval-valued fuzzy probability into consideration; in 2016, Lower, Magott and Skorupski presented a new approach as previous research in analyzing Air Traffic Incidents has focused more on defining accident occurrence probabilities in [27]; in [28], Chutia and Datta proposed the fuzzy random variable valued Gumbel, Weibull and Gaussian functions, and discussed fundamental properties of these functions in the fuzzy probability space; in 2017, Coletti, Petturiti and Vantaggi introduced the concept of possibility of a fuzzy event, and provided a comparison with the probability of a fuzzy event in [29].

While many results have been obtained in the theory and application of the combination of random theory and fuzzy set theory, the work has not yet reached a perfect degree. In the aspect of the theoretical research results, the researchers generally study the combination of the two theory only from the point of view of mathematics (only from the mathematical theory itself), this leads to that the obtained theoretical results may look (from the point of view of mathematics) very beautiful, but they lack application background, and it is difficult to get real application in engineering or practical problems. In the aspect of the applied research work, researchers often only focus on an isolated specific problem, the used theory (of combination of random theory and fuzzy set theory) is still in the initial stage, lacking in depth, and the obtained results are also lacking in systematicness.

In order to establish the systematic theory of "random fuzzy sets" and "random fuzzy numbers" with strong usability, in this paper, we do some preliminary research work. From the introduction of basic fuzzy events, we give some concepts such as basic fuzzy event space, fuzzy events, probability distribution on basic fuzzy event space, probability fuzzy space and so on, investigate their related properties, and propose some specific models of probability distribution of probability fuzzy space based on a known probability space, which have a strong application background. Specific arrangements are as follows: In Section 2, we briefly review some basic notions and definitions which will be used in this paper; in Section 3, we define the concepts of basic fuzzy event, fuzzy event and basic fuzzy event space, investigate related properties, and obtain some results that will be used in the next section; in Section 4, we introduce the definitions of the probability function about fuzzy events and probability fuzzy space, obtain some properties of the defined probability function, propose some models of probability distribution of probability fuzzy space based on a known probability space, and give some examples to show the using of the proposed models of probability distribution. In Section 5, we make a summary of this paper.

2. Basic Definition and Notation

Let Ψ be nonempty set (in this paper, we denote the empty set by ϕ). We denote the collection of all subsets of Ψ by 2^{Ψ}. A mapping $\mu: \Psi \to [0,1]$ is called a fuzzy subset (in short, a fuzzy set) of Ψ. We denote the collection of all fuzzy sets of Ψ by $\mathcal{F}(\Psi)$.

For a fuzzy set μ of of Ψ, we denote its r-level set $\{x \in \Psi : \mu(x) \geq r\}$ by $[\mu]^r$ for any $r \in (0,1]$, i.e., $[\mu]^r = \{x \in \Psi : \mu(x) \geq r\}$, and denote its strong r-level set $\{x \in \Psi : \mu(x) > r\}$ by $(\mu)^r$ for any $r \in [0,1]$, i.e., $(\mu)^r = \{x \in \Psi : \mu(x) > r\}$. By supp$\mu$ we denote the support of μ, i.e., the set $(\mu)^0 = \{x \in \Psi : \mu(x) > 0\}$.

Let Ω be a basic event space. If $\sigma(\Omega) \subset 2^{\Omega}$ be a σ-algebra, i.e., satisfies the following properties:

(1) $\phi \in \sigma(\Omega)$;
(2) $A \in \sigma(\Omega)$ if and only if $A^c \in \sigma(\Omega)$, where A^c is the complement of A;
(3) $\cap_{i=1}^{\infty} A_i \in \sigma(\Omega)$ for any $A_i \in \sigma(\Omega)$, $i = 1, 2, \cdots$.

then we say $\sigma(\Omega)$ is an *event set*, and call A an *event* if $A \in \sigma(\Omega)$.

If $\sigma(\Omega) \subset 2^{\Omega}$ be a algebra, i.e., the Condition (3) $\cap_{i=1}^{\infty} A_i \in \sigma(\Omega)$ for any $A_i \in \sigma(\Omega)$, $i = 1, 2, \cdots$ is replaced with (3') $A_1 \cap A_2 \in \sigma(\Omega)$ for any $A_1, A_2 \in \sigma(\Omega)$, then we say $\sigma(\Omega)$ is a finite intersection event set.

Let R be the real field. For basic event space Ω and event set $\sigma(\Omega)$, if mapping $P : \sigma(\Omega) \to R$ satisfies the axioms of Kolmogorov:

(1) $P(A) \geq 0$ for any $A \in \sigma(\Omega)$;
(2) $P(\Omega) = 1$;
(3) $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ for any $A_i \in \sigma(\Omega)$ $(i = 1, 2, \cdots)$ with $A_m \cap A_n = \phi$ $(m \neq n$ and $m, n = 1, 2, \cdots)$,

then we call P a *probability distribution function* on $\sigma(\Omega)$, and say $(\Omega, \sigma(\Omega), P)$ is a probability space.

If $\sigma(\Omega)$ is a finite intersection event set, and the Condition (3) $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ for any $A_i \in \sigma(\Omega)$ $(i = 1, 2, \cdots)$ with $A_m \cap A_n = \phi$ $(m \neq n$ and $m, n = 1, 2, \cdots)$ is replaced with (3') $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ for any $A_1, A_2 \in \sigma(\Omega)$ with $A_1 \cap A_2 = \phi$, then we call P a finitely additive probability distribution function on $\sigma(\Omega)$, and say $(\Omega, \sigma(\Omega), P)$ is a finitely additive probability space.

3. Basic Fuzzy Event Space

In order to establish the relevant theory of probability fuzzy space, in this section, we are going to give the following concepts of fuzzy basic events and fuzzy basic event space, and investigate their related properties:

Definition 1. *Let Ω be a basic event space. For $\omega \in \Omega$ and $r \in [0, 1]$, we define fuzzy set $(\omega, r) : \Omega \to [0, 1]$ of Ω as:*

$$(\omega, r)(\psi) = \begin{cases} r, & \psi = \omega \\ 0, & \psi \neq \omega \end{cases} \quad \text{for any } \psi \in \Omega$$

and call it a basic fuzzy event of Ω. We denote $\tilde{\Omega} = \{(\omega, r) : \omega \in \Omega, r \in (0, 1]\}$, and call $\tilde{\Omega}$ the basic fuzzy event space (generated by Ω).

It is obvious that $(\omega, r) \in \tilde{\Omega} \Longleftrightarrow \omega \in \Omega, r \in (0, 1]$. Therefore we can establish a one-to-one correspondence between $\tilde{\Omega}$ and $\Omega \times (0, 1]$, so we can directly use $\Omega \times (0, 1]$ to represent $\tilde{\Omega}$, i.e., we can denote $\tilde{\Omega} = \Omega \times (0, 1]$, where $\Omega \times (0, 1]$ is the Cartesian product of Ω and $(0, 1]$.

Definition 2. *Let Ω be a basic event space. If $\tilde{A} \subset \tilde{\Omega}$ (i.e., $\tilde{A} \in 2^{\tilde{\Omega}}$), then we say $\tilde{A}$ is a fuzzy event, and call the set $\{\omega \in \Omega : \text{ there exists } r \in (0, 1] \text{ such that } (\omega, r) \in \tilde{A}\}$ (denoted by supp$\tilde{A}$ or supp$(\tilde{A})$) basic event support set (in short, support) of $\tilde{A}$.*

It is obvious that for $\omega \in \Omega$, $\omega \in \text{supp}\tilde{A}$ if and only if there exists $r_\omega \in (0, 1]$ such that $(\omega, r_\omega) \in \tilde{A}$.

Definition 3. *Let Ω be a basic event space. We define mapping $\mathbb{E} : 2^{\tilde{\Omega}} \to \mathcal{F}(\Omega)$ as*

$$\mathbb{E}(\tilde{A}) = \begin{cases} \cup_{\tilde{\omega} \in \tilde{A}} \tilde{\omega}, & \tilde{A} \in 2^{\tilde{\Omega}} \ (\text{i.e., } \tilde{A} \subset \tilde{\Omega}) \text{ with } \tilde{A} \neq \phi \\ \phi & \tilde{A} = \phi \end{cases}$$

and call $\mathbb{E}$ canonical mapping of fuzzy events (with respect to Ω).

Remark 1. *Let Ω be a basic event space, $\tilde{A} \in 2^{\tilde{\Omega}}$. It is obvious that supp$\mathbb{E}(\tilde{A}) =$ supp$\tilde{A}$. In addition, for convenience, we denote $[\mathbb{E}(\tilde{A})]^r$ by $[\tilde{A}]^r$ (resp. $(\mathbb{E}(\tilde{A}))^r$ by $(\tilde{A})^r$) for any $r \in [0,1]$.*

Remark 2. *Let Ω be a basic event space. For a $\tilde{A} \in 2^{\tilde{\Omega}}$ (i.e., $\tilde{A} \subset \tilde{\Omega}$, we see that $\tilde{A}$ is a classical set (collection) whose elements are some basic fuzzy events. However, according to the canonical mapping $\mathbb{E}$, we can regard $\tilde{A}$ as a fuzzy set of Ω (i.e., an element in 2^{Ω}).*

For two nonempty sets A, B, we denote $B \setminus A = \{x : x \in B, x \notin A\}$.

Proposition 1. *Let Ω be a basic event space. We have*

(1) $\mathbb{E}(\tilde{A}) \subset \mathbb{E}(\tilde{B})$ *for any* $\tilde{A}, \tilde{B} \subset \tilde{\Omega}$ *with* $\tilde{A} \subset \tilde{B}$;
(2) *for any* $\tilde{A}, \tilde{B} \subset \tilde{\Omega}$ *with* $\tilde{A} \subset \tilde{B}$ *and* $\omega \in$ supp$\tilde{B}$, $\left(\mathbb{E}(\tilde{A})\right)(\omega) \neq \left(\mathbb{E}(\tilde{B})\right)(\omega) \Longleftrightarrow \omega \in$ supp$(\tilde{B} \setminus \tilde{A})$;
(3) *for any* $\tilde{A}, \tilde{B} \subset \tilde{\Omega}$, $\mathbb{E}(\tilde{A}) \subset \mathbb{E}(\tilde{B}) \Longleftrightarrow [\tilde{A}]^r \subset [\tilde{B}]^r$ *(*$r \in [0,1]$*)* $\Longleftrightarrow (\tilde{A})^r \subset (\tilde{B})^r$ *(*$r \in (0,1]$*)*;
(4) *for any* $\tilde{A}, \tilde{B} = \tilde{\Omega}$, $\mathbb{E}(\tilde{A}) = \mathbb{E}(\tilde{B}) \Longleftrightarrow [\tilde{A}]^r = [\tilde{B}]^r$ *(*$r \in [0,1]$*)* $\Longleftrightarrow (\tilde{A})^r = (\tilde{B})^r$ *(*$r \in (0,1]$*)*;
(5) $\mathbb{E}(\cup_{i=1}^{\infty}\tilde{A}_i) = \cup_{i=1}^{\infty}\mathbb{E}(\tilde{A}_i)$ *for any* $\tilde{A}_i \subset \tilde{\Omega}$, $i = 1, 2, \cdots$.

Proof. (1) For any $\omega \in \Omega$, from $\tilde{A} \subset \tilde{B}$, we see that $\{\tilde{\omega}(\omega) : \tilde{\omega} \in \tilde{A}\} \subset \{\tilde{\omega}(\omega) : \tilde{\omega} \in \tilde{B}\}$. Therefore, by the definition of canonical mapping $\mathbb{E}$, we have that $\mathbb{E}(\tilde{A})(\omega) = (\cup_{\tilde{\omega} \in \tilde{A}}\tilde{\omega})(\omega) = \sup\{\tilde{\omega}(\omega) : \tilde{\omega} \in \tilde{A}\} \leq \sup\{\tilde{\omega}(\omega) : \tilde{\omega} \in \tilde{B}\} = (\cup_{\tilde{\omega} \in \tilde{B}}\tilde{\omega})(\omega) = \mathbb{E}(\tilde{B})(\omega)$.

(2) Let $\tilde{A} \subset \tilde{B} \in 2^{\tilde{\Omega}}$ and $\omega \in$ supp$\tilde{B}$. Then, $\left(\mathbb{E}(\tilde{A})\right)(\omega) \neq \left(\mathbb{E}(\tilde{B})\right)(\omega) \Longleftrightarrow$ there exists $r_\omega \in (0,1]$ such that $(\omega, r_\omega) \in \tilde{B}$ and $(\omega, r_\omega) \notin \tilde{A} \Longleftrightarrow$ there exists $r_\omega \in (0,1]$ such that $(\omega, r_\omega) \in \tilde{B} \setminus \tilde{A} \Longleftrightarrow \omega \in$ supp$(\tilde{B} \setminus \tilde{A})$;

(3) By Remark (1), we see that $[\tilde{A}]^r = [\mathbb{E}(\tilde{A})]^r$ and $[\tilde{B}]^r = [\mathbb{E}(\tilde{B})]^r$ ($r \in [0,1]$), $(\tilde{A})^r = (\mathbb{E}(\tilde{A}))^r$ and $(\tilde{B})^r = (\mathbb{E}(\tilde{B}))^r$ ($r \in (0,1]$). So from $\mathbb{E}(\tilde{A}) \subset \mathbb{E}(\tilde{B}) \Longleftrightarrow [\mathbb{E}(\tilde{A})]^r \subset [\mathbb{E}(\tilde{B})]^r$ ($r \in [0,1]$) $\Longleftrightarrow (\mathbb{E}(\tilde{A}))^r \subset (\mathbb{E}(\tilde{B}))^r$ ($r \in (0,1]$), we know that Conclusion (3) holds;

(4) By the Conclusion (3), we can directly obtain the Conclusion (4);

(5) To prove the Conclusion (5) of the proposition, we only need show that $\mathbb{E}(\cup_{i=1}^{\infty}\tilde{A}_i)(\omega) = \cup_{i=1}^{\infty}\mathbb{E}(\tilde{A}_i)(\omega)$ for any $\omega \in \Omega$. In fact,

$$\begin{aligned}
\mathbb{E}(\cup_{i=1}^{\infty}\tilde{A}_i)(\omega) &= (\cup_{\tilde{\omega} \in \cup_{i=1}^{\infty}\tilde{A}_i}\tilde{\omega})(\omega) \\
&= \sup\{\tilde{\omega}(\omega) : \tilde{\omega} \in \cup_{i=1}^{\infty}\tilde{A}_i\} \\
&= \sup\left\{\sup\{\tilde{\omega}(\omega) : \tilde{\omega} \in \tilde{A}_1\}, \sup\{\tilde{\omega}(\omega) : \tilde{\omega} \in \tilde{\omega} \in \tilde{A}_2\}, \cdots\right\} \\
&= \sup_{i=1}^{\infty}\left\{\sup\{\tilde{\omega}(\omega) : \tilde{\omega} \in \tilde{A}_i\}\right\} \\
&= \sup_{i=1}^{\infty}\left\{(\cup_{\tilde{\omega} \in \tilde{A}_i}\tilde{\omega})(\omega)\right\} \\
&= \sup_{i=1}^{\infty}\left\{\mathbb{E}(\tilde{A}_i)(\omega)\right\} \\
&= \cup_{i=1}^{\infty}\mathbb{E}(\tilde{A}_i)(\omega)
\end{aligned}$$

□

Remark 3. *The canonical mapping $\mathbb{E}$ is not one-to-one mapping (see the following Example 1).*

Example 1. *let $\Omega = \{a, b\}$, $\tilde{A} = \{(a, 0.8), (a, 0.5), (b, 0.9),\}$ and $\tilde{B} = \{(a, 0.8), (b, 0.6), (b, 0.9)\}$. Then $\tilde{A}, \tilde{B} \in 2^{\tilde{\Omega}}$ and $\tilde{A} \neq \tilde{B}$. However, from $\mathbb{E}(\tilde{A})(a) = (\cup_{\tilde{\omega} \in \tilde{A}}\tilde{\omega})(a) = \max\{(a, 0.8)(a), (a, 0.5)(a), (b, 0.9)(a)\} = \max\{0.8, 0.5, 0\} = 0.8 = \max\{0.8, 0, 0\} = \max\{(a, 0.8)(a), (b, 0.6)(a), (b, 0.9)(a)\} = (\cup_{\tilde{\omega} \in \tilde{B}}\tilde{\omega})(a) = \mathbb{E}(\tilde{B})(a)$ and $\mathbb{E}(\tilde{A})(b) = (\cup_{\tilde{\omega} \in \tilde{A}}\tilde{\omega})(b) = \max\{(a, 0.8)(b), (a, 0.5)(b), (b, 0.9)(b)\} = \max\{0, 0, 0.9\} = 0.9 = \max\{0, 0.6, 0.9\} = \max\{(a, 0.8)(b), (b, 0.6)(b), (b, 0.9)(b)\} = (\cup_{\tilde{\omega} \in \tilde{B}}\tilde{\omega})(b) = \mathbb{E}(\tilde{B})(b)$, we know $\mathbb{E}(\tilde{A}) = \mathbb{E}(\tilde{B})$.*

Proposition 2. *Let Ω be a basic event space, $\tilde{A} \in 2^{\tilde{\Omega}}$. Then*

$$\left(\mathbb{E}(\tilde{A})\right)(\omega) = \begin{cases} \sup\{r \in (0,1] : (\omega, r) \in \tilde{A}\}, & \omega \in supp\tilde{A} \\ 0, & \omega \notin supp\tilde{A} \end{cases}$$

for any $\omega \in \Omega$.

Proof. For any $\omega \in \Omega$, we have that $\left(\mathbb{E}(\tilde{A})\right)(\omega) = (\cup_{\tilde{\psi} \in \tilde{A}} \tilde{\psi})(\omega) = \sup\{\tilde{\psi}(\omega) : \tilde{\psi} \in \tilde{A}\} = \sup\{(\psi, r)(\omega) : (\psi, r) \in \tilde{A}\}$. Therefore if $\omega \in$ supp$\tilde{A}$, then there exists $r_0 \in (0,1]$ such that $(\omega, r_0) \in \tilde{A}$. Therefore we see that $\left(\mathbb{E}(\tilde{A})\right)(\omega) = \sup\{(\psi, r)(\omega) : (\psi, r) \in \tilde{A}\} = \sup\{(\omega, r)(\omega) : (\omega, r) \in \tilde{A}\} = \sup\{r : (\omega, r) \in \tilde{A}\}$. If $\omega \notin$ supp$\tilde{A}$, i.e., $(\omega, r) \notin \tilde{A}$ for any $r \in (0,1]$, then we see that $\left(\mathbb{E}(\tilde{A})\right)(\omega) = \sup\{(\psi, r)(\omega) : (\psi, r) \in \tilde{A}\} = \sup\{0 : (\psi, r) \in \tilde{A}\} = 0$. □

By Proposition 2, we can easily obtain the following corollary:

Corollary 1. *Let Ω be a basic event space. Then $\mathbb{E}(\tilde{\Omega}) = \Omega$ and $\mathbb{E}(\phi) = \phi$.*

Let Ω be a basic event space, $\tilde{A} \subset \tilde{B} \in 2^{\tilde{\Omega}}$ and $\omega \in$ supp$\tilde{B}$. If $\left(\mathbb{E}(\tilde{A})\right)(\omega) \neq \left(\mathbb{E}(\tilde{B})\right)(\omega)$, then $\omega \in$ supp$(\tilde{B} \setminus \tilde{A})$ (by the Conclusion (2) of Proposition 1), so, by $\tilde{A} \subset \tilde{B}$, we have that $\mathbb{E}(\tilde{B} \setminus \tilde{A})(\omega) = \sup\{r \in (0,1] : (\omega, r) \in \tilde{B} \setminus \tilde{A}\} = \sup\{r \in (0,1] : (\omega, r) \in \tilde{B}\} = \mathbb{E}(\tilde{B})(\omega)$; If $\left(\mathbb{E}(\tilde{A})\right)(\omega) = \left(\mathbb{E}(\tilde{B})\right)(\omega)$, then $\omega \notin$ supp$(\tilde{B} \setminus \tilde{A})$ (by the Conclusion (2) of Proposition 1), so, by Proposition 2, we have that $\mathbb{E}(\tilde{B} \setminus \tilde{A})(\omega) = 0$. Therefore, we have the following corollary:

Corollary 2. *Let Ω be a basic event space, $\tilde{A} \subset \tilde{B} \in 2^{\tilde{\Omega}}$. Then for any $\omega \in \Omega$, we have that*

$$\left(\mathbb{E}(\tilde{B} \setminus \tilde{A})\right)(\omega) = \begin{cases} \left(\mathbb{E}(\tilde{B})\right)(\omega), & \left(\mathbb{E}(\tilde{A})\right)(\omega) \neq \left(\mathbb{E}(\tilde{B})\right)(\omega) \\ 0, & \left(\mathbb{E}(\tilde{A})\right)(\omega) = \left(\mathbb{E}(\tilde{B})\right)(\omega) \end{cases}$$

For the convenience of the following discussion, we give the following concepts associated with fuzzy events:

Definition 4. *Let Ω be a basic event space and $\tilde{A} \subset \tilde{\Omega}$ (i.e., $\tilde{A} \in 2^{\tilde{\Omega}}$). If for each fixed $\omega_0 \in \Omega$, the r (if it exists) with $r \in (0,1]$ and $(\omega_0, r) \in \tilde{A}$ is unique (i.e., $\{r \in (0,1] : \tilde{\omega}_0 = (\omega_0, r) \in \tilde{A}\}$ is empty set or a single point set), then we call $\tilde{A}$ a simple fuzzy event. We denote $S(2^{\tilde{\Omega}}) = \{\tilde{A} \subset \tilde{\Omega} : \tilde{A}$ is simple fuzzy event$\}$, and call $S(2^{\tilde{\Omega}})$ simple fuzzy event space.*

For a basic event space Ω and a real number $r \in (0,1]$, we denote $S^r(\tilde{\Omega}) = \{(\omega, r) : \omega \in \Omega\}$.

Remark 4. *It is obvious that $S^r(\tilde{\Omega}) \in S(2^{\tilde{\Omega}})$ for any $r \in (0,1]$, and $\mathbb{E}(\tilde{A}) \subset \mathbb{E}(S^1(\tilde{\Omega})) = \Omega \in \mathcal{F}(\Omega)$ for any $\tilde{A} \in S(2^{\tilde{\Omega}})$.*

Restricting the canonical mapping $\mathbb{E} : 2^{\tilde{\Omega}} \to \mathcal{F}(\Omega)$ on $S(2^{\tilde{\Omega}})$, we can obtain a mapping (denoted by $\mathbb{E}_S$) from $S(2^{\tilde{\Omega}})$ into $\mathcal{F}(\Omega)$.

Proposition 3. *Let Ω be a basic event space, $\tilde{A} \in S(2^{\tilde{\Omega}})$. Then*

$$\left(\mathbb{E}(\tilde{A})\right)(\omega) = \left(\mathbb{E}_S(\tilde{A})\right)(\omega) = \begin{cases} r, & (\omega, r) \in \tilde{A} \\ 0, & (\omega, r) \notin \tilde{A} \end{cases}$$

Proof. If $(\omega, r) \in \tilde{A}$ and $\tilde{A} \in S(2^{\tilde{\Omega}})$, we know that $\left(\mathbb{E}(\tilde{A})\right)(\omega) = \left(\mathbb{E}_S(\tilde{A})\right)(\omega) = (\cup_{\tilde{\omega} \in \tilde{A}} \tilde{\omega})(\omega) = \sup\{\tilde{\omega}(\omega) : \tilde{\omega} \in \tilde{A}\} = \sup\{(\psi, s)(\omega) : (\psi, s) \in \tilde{A}\} = \sup\{(\omega, s)(\omega) : (\omega, s) \in \tilde{A}\} = \sup\{(\omega, r)(\omega) : (\omega, r) \in \tilde{A}\} = \sup\{r : (\omega, r) \in \tilde{A}\} = r$. □

About the mapping $\mathbb{E}_S: \ S(2^{\tilde{\Omega}}) \to \mathcal{F}(\Omega)$, we have the following result:

Proposition 4. *Let Ω be a basic event space. Then $\mathbb{E}_S$ is an injective mapping (i.e., one-to-one mapping).*

Proof. Let $\tilde{A}, \tilde{B} \in S(2^{\tilde{\Omega}})$ with $\tilde{A} \neq \tilde{B}$. To prove that the proposition is established, we only need show that $\mathbb{E}_S(\tilde{A}) \neq \mathbb{E}_S(\tilde{B})$, i.e., only need show that there exists $\omega_0 \in \Omega$ such that $\left(\mathbb{E}_S(\tilde{A})\right)(\omega_0) \neq \left(\mathbb{E}_S(\tilde{B})\right)(\omega_0)$. In fact, from $\tilde{A} \neq \tilde{B}$, we know that there exists $(\omega_1, r_1) \in \tilde{A}$ such that $(\omega_1, r_1) \notin \tilde{B}$, or there exists $(\omega_2, r_2) \in \tilde{B}$ such that $(\omega_2, r_2) \notin \tilde{A}$.

(1) As there exists $(\omega_1, r_1) \in \tilde{A}$ such that $(\omega_1, r_1) \notin \tilde{B}$, from $(\omega_1, r_1) \in \tilde{A}$, we have that $\left(\mathbb{E}_S(\tilde{A})\right)(\omega_1) = r_1$ by Proposition 3; From $(\omega_1, r_1) \notin \tilde{B}$, we can prove that $\left(\mathbb{E}_S(\tilde{B})\right)(\omega_1) \neq r_1$ (in fact, if there exists $s_0 \in (0,1]$ such that $(\omega_1, s_0) \in \tilde{B}$, then $\left(\mathbb{E}_S(\tilde{B})\right)(\omega_1) = s_0 \neq r_1$; if for any $r \in (0,1]$, $(\omega_1, r) \notin \tilde{B}$, then $\left(\mathbb{E}_S(\tilde{B})\right)(\omega_1) = 0 < r_1$). Therefore we see that $\left(\mathbb{E}_S(\tilde{A})\right)(\omega_1) \neq \left(\mathbb{E}_S(\tilde{B})\right)(\omega_1)$. Thus, as long as we take $\omega_0 = \omega_1$, we have $\left(\mathbb{E}_S(\tilde{A})\right)(\omega_0) \neq \left(\mathbb{E}_S(\tilde{B})\right)(\omega_0)$;

(2) As there exists $(\omega_2, r_2) \in \tilde{B}$ such that $(\omega_2, r_2) \notin \tilde{A}$, in the same way, we can show that as long as we take $\omega_0 = \omega_2$, we have $\left(\mathbb{E}_S(\tilde{A})\right)(\omega_0) \neq \left(\mathbb{E}_S(\tilde{B})\right)(\omega_0)$. □

Proposition 5. *Let Ω be a basic event space. Then $\mathbb{E}_S$ is a surjection (i.e., surjective mapping).*

Proof. In order to complete the proof of the proposition, we only prove that for any $\mu \in \mathcal{F}(\Omega)$, there exists $\tilde{A}_\mu \in S(2^{\tilde{\Omega}})$ such that $\mathbb{E}_S(\tilde{A}_\mu) = \mu$. For the fixed $\mu \in \mathcal{F}(\Omega)$, we take $\tilde{A}_\mu = \{(\omega, \mu(\omega)) : \omega \in \text{supp}\mu \subset \Omega\}$, then $\tilde{A}_\mu \in S(2^{\tilde{\Omega}})$. In the following, we show that $\mathbb{E}_S(\tilde{A}_\mu) = \mu$, i.e., $\left(\mathbb{E}_S(\tilde{A}_\mu)\right)(\psi) = \mu(\psi)$ for any $\psi \in \Omega$:

(1) As $\mu(\psi) = 0$, i.e., $\psi \notin \text{supp}\mu$, then $(\psi, r) \notin \tilde{A}\mu$ for any $r \in (0,1]$, so by Proposition 2, we have that $\left(\mathbb{E}_S(\tilde{A}_\mu)\right)(\psi) = 0 = \mu(\psi)$;

(2) As $\mu(\psi) \neq 0$, i.e., $\psi \in \text{supp}\mu$, then $(\psi, \mu(\psi)) \in \tilde{A}\mu$, so by Proposition 3, we have that $\left(\mathbb{E}_S(\tilde{A}_\mu)\right)(\psi) = \mu(\psi)$. □

Remark 5. *By Propositions 4 and 5, we see that there exists a one-to-one correspondence $\mathbb{E}_S$ from $S(2^{\tilde{\Omega}})$ onto $\mathcal{F}(\Omega)$. So we can regard $S(2^{\tilde{\Omega}})$ and $\mathcal{F}(\Omega)$ as the same.*

Proposition 6. *Let Ω be a basic event space. For the inverse mapping $\mathbb{E}_S^{-1}: \ \mathcal{F}(\Omega) \to S(2^{\tilde{\Omega}})$, we have that $\mathbb{E}_S^{-1}(\mu) = \{(\omega, \mu(\omega)) : \ \omega \in \Omega, \ \mu(\omega) > 0\}$.*

Proof. Defining mapping $\mathbb{G}: \ \mathcal{F}(\Omega) \to S(2^{\tilde{\Omega}})$ as $\mathbb{G}(\mu) = \{(\omega, \mu(\omega)) : \ \omega \in \Omega, \ \mu(\omega) > 0\}$ for any $\mu \in \mathcal{F}(\Omega)$, then by the proof of Proposition 5, we see that $\mathbb{E}_S(\mathbb{G}(\mu)) = \mathbb{E}_S(\{(\omega, \mu(\omega)) : \ \omega \in \Omega, \ \mu(\omega) > 0\}) = \mu$ for any $\mu \in \mathcal{F}(\Omega)$, so $\mathbb{E}_S^{-1} = \mathbb{G}$. □

Proposition 7. *Let Ω be a basic event space. Then for any $\tilde{A} \in 2^{\tilde{\Omega}}$, $\mathbb{E}_S^{-1}\left(\mathbb{E}(\tilde{A})\right) \in S(2^{\tilde{\Omega}})$, and we have that*

$$\mathbb{E}_S^{-1}\left(\mathbb{E}(\tilde{A})\right) = \{(\omega, r_{\tilde{A}}(\omega)) : \ \omega \in supp\tilde{A}\}$$

where, $r_{\tilde{A}}(\omega) = \sup\{r \in (0,1] : \ (\omega, r) \in \tilde{A}\}$ for $\omega \in supp\tilde{A}$.

Proof. For each fixed $\tilde{A} \in 2^{\tilde{\Omega}}$ and $\omega \in \text{supp}\tilde{A}$, $r_{\tilde{A}}(\omega) = \sup\{r \in (0,1] : \ (\omega, r) \in \tilde{A}\}$ is unique, so $\{(\omega, r_{\tilde{A}}(\omega)) : \ \omega \in \text{supp}\tilde{A}\} \in S(2^{\tilde{\Omega}})$. Therefore, we just have to prove $\mathbb{E}_S^{-1}\left(\mathbb{E}(\tilde{A})\right) = \{(\omega, r_{\tilde{A}}(\omega)) : \omega \in \text{supp}\tilde{A}\}$, i.e., only need to show that $\mathbb{E}(\tilde{A}) = \mathbb{E}_S\left(\{(\omega, r_{\tilde{A}}(\omega)) : \ \omega \in \text{supp}\tilde{A}\}\right)$. In fact, for any $\omega \in \Omega$, by Propositions 2 and 3, we have that

$$
\begin{aligned}
\left(\mathbb{E}(\tilde{A})\right)(\omega) &= \begin{cases} r_{\tilde{A}}(\omega), & \omega \in \ \mathrm{supp}\tilde{A} \\ 0, & \omega \notin \ \mathrm{supp}\tilde{A} \end{cases} \\
&= \begin{cases} r_{\tilde{A}}(\omega), & (\omega, r_{\tilde{A}}(\omega)) \in \{(\omega, r_{\tilde{A}}(\omega)) :\ \omega \in \mathrm{supp}\tilde{A}\} \\ 0, & (\omega, r_{\tilde{A}}(\omega)) \notin \{(\omega, r_{\tilde{A}}(\omega)) :\ \omega \in \mathrm{supp}\tilde{A}\} \end{cases} \\
&= \left(\mathbb{E}_S(\{(\omega, r_{\tilde{A}}(\omega)) :\ \omega \in \mathrm{supp}\tilde{A}\})\right)(\omega)
\end{aligned}
$$

so $\mathbb{E}(\tilde{A}) = \mathbb{E}_S\left(\{(\omega, r_{\tilde{A}}(\omega)) :\ \omega \in \mathrm{supp}\tilde{A}\}\right)$ holds □

Definition 5. *Let Ω be a basic event space, $\tilde{A} \in 2^{\tilde{\Omega}}$. We call $\mathbb{E}_S^{-1}\left(\mathbb{E}(\tilde{A})\right)$ the simplistic fuzzy event of fuzzy event $\tilde{A}$, and denote it by $\tilde{A}_S$.*

Example 2. *Let $\Omega = \{\omega_1, \omega_2\}$, $\tilde{A} = \left\{(\omega_1, r), (\omega_2, s) \in \tilde{\Omega} :\ r \in [\frac{1}{3}, \frac{2}{3}], r \in [\frac{1}{4}, \frac{3}{4})\right\}$. Then $\tilde{A} \in 2^{\tilde{\Omega}}$, and by Proposition 7, we have $\tilde{A}_S = \{(\omega_1, \frac{2}{3}), (\omega_2, \frac{3}{4})\}$.*

Proposition 8. *Let Ω be a basic event space. Then $(\tilde{A} \cup \tilde{B})_S = \tilde{A}_S \cup \tilde{B}_S$ for any $\tilde{A}, \tilde{B} \in 2^{\tilde{\Omega}}$ with $supp(\tilde{A}) \cap supp(\tilde{B}) = \phi$.*

Proof. Let $\tilde{A}, \tilde{B} \in 2^{\tilde{\Omega}}$. We denote

$$
r_{\tilde{A}}(\omega) = \begin{cases} \sup\{r \in (0,1] :\ (\omega, r) \in \tilde{A}\}, & \omega \in \mathrm{supp}\tilde{A} \\ 0, & \omega \notin \mathrm{supp}\tilde{A} \end{cases}
$$

$$
r_{\tilde{B}}(\omega) = \begin{cases} \sup\{r \in (0,1] :\ (\omega, r) \in \tilde{B}\}, & \omega \in \mathrm{supp}\tilde{B} \\ 0, & \omega \notin \mathrm{supp}\tilde{B} \end{cases}
$$

and

$$
r_{\tilde{A}\cup\tilde{B}}(\omega) = \begin{cases} \sup\{r \in (0,1] :\ (\omega, r) \in \tilde{A} \cup \tilde{B}\}, & \omega \in \mathrm{supp}(\tilde{A} \cup \tilde{B}) \\ 0, & \omega \notin \mathrm{supp}(\tilde{A} \cup \tilde{B}) \end{cases}
$$

for any $\omega \in \Omega$. From $\sup\{r \in (0,1] : (\omega, r) \in \tilde{A} \cup \tilde{B}\} = \max\left\{\sup\{r \in (0,1] :\ (\omega, r) \in \tilde{A}\},\ \sup\{r \in (0,1] :\ (\omega, r) \in \tilde{B}\}\right\}$ and $\mathrm{supp}(\tilde{A}) \cap \mathrm{supp}(\tilde{B}) = \phi$, we can see that

$$
r_{\tilde{A}\cup\tilde{B}}(\omega) = \begin{cases} r_{\tilde{A}}, & \omega \in \mathrm{supp}(\tilde{A}) \\ r_{\tilde{B}}, & \omega \in \mathrm{supp}(\tilde{B}) \end{cases}
$$

Therefore, by the definition of the simplistic fuzzy event of fuzzy event and Proposition 7, we have that

$$
\begin{aligned}
\tilde{A}_S \cup \tilde{B}_S &= \mathbb{E}_S^{-1}\left(\mathbb{E}(\tilde{A})\right) \cup \mathbb{E}_S^{-1}\left(\mathbb{E}(\tilde{B})\right) \\
&= \{(\omega, r_{\tilde{A}}(\omega)) :\ \omega \in \mathrm{supp}\tilde{A}\} \cup \{(\omega, r_{\tilde{B}}(\omega)) :\ \omega \in \mathrm{supp}\tilde{B}\} \\
&= \{(\omega, r_{\tilde{A}\cup\tilde{B}}(\omega)) :\ \omega \in \mathrm{supp}\tilde{A}\} \cup \{(\omega, r_{\tilde{A}\cup\tilde{B}}(\omega)) :\ \omega \in \mathrm{supp}\tilde{B}\} \\
&= \{(\omega, r_{\tilde{A}\cup\tilde{B}}(\omega)) :\ \omega \in \mathrm{supp}\tilde{A} \cup \mathrm{supp}\tilde{B}\} \\
&= \{(\omega, r_{\tilde{A}\cup\tilde{B}}(\omega)) :\ \omega \in \mathrm{supp}(\tilde{A} \cup \tilde{B})\} \\
&= \mathbb{E}_S^{-1}\left(\mathbb{E}(\tilde{A} \cup \tilde{B})\right) \\
&= (\tilde{A} \cup \tilde{B})_S
\end{aligned}
$$

□

4. Probability Fuzzy Space

Let Ω be a basic event space. By the definitions of basic fuzzy event space $\tilde{\Omega}$ (generated by Ω), r-level set of fuzzy set and strong r-level set of fuzzy set and Remark 2, we can see that for any $\tilde{A} \in 2^{\tilde{\Omega}}$, $\mathrm{supp}\tilde{A} \in 2^{\Omega}$, and $[\tilde{A}]^r, (\tilde{A})^r \in 2^{\Omega}$ $(r \in [0,1])$.

It is known that for a probability space $(\Omega, \sigma(\Omega), P)$ (where, Ω is a basic event space, $\sigma(\Omega) \subset 2^{\Omega}$ is a event space, $P: \sigma(\Omega) \to [0,1]$ is a probability function), the event space $\sigma(\Omega)$ should keep the closeness of operations of union, intersection and difference. However, due to the complexity of the structure of $2^{\tilde{\Omega}}$, it is often difficult to make a subset (i.e., a collection of some fuzzy events) of $2^{\tilde{\Omega}}$ keeping the closeness of operations of union, intersection and difference. Therefore, when we introduce a probability fuzzy space $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P})$, we do not claim that $\sigma(\tilde{\Omega})$ $(\subset 2^{\tilde{\Omega}})$ keeps the closeness of operations of union, intersection and difference; we only claim that $\sigma(\tilde{\Omega})$ satisfies supp$\tilde{A}$, $[\tilde{A}]^r$ and $(\tilde{A})^r \in \sigma(\Omega)$ for any $\tilde{A} \in \sigma(\tilde{\Omega})$ and $r \in [0,1]$.

Owing to the complexity of structures of $\tilde{\Omega}$ and $2^{\tilde{\Omega}}$, and the non-closeness of operations of union, intersection and difference of $\sigma(\tilde{\Omega})$, when we introduce a probability fuzzy space $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P})$, if we still let probability function $\tilde{P}$ satisfy $\tilde{P}(\tilde{\Omega}) = 1$ and $0 \le \tilde{P}(\tilde{A}) \le 1$ for any $\tilde{A} \in \sigma(\tilde{\Omega})$, and the additivity of $\tilde{P}$: $\tilde{P}(\cup_{i=1}^{\infty} \tilde{A}_i) = \sum_{i=1}^{\infty} \tilde{P}(\tilde{A}_i)$ for any $\tilde{A}_i \in \sigma(\tilde{\Omega})$ $(i = 1, 2 \cdots)$ with $\tilde{A}_m \cap \tilde{A}_n = \phi$ $(m \neq n$ and $m, n = 1, 2, \cdots)$ like we define probability space $(\Omega, \sigma(\Omega), P)$, then we can not guarantee the rationality of probability function $\tilde{P}$ (see the following Example 3).

Example 3. *If we defined $\Omega = \{\omega_1, \omega_2\}$, $\sigma(\Omega) = 2^{\Omega} = \{\phi, \{\omega_1\}, \{\omega_2\}, \Omega\}$ and $P(\omega_1) = (\omega_2) = \frac{1}{2}$, then $(\Omega, \sigma(\Omega), P)$ is a probability space. By the definition of $\tilde{\Omega}$, we see that $\tilde{\Omega} = \{(\omega_1, r), (\omega_2, s) : r, s \in (0,1]\}$. Let*

$$\sigma(\tilde{\Omega}) = \{\phi, \tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4, \tilde{A}_5, \tilde{A}_6, \tilde{\Omega}\}$$

where $\tilde{A}_1 = \{(\omega_1, \frac{4}{5})\}$, $\tilde{A}_2 = \{(\omega_1, \frac{3}{5})\}$, $\tilde{A}_3 = \{(\omega_2, \frac{1}{2})\}$, $\tilde{A}_4 = \{(\omega_1, \frac{4}{5}), (\omega_1, \frac{3}{5})\}$, $\tilde{A}_5 = \{(\omega_1, \frac{4}{5}), (\omega_2, \frac{1}{2})\}$ and, $\tilde{A}_6 = \{(\omega_1, \frac{3}{5}), (\omega_2, \frac{1}{2})\}$. Then $\sigma(\tilde{\Omega})$ satisfies supp$\tilde{A}$, $[\tilde{A}]^r$ and $(\tilde{A})^r \in \sigma(\Omega)$ for any $\tilde{A} \in \sigma(\tilde{\Omega})$ and $r \in [0,1]$. If we define $\tilde{P}: \sigma(\tilde{\Omega}) \to R$ as $\tilde{P}(\phi) = 0$, $\tilde{P}(\tilde{A}_1) = \frac{1}{3}$, $\tilde{P}(\tilde{A}_2) = \frac{1}{3}$, $\tilde{P}(\tilde{A}_3) = \frac{1}{2}$, $\tilde{P}(\tilde{A}_4) = \frac{2}{3}$, $\tilde{P}(\tilde{A}_5) = \frac{5}{6}$, $\tilde{P}(\tilde{A}_6) = \frac{5}{6}$ and $\tilde{P}(\tilde{\Omega}) = 1$, then $\tilde{P}: \sigma(\tilde{\Omega}) \to R$ satisfies $\tilde{P}(\cup_{i=1}^{\infty} \tilde{A}_i) = \sum_{i=1}^{\infty} \tilde{P}(\tilde{A}_i)$ for any $\tilde{A}_i \in \sigma(\tilde{\Omega})$ $(i = 1, 2 \cdots)$ with $\tilde{A}_m \cap \tilde{A}_n = \phi$ $(m \neq n$ and $m, n = 1, 2, \cdots)$. Generally speaking, $\tilde{A}_1 = \{(\omega_1, \frac{4}{5})\}$ and $\tilde{A}_2 = \{(\omega_1, \frac{3}{5})\}$ should be regarded as basic fuzzy evens $(\omega_1, \frac{4}{5})$ and $(\omega_1, \frac{3}{5})$. Therefore, from $(\omega_1, \frac{4}{5}) \cup (\omega_1, \frac{3}{5}) = (\omega_1, \frac{4}{5})$, we consider that $\tilde{P}(\tilde{A}_4) = \tilde{P}(\tilde{A}_1 \cup \tilde{A}_2) = \tilde{P}\left(\{(\omega_1, \frac{4}{5})\} \cup \{(\omega_1, \frac{3}{5})\}\right)$ should be defined as $\tilde{P}\left(\{(\omega_1, \frac{4}{5})\}\right) = \tilde{P}(\tilde{A}_1) = \frac{1}{3}$. However, this is not consistent with the definition $\tilde{P}(\tilde{A}_4) = \frac{2}{3}$. So we think that such defined $\tilde{P}$ has some irrationality.

From the above analysis, we know that in order to define a rational probability function $\tilde{P}$ on $\sigma(\tilde{\Omega})$, we have to change this Condition (that is: $\tilde{P}(\cup_{i=1}^{\infty} \tilde{A}_i) = \sum_{i=1}^{\infty} \tilde{P}(\tilde{A}_i)$ for any $\tilde{A}_i \in \sigma(\tilde{\Omega})$ $(i = 1, 2 \cdots)$ with $\tilde{A}_m \cap \tilde{A}_n = \phi$ $(m \neq n$ and $m, n = 1, 2, \cdots))$ to make it reasonable. Considering the complexity of the structures of $\tilde{\Omega}$ and $\sigma(\tilde{\Omega})$, we propose to replace the irrational additivity of $\tilde{P}$ with the following rational conditions: (1) $\tilde{P}(\tilde{A}) \le \tilde{P}(\tilde{B})$ for any $\tilde{A}, \tilde{B} \in \sigma(\tilde{\Omega})$ with $\mathbb{E}(\tilde{A}) \subset \mathbb{E}(\tilde{B})$; (2) $\tilde{P}(\cup_{i=1}^{\infty} \tilde{A}_i) = \sum_{i=1}^{\infty} \tilde{P}(\tilde{A}_i)$ for any $\tilde{A}_i \in \sigma(\tilde{\Omega})$ $(i = 1, 2 \cdots)$ with $\mathbb{E}(\tilde{A}_m) \cap \mathbb{E}(\tilde{A}_n) = \phi$ $(m \neq n$ and $m, n = 1, 2, \cdots)$.

Definition 6. *Let Ω be a basic event space, $\tilde{\Omega}$ be the basic fuzzy event space generated by Ω, and $\sigma(\tilde{\Omega}) \subset 2^{\tilde{\Omega}}$ with $\phi, \tilde{\Omega} \in \sigma(\tilde{\Omega})$. If $\tilde{P}: \sigma(\tilde{\Omega}) \to R$ satisfies*

(1) $0 \le \tilde{P}(\tilde{A})$;
(2) $\tilde{P}(\tilde{\Omega}) = 1$;
(3) $\tilde{P}(\tilde{A}) \le \tilde{P}(\tilde{B})$ *for any* $\tilde{A}, \tilde{B} \in \sigma(\tilde{\Omega})$ *with* $\mathbb{E}(\tilde{A}) \subset \mathbb{E}(\tilde{B})$;
(4) $\tilde{P}(\cup_{i=1}^{\infty} \tilde{A}_i) = \sum_{i=1}^{\infty} \tilde{P}(\tilde{A}_i)$ *for any* $\tilde{A}_i \in \sigma(\tilde{\Omega})$ $(i = 1, 2 \cdots)$ *with* $\mathbb{E}(\tilde{A}_m) \cap \mathbb{E}(\tilde{A}_n) = \phi$ $(m \neq n$ *and* $m, n = 1, 2, \cdots)$,

then we call $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P})$ a probability fuzzy space.

Remark 6. *(i) From the Conditions 2 and 3 in Definition 6, we see that $\tilde{P}(\tilde{A}) \le 1$ for any $\tilde{A} \in \sigma(\tilde{\Omega})$.*

(ii) *If the Condition (4) $\tilde{P}(\cup_{i=1}^{\infty}\tilde{A}_i) = \sum_{i=1}^{\infty}\tilde{P}(\tilde{A}_i)$ for any $\tilde{A}_i \in \sigma(\tilde{\Omega})$ $(i = 1, 2\cdots)$ with $\mathbb{E}(\tilde{A}_m) \cap \mathbb{E}(\tilde{A}_n) = \phi$ $(m \neq n$ and $m, n = 1, 2, \cdots)$ is replaced with Condition (4') $P(\tilde{A}_1 \cup \tilde{A}_2) = P(\tilde{A}_1) + P(\tilde{A}_2)$ for any $\tilde{A}_1, \tilde{A}_2 \in \sigma(\tilde{\Omega})$ with $\mathbb{E}(\tilde{A}_1) \cap \mathbb{E}(\tilde{A}_2) = \phi$, then we call $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P})$ a finitely additive probability fuzzy space.*

Lemma 1. *Let $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P})$ be a probability fuzzy space. For any $\tilde{A}, \tilde{B} \in S(2^{\tilde{\Omega}})$ with $\tilde{A} \subset \tilde{B}$, we have that $supp(\tilde{B} \setminus \tilde{A}) = supp\tilde{B} \setminus supp\tilde{A}$.*

Proof. For any $\omega \in \mathrm{supp}(\tilde{B} \setminus \tilde{A})$, by Definition 2, we see that there exists $r_\omega \in (0, 1]$ such that $(\omega, r_\omega) \in \tilde{B} \setminus \tilde{A}$, i.e., $(\omega, r_\omega) \in \tilde{B}$ and $(\omega, r_\omega) \notin \tilde{A}$. It is implied that $\omega \in \mathrm{supp}\tilde{B}$ (by Definition 2) and $\omega \notin \mathrm{supp}\tilde{A}$ (if not, then there exists $\grave{r} \in (0, 1]$ with $\grave{r} \neq r_\omega$ such that $(\omega, \grave{r}) \in \tilde{A} \subset \tilde{B}$). This is contradictory to $(\omega, r_\omega) \in \tilde{B} \in S(2^{\tilde{\Omega}})$), i.e., $\omega \in \mathrm{supp}\tilde{B} \setminus \mathrm{supp}\tilde{A}$. Thus, we have proved $\mathrm{supp}(\tilde{B} \setminus \tilde{A}) \subset \mathrm{supp}\tilde{B} \setminus \mathrm{supp}\tilde{A}$

Conversely, for any $\omega \in \mathrm{supp}\tilde{B} \setminus \mathrm{supp}\tilde{A}$, we know that $\omega \in \mathrm{supp}\tilde{B}$ and $\omega \notin \mathrm{supp}\tilde{A}$, i.e., there exists $r_\omega \in (0, 1]$ such that $(\omega, r_\omega) \in \tilde{B}$, and $(\omega, r) \notin \tilde{A}$ for any $r \in (0, 1]$. Therefore, we see that $(\omega, r_\omega) \in \tilde{B}$ and $(\omega, r_\omega) \notin \tilde{A}$, i.e., $(\omega, r_\omega) \in \tilde{B} \setminus \tilde{A}$, so $\omega \in \mathrm{supp}(\tilde{B} \setminus \tilde{A})$. Thus, we have proved $\mathrm{supp}(\tilde{B} \setminus \tilde{A}) \supset \mathrm{supp}\tilde{B} \setminus \mathrm{supp}\tilde{A}$. □

Proposition 9. *Let $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P})$ be a probability fuzzy space (or a finitely additive probability fuzzy space). Then*

(1) $\tilde{P}(\phi) = 0$;
(2) $\tilde{P}(\cup_{i=1}^{N}\tilde{A}_i) = \sum_{i=1}^{N}\tilde{P}(\tilde{A}_i)$ *for any positive integer N and $\tilde{A}_i \in \sigma(\tilde{\Omega})$ $(i = 1, 2\cdots, N)$ with $\mathbb{E}(\tilde{A}_m) \cap \mathbb{E}(\tilde{A}_n) = \phi$ $(m \neq n$ and $m, n = 1, 2, \cdots, N)$;*
(3) $\tilde{P}(\tilde{A}) \leq \tilde{P}(\tilde{B})$ *for any $\tilde{A}, \tilde{B} \in \sigma(\tilde{\Omega})$ with $\tilde{A} \subset \tilde{B}$;*
(4) $\tilde{P}(\tilde{A}) = \tilde{P}(\tilde{B})$ *for any $\tilde{A}, \tilde{B} \in \sigma(\tilde{\Omega})$ with $\mathbb{E}(\tilde{A}) = \mathbb{E}(\tilde{B})$;*
(5) $\tilde{P}(\tilde{A}) = \tilde{P}(\tilde{A}_S)$ *(i.e., $\tilde{P}\left(\mathbb{E}_S^{-1}(\mathbb{E}(\tilde{A}))\right)$ for any $\tilde{A} \in \sigma(\tilde{\Omega})$;*
(6) $\tilde{P}(\tilde{B} \setminus \tilde{A}) = \tilde{P}(\tilde{B}) - \tilde{P}(\tilde{A})$ *for any $\tilde{A}, \tilde{B} \in \sigma(\tilde{\Omega})$ with $\tilde{A}, \tilde{B} \in S(2^{\tilde{\Omega}})$ and $\tilde{A} \subset \tilde{B}$, where $\tilde{B} \setminus \tilde{A} = \{(\omega, r) \in 2^{\tilde{\Omega}} : (\omega, r) \in \tilde{B}, (\omega, r) \notin \tilde{A}\}$;*
(7) $\tilde{P}(\tilde{A}^{SC}) = \tilde{P}(S^r(\tilde{\Omega})) - \tilde{P}(\tilde{A})$ *for any $\tilde{A}, S^r(\tilde{\Omega}) \in \sigma(\tilde{\Omega})$ with $\tilde{A} \subset S^r(\tilde{\Omega})$, where $\tilde{A}^{SC} = S^r(\tilde{\Omega}) \setminus \tilde{A}$;*

Proof. (1) Let $\tilde{A}_i = \phi$ $(i = 1, 2, \cdots)$. Then $\tilde{A}_i \in \sigma(\tilde{\Omega})$ $(i = 1, 2, \cdots)$ and $\mathbb{E}(\tilde{A}_m) \cap \mathbb{E}(\tilde{A}_n) = \phi$ $(m \neq n$ and $m, n = 1, 2, \cdots)$, so by the Condition (4) in Definition 6, we have that $\tilde{P}(\phi) = \tilde{P}(\cup_{i=1}^{\infty}\tilde{A}_i) = \sum_{i=1}^{\infty}\tilde{P}(\tilde{A}_i) = \tilde{P}(\phi) + \tilde{P}(\phi) + \cdots + \tilde{P}(\phi) + \cdots$. It implies $\tilde{P}(\phi) = 0$;

(2) Let N is a positive integer, $\tilde{A}_i \in \sigma(\tilde{\Omega})$ $(i = 1, 2\cdots, N)$ with $\mathbb{E}(\tilde{A}_m) \cap \mathbb{E}(\tilde{A}_n) = \phi$ $(m \neq n$ and $m, n = 1, 2, \cdots, N)$, and $\tilde{A}_i = \phi$ $(i = N+1, N+2, \cdots)$. Then $\tilde{P}(\cup_{i=1}^{N}\tilde{A}_i) = \tilde{P}(\cup_{i=1}^{\infty}\tilde{A}_i) = \sum_{i=1}^{\infty}\tilde{P}(\tilde{A}_i) = \sum_{i=1}^{N}\tilde{P}(\tilde{A}_i) + 0 + \cdots + 0 + \cdots = \sum_{i=1}^{N}\tilde{P}(\tilde{A}_i)$;

(3) From $\tilde{A} \subset \tilde{B}$, by Conclusion (1) of Proposition 1, we see $\mathbb{E}(\tilde{A}) \subset \mathbb{E}(\tilde{B})$. Therefore, by Condition (3) of Definition 6, we directly know that Conclusion (3) of the proposition holds;

(4) $\mathbb{E}(\tilde{A}) = \mathbb{E}(\tilde{B})$ implies that $\mathbb{E}(\tilde{A}) \subset \mathbb{E}(\tilde{B})$ and $\mathbb{E}(\tilde{B}) \subset \mathbb{E}(\tilde{A})$. So by the Condition (3) in Definition 6, we see that $\tilde{P}(\tilde{A}) \leq \tilde{P}(\tilde{B})$ and $\tilde{P}(\tilde{B}) \leq \tilde{P}(\tilde{A})$, so we have that $\tilde{P}(\tilde{A}) = \tilde{P}(\tilde{B})$;

(5) For any $\tilde{A} \in \sigma(\tilde{\Omega})$, we have that $\mathbb{E}(\tilde{A}) = \left(\mathbb{E}_S\mathbb{E}_S^{-1}\right)(\mathbb{E}(\tilde{A})) = \mathbb{E}_S\left(\mathbb{E}_S^{-1}(\mathbb{E}(\tilde{A}))\right) = \mathbb{E}\left(\mathbb{E}_S^{-1}(\mathbb{E}(\tilde{A}))\right)$. So by the Conclusion (4), we see that $\tilde{P}(\tilde{A}) = \tilde{P}\left(\mathbb{E}_S^{-1}(\mathbb{E}(\tilde{A}))\right)$;

(6) From $\tilde{A}, \tilde{B} \in S(2^{\tilde{\Omega}})$ and $\tilde{A} \subset \tilde{B}$, by Lemma 1, we can show that $\mathrm{supp}\tilde{A} \cap \mathrm{supp}(\tilde{B} \setminus \tilde{A}) = \phi$ (if not, then there exist $\omega \in \mathrm{supp}\tilde{A}$ and $\omega \in \mathrm{supp}(\tilde{B} \setminus \tilde{A})$. By Lemma 1, we have that $\omega \in \mathrm{supp}\tilde{A}$ and $\omega \in \mathrm{supp}\tilde{B} \setminus \mathrm{supp}\tilde{A}$, so we obtain contradictory conclusions: $\omega \in \mathrm{supp}\tilde{A}$ and $\omega \notin \mathrm{supp}\tilde{A}$). By Proposition 2, we have that

$$
\begin{aligned}
&\left(\mathbb{E}(\tilde{A}) \cap \mathbb{E}(\tilde{B} \setminus \tilde{A})\right)(\omega) \\
&= \min\left\{\mathbb{E}(\tilde{A})(\omega),\ \mathbb{E}(\tilde{B} \setminus \tilde{A})(\omega)\right\} \\
&= \begin{cases} \min\{r_\omega,\ 0\}, & \omega \in \mathrm{supp}\tilde{A} \text{ (it implies } \omega \notin \mathrm{supp}(\tilde{B} \setminus \tilde{A}) \text{ by } \mathrm{supp}\tilde{A} \cap \mathrm{supp}(\tilde{B} \setminus \tilde{A}) = \phi) \\ \min\left\{0,\ \mathbb{E}(\tilde{B} \setminus \tilde{A})(\omega)\right\}, & \omega \notin \mathrm{supp}\tilde{A} \end{cases} \\
&= 0
\end{aligned}
$$

for any $\omega \in \Omega$, so $\mathbb{E}(\tilde{A}) \cap \mathbb{E}(\tilde{B} \setminus \tilde{A}) = \phi$. Therefore, by Conclusion (2), we have that $\tilde{P}(\tilde{B}) = \tilde{P}\left(\tilde{A} \cup (\tilde{B} \setminus \tilde{A})\right) = \tilde{P}(\tilde{A}) + \tilde{P}(\tilde{B} \setminus \tilde{A})$, so $\tilde{P}(\tilde{B} \setminus \tilde{A}) = \tilde{P}(\tilde{B}) - \tilde{P}(\tilde{A})$;

(7) For any fixed $r \in (0, 1]$, by the definition of $S^r(\tilde{\Omega})$, and from $\tilde{A} \subset S^r(\tilde{\Omega})$, we see that $S^r(\tilde{\Omega}) \in S(2^{\tilde{\Omega}})$ and $\tilde{A} \in S(2^{\tilde{\Omega}})$. Therefore, by the Conclusion (6), we can directly obtain the Conclusion (7). □

Theorem 1. *Let $(\Omega, \sigma(\Omega), P)$ be a probability space, and $\sigma(\tilde{\Omega}) = \{\tilde{A} \in 2^{\tilde{\Omega}} :\ (\tilde{A})^r \in \sigma(\Omega)$ for any $r \in (0, 1]\}$. For fixed $r \in (0, 1)$, if $\tilde{P}_{(r)} :\ \sigma(\tilde{\Omega}) \to R$ is defined by*

$$\tilde{P}_{(r)}(\tilde{A}) = P((\tilde{A})^r), \text{ for any } \tilde{A} \in \sigma(\tilde{\Omega}) \tag{1}$$

then $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P}_{(r)})$ is a probability fuzzy space (we call it strong-r-probability fuzzy space generated by $(\Omega, \sigma(\Omega), P)$).

Proof. Since $(\Omega, \sigma(\Omega), P)$ is a probability space, we have that (1) $0 \le P(A) \le 1$ for any $A \in \sigma(\Omega)$; (2) $P(\Omega) = 1$; (3) $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ for any $A_i \in \sigma(\Omega)$ with $A_m \cap A_n = \phi$ ($m \neq n$ and $m, n = 1, 2, \cdots, \infty$).

(1) For $\tilde{P}_{(r)}$, $r \in (0, 1)$, from the definition of $\tilde{P}_{(r)}$, we see that the Condition (1) in Definition 6 holds;

(2) For $\tilde{P}_{(r)}$, $r \in (0, 1)$, we have that $\tilde{P}_{(r)}(\tilde{\Omega}) = P((\tilde{\Omega})^r) = P([\mathbb{E}(\tilde{\Omega})]^r) = P(\Omega) = 1$, so the Condition (2) in Definition 6 also holds;

(3) Let $\tilde{A}, \tilde{B} \in \sigma(\tilde{\Omega})$ with $\mathbb{E}(\tilde{A}) \subset \mathbb{E}(\tilde{B})$. By the Conclusion (2) of Proposition 1, we see that $[\tilde{A}]^r \subset [\tilde{B}]^r$ ($r \in (0, 1)$). Therefore, $\tilde{P}_{(r)}(\tilde{A}) = P((\tilde{A})^r) \le P((\tilde{B})^r) = \tilde{P}(\tilde{B})$ ($r \in (0, 1)$), so the Condition (3) in Definition 6 also holds;

(4) Let $\tilde{A}_i \in \sigma(\tilde{\Omega})$ ($i = 1, 2 \cdots$) with $\mathbb{E}(\tilde{A}_m) \cap \mathbb{E}(\tilde{A}_n) = \phi$ ($m \neq n$ and $m, n = 1, 2, \cdots$). From $\tilde{A}_i \in \sigma(\tilde{\Omega})$ and $\mathbb{E}(\tilde{A}_m) \cap \mathbb{E}(\tilde{A}_n) = \phi$, we know that $(\tilde{A}_i)^r \in \sigma(\Omega)$ and $\mathrm{supp}\mathbb{E}(\tilde{A}_m) \cap \mathrm{supp}\mathbb{E}(\tilde{A}_n) = \phi$, so $(\tilde{A}_m)^r \cap (\tilde{A}_n)^r = (\mathbb{E}(\tilde{A}_m))^r \cap (\mathbb{E}(\tilde{A}_n))^r = \phi$ for any $r \in (0, 1)$ ($m \neq n$ and $m, n = 1, 2, \cdots$). Therefore we have that $\tilde{P}_{(r)}(\cup_{i=1}^{\infty} \tilde{A}_i) = P((\cup_{i=1}^{\infty} \tilde{A}_i)^r) = P(\cup_{i=1}^{\infty} (\tilde{A}_i)^r) = \sum_{i=1}^{\infty} P((\tilde{A}_i)^r) = \sum_{i=1}^{\infty} \tilde{P}_r(\tilde{A}_i)$. Thus we have shown that the Condition (4) in Definition 6 also holds.

By the Definition 6, we see that for any $r \in [0, 1]$, $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P}_r)$ is a probability fuzzy space. □

Theorem 2. *Let Ω be a finite set, $(\Omega, \sigma(\Omega), P)$ be a probability space, and $\sigma(\tilde{\Omega}) = \{\tilde{A} \in 2^{\tilde{\Omega}} :\ [\tilde{A}]^r \in \sigma(\Omega)$ for any $r \in [0, 1]\}$. For any $r \in (0, 1]$, if $\tilde{P}_{[r]} :\ \sigma(\tilde{\Omega}) \to R$ is defined by*

$$\tilde{P}_{[r]}(\tilde{A}) = P([\tilde{A}]^r), \text{ for any } \tilde{A} \in \sigma(\tilde{\Omega}) \tag{2}$$

then $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P}_{[r]})$ is a probability fuzzy space (we call it the r-probability fuzzy space generated by $(\Omega, \sigma(\Omega), P)$).

Proof. The proof of the theorem is similar to the proof of Theorem 1, and we omit it. □

Remark 7. *In the proof of the conclusion 4 of Theorem 1, the fact $(\cup_{i=1}^{\infty} \tilde{A}_i)^r = \cup_{i=1}^{\infty} (\tilde{A}_i)^r$ (that is well known) is used. However, it is also well known that $[\cup_{i=1}^{\infty} \tilde{A}_i]^r = \cup_{i=1}^{\infty} [\tilde{A}_i]^r$ is not true (it is just right in finite cases, i.e., $[\cup_{i=1}^{N} \tilde{A}_i]^r = \cup_{i=1}^{N} [\tilde{A}_i]^r$ is true, where N is a positive integer), so, generally speaking, the condition "Ω is a finite set" cannot be missed in Theorem 2. However, since $[\cup_{i=1}^{N} \tilde{A}_i]^r = \cup_{i=1}^{N} [\tilde{A}_i]^r$ is true, we can see the following result holds:*

Theorem 3. *Let $(\Omega, \sigma(\Omega), P)$ be a probability space, and $\sigma(\tilde{\Omega}) = \{\tilde{A} \in 2^{\tilde{\Omega}} : [\tilde{A}]^r \in \sigma(\Omega)$ for any $r \in [0,1]\}$. For any $r \in (0,1]$, if $\tilde{P}_{[r]} : \sigma(\tilde{\Omega}) \to R$ is defined by Equality (2), then $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P}_{[r]})$ is a finitely additive probability fuzzy space (we call it the finitely additive r-probability fuzzy space generated by $(\Omega, \sigma(\Omega), P)$).*

Example 4. *By statistical methods, we obtain the possibility that a person may be in target shooting as follows: The possibility of hitting the i ring is $\frac{1}{12}$ ($i = 1, 2, \cdots, 10$); the possibility of missing the target (we say to hit the 0 ring) is $\frac{1}{6}$. Let $\omega_i =$"hitting the i ring", $i = 0, 1, 2, \cdots, 10$, $\Omega = \{\omega_0, \omega_1, \omega_2, \cdots, \omega_{10}\}$, $\sigma(\Omega) = 2^{\Omega}$ and $P : 2^{\Omega} \to [0,1]$ be defined as $P(\omega_0) = \frac{1}{6}$ and $P(\omega_i) = \frac{1}{12}$, $i = 1, 2, \cdots, 10$. Then $(\Omega, \sigma(\Omega), P)$ is a probability space, that characterizes the possibility that the person may be in target shooting. Let $\sigma(\tilde{\Omega}) = 2^{\tilde{\Omega}}$, then $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P}_{[r]})$ is the r-probability fuzzy space generated by $(\Omega, \sigma(\Omega), P)$.*

If we use discrete fuzzy numbers $\tilde{A}_1 = \{(\omega_7, \frac{1}{4}), (\omega_8, \frac{3}{4}), (\omega_9, 1), (\omega_{10}, 1)\}$, $\tilde{A}_2 = \{(\omega_8, \frac{1}{4}), (\omega_9, \frac{3}{4}), (\omega_{10}, 1)\}$ and $\tilde{A}_3 = \{(\omega_9, \frac{1}{4}), (\omega_{10}, 1)\}$ represent fuzzy events "Better hitting result", "Good hitting result" and "Very good hitting result", respectively. Then $\tilde{P}_{[0.75]}(\tilde{A}_1) = P\{\omega_8, \omega_9, \omega_{10}\} = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{1}{4}$, $\tilde{P}_{[0.75]}(\tilde{A}_2) = P\{\omega_9, \omega_{10}\} = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$ and $\tilde{P}_{[0.75]}(\tilde{A}_3) = P\{\omega_{10}\} = \frac{1}{12}$ express, respectively, that at the confidence level value 0.75, the probability that the person would get better shooting result in one shooting is $\frac{1}{4}$, that at the confidence level value 0.75, the probability that the person would get good shooting result in one shooting is $\frac{1}{6}$, and that at the confidence level value 0.75, the probability that the person would get very good shooting result in one shooting is $\frac{1}{12}$.

Of course, these results change with the change of the level value. The level value characterizes the reliability of these results. For example, if we take the level value as 0.95, then we have that $\tilde{P}_{[0.95]}(\tilde{A}_1) = P\{\omega_9, \omega_{10}\} = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$, $\tilde{P}_{[0.95]}(\tilde{A}_2) = P\{\omega_{10}\} = \frac{1}{12}$ and $\tilde{P}_{[0.95]}(\tilde{A}_3) = P\{\omega_{10}\} = \frac{1}{12}$. It tells us that at the confidence level value 0.95, the probability that the person would get a better shooting result in one shooting is $\frac{1}{6}$; the probability that the person would get a good shooting result in one shooting is $\frac{1}{12}$; and the probability that the person would get a very good shooting result in one shooting is $\frac{1}{12}$.

Theorem 4. *Let Ω be a countable set (including finite set), $(\Omega, \sigma(\Omega), P)$ be a probability space, and $\sigma(\tilde{\Omega}) = \{\tilde{A} \in 2^{\tilde{\Omega}} : supp\tilde{A} \in \sigma(\Omega)\}$. If $\tilde{P} : \sigma(\tilde{\Omega}) \to R$ is defined as*

$$\tilde{P}(\tilde{A}) = \sum_{\omega \in supp\tilde{A}} P(\omega) \left(\mathbb{E}(\tilde{A})\right)(\omega), \quad \text{for any } \tilde{A} \in \sigma(\tilde{\Omega}) \tag{3}$$

then $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P})$ is a probability fuzzy space.

Proof. (1) For any $\tilde{A} \in \sigma(\tilde{\Omega})$, from $\left(\mathbb{E}(\tilde{A})\right)(\omega) \leq 1$ for any $\omega \in \mathrm{supp}\tilde{A}$, we have that

$$\begin{aligned} \tilde{P}(\tilde{A}) &= \textstyle\sum_{\omega \in \mathrm{supp}\tilde{A}} P(\omega) \left(\mathbb{E}(\tilde{A})\right)(\omega) \\ &\leq \textstyle\sum_{\omega \in \mathrm{supp}\tilde{A}} P(\omega) \\ &= P(\mathrm{supp}\tilde{A}) \\ &\leq 1 \end{aligned}$$

On the other hand, by Equality (3), the correctness of $\tilde{P}(\tilde{A}) \geq 0$ is obvious. Therefore, the Condition (1) in Definition 6 holds;

(2) $\tilde{P}(\tilde{\Omega}) = \sum_{\omega \in \mathrm{supp}\tilde{\Omega}} P(\omega) \left(\mathbb{E}(\tilde{\Omega})\right)(\omega) = \sum_{\omega \in \Omega} P(\omega)\Omega(\omega) = \sum_{\omega \in \Omega} P(\omega) = P(\Omega) = 1$, so the Condition (2) in Definition 6 holds;

(3) Let $\tilde{A}, \tilde{B} \in \sigma(\tilde{\Omega})$ with $\mathbb{E}(\tilde{A}) \subset \mathbb{E}(\tilde{B})$.

$$\begin{aligned}\tilde{P}(\tilde{A}) &= \textstyle\sum_{\omega \in \mathrm{supp}\tilde{A}} P(\omega)\left(\mathbb{E}(\tilde{A})\right)(\omega) \\ &= \textstyle\sum_{\omega \in \mathrm{supp}\tilde{B}} P(\omega)\left(\mathbb{E}(\tilde{A})\right)(\omega) \\ &\leq \textstyle\sum_{\omega \in \mathrm{supp}\tilde{B}} P(\omega)\left(\mathbb{E}(\tilde{B})\right)(\omega) \\ &= \tilde{P}(\tilde{B})\end{aligned}$$

so the Condition (3) in Definition 6 holds;

(4) Let $\tilde{A}_i \in \sigma(\tilde{\Omega})$ $(i = 1, 2\cdots)$ with $\mathbb{E}(\tilde{A}_m) \cap \mathbb{E}(\tilde{A}_n) = \phi$ $(m \neq n$ and $m, n = 1, 2, \cdots)$. From $\tilde{A}_i \in \sigma(\tilde{\Omega})$ and $\mathbb{E}(\tilde{A}_m) \cap \mathbb{E}(\tilde{A}_n) = \phi$, we know that $(\tilde{A}_i)^r \in \sigma(\Omega)$ and $\mathrm{supp}\mathbb{E}(\tilde{A}_m) \cap \mathrm{supp}\mathbb{E}(\tilde{A}_n) = \phi$, so $(\tilde{A}_m)^r \cap (\tilde{A}_n)^r = (\mathbb{E}(\tilde{A}_m))^r \cap (\mathbb{E}(\tilde{A}_n))^r = \phi$ for any $r \in (0, 1)$ $(m \neq n$ and $m, n = 1, 2, \cdots)$. Therefore we have that

$$\begin{aligned}\tilde{P}(\cup_{i=1}^{\infty}\tilde{A}_i) &= \textstyle\sum_{\omega \in \mathrm{supp}(\cup_{i=1}^{\infty}\tilde{A}_i)} P(\omega)\left(\mathbb{E}(\cup_{i=1}^{\infty}\tilde{A}_i)\right)(\omega) \\ &= \textstyle\sum_{\omega \in \Omega} P(\omega)\left(\mathbb{E}(\cup_{i=1}^{\infty}\tilde{A}_i)\right)(\omega) \\ &= \textstyle\sum_{\omega \in \Omega} P(\omega)\left(\cup_{i=1}^{\infty}\mathbb{E}(\tilde{A}_i)\right)(\omega) \\ &= \textstyle\sum_{\omega \in \mathrm{supp}(\cup_{j=1}^{\infty}\mathbb{E}(\tilde{A}_j))} P(\omega)\left(\cup_{i=1}^{\infty}\mathbb{E}(\tilde{A}_i)\right)(\omega) \\ &= \textstyle\sum_{j=1}^{\infty}\sum_{\omega \in \mathrm{supp}\mathbb{E}(\tilde{A}_j)} P(\omega)\left(\cup_{i=1}^{\infty}\mathbb{E}(\tilde{A}_i)\right)(\omega) \\ &= \textstyle\sum_{j=1}^{\infty}\sum_{\omega \in \mathrm{supp}\mathbb{E}(\tilde{A}_j)} P(\omega)\left(\mathbb{E}(\tilde{A}_j)\right)(\omega) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{\omega \in \mathrm{supp}(\tilde{A}_i)} P(\omega)\left(\mathbb{E}(\tilde{A}_i)\right)(\omega) \\ &= \textstyle\sum_{i=1}^{\infty}\tilde{P}(\tilde{A}_i)\end{aligned}$$

Thus we have shown that the Condition (4) in Definition 6 also holds.

By the Definition 6, we see that $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P})$ is a probability fuzzy space. □

Definition 7. *Let Ω be a countable set (including finite set), $(\Omega, \sigma(\Omega), P)$ be a probability space. If $\sigma(\tilde{\Omega})$ and $\tilde{P} : \sigma(\tilde{\Omega}) \to R$ is defined as in Theorem 4, then we call $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P})$ the probability fuzzy space generated by $(\Omega, \sigma(\Omega), P)$, and denote the probability function and the generated probability fuzzy space as $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P}_G)$ and $\tilde{P}_G$, respectively.*

Remark 8. *By the definition of $\tilde{P}_G$ (see the Formula (3)), we directly see that*

$$\tilde{P}_G(\tilde{A}) = \tilde{P}_G(\tilde{B}) \ \textit{ for any } \tilde{A}, \tilde{B} \in 2^{\tilde{\Omega}} \textit{ with } \mathbb{E}(\tilde{A}) = \mathbb{E}(\tilde{B})$$

Example 5. *In Example 4, let $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P}_G)$ be the probability fuzzy space generated by $(\Omega, \sigma(\Omega), P)$. Then by Equality (3), we have that $\tilde{P}_G(\tilde{A}_1) = \frac{1}{12} \cdot \frac{1}{4} + \frac{1}{12} \cdot \frac{3}{4} + \frac{1}{12} \cdot 1 + \frac{1}{12} \cdot 1 = \frac{1}{4}$, $\tilde{P}_G(\tilde{A}_2) = \frac{1}{12} \cdot \frac{1}{4} + \frac{1}{12} \cdot \frac{3}{4} + \frac{1}{12} \cdot 1 = \frac{1}{6}$ and $\tilde{P}_G(\tilde{A}_3) = \frac{1}{12} \cdot \frac{1}{4} + \frac{1}{12} \cdot 1 = \frac{5}{48}$, that express, respectively, that the probability that the person would get better shooting result in one shooting is $\frac{1}{4}$, that the probability that the person would get good shooting result in one shooting is $\frac{1}{6}$, and that the probability that the person would get very good shooting result in one shooting is $\frac{5}{48}$.*

Since $\tilde{P}_G$, $\tilde{P}_{[r]}$ $(r \in [0, 1])$ and $\tilde{P}_{(r)}$ $(r \in (0, 1))$ are all especial probability functions, not only the conclusions (1)–(7) all hold for them, but also they posses the following property:

Proposition 10. *Let Ω be a countable set (including finite set), $(\Omega, \sigma(\Omega), P)$ be a probability space and $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P}_G)$ be the generated probability fuzzy space. Then*

(1) $\tilde{P}_G(\tilde{B} \setminus \tilde{A}) = \tilde{P}_G(\tilde{B}) - \tilde{P}_G(\tilde{A}_{\mathbb{E}(\tilde{B})})$ *for any* $\tilde{A}, \tilde{B} \in 2^{\tilde{\Omega}}$ *with* $\tilde{A} \subset \tilde{B}$, *where* $\tilde{A}_{\mathbb{E}(\tilde{B})} = \{(\omega, r) \in \tilde{A} : \left(\mathbb{E}(\tilde{B})\right)(\omega) = \left(\mathbb{E}(\tilde{A})\right)(\omega),\ r \in (0, 1]\}$ *(it obviously implies that* $\mathrm{supp}\tilde{A}_{\mathbb{E}(\tilde{B})} = \{\omega \in \mathrm{supp}\tilde{A} : \left(\mathbb{E}(\tilde{B})\right)(\omega) = \left(\mathbb{E}(\tilde{A})\right)(\omega)\}$*);*

(2) $\tilde{P}_G(\tilde{A}^C) = 1 - \tilde{P}_G(\tilde{A}_{[1]}) = 1 - \sum_{(\mathbb{E}(\tilde{A}))(\omega)=1} P(\omega)$ *for any* $\tilde{A} \in 2^{\tilde{\Omega}}$, *where* $\tilde{A}^C = \tilde{\Omega} \setminus \tilde{A}$, $\tilde{A}_{[1]} = \{(\omega, r) \in \tilde{A} : (\mathbb{E}(\tilde{A}))(\omega) = 1,\ r \in (0,1]\}$ *(it obviously implies that supp*$\tilde{A}_{[1]} = \{\omega \in$ *supp*$\tilde{A}$: $(\mathbb{E}(\tilde{A}))(\omega) = 1\}$*).*

Proof. (1) By the definition of $\tilde{P}_G$ and Corollary 2, we have that

$$\begin{aligned}
\tilde{P}_G(\tilde{B}) &= \textstyle\sum_{\omega\in \mathrm{supp}\tilde{B}} P(\omega)\left(\mathbb{E}(\tilde{B})\right)(\omega) \\
&= \textstyle\sum_{\omega\in\Omega} P(\omega)\left(\mathbb{E}(\tilde{B})\right)(\omega) \\
&= \textstyle\sum_{(\mathbb{E}(\tilde{B}))(\omega)\neq(\mathbb{E}(\tilde{A}))(\omega)} P(\omega)\left(\mathbb{E}(\tilde{B})\right)(\omega) + \sum_{(\mathbb{E}(\tilde{B}))(\omega)=(\mathbb{E}(\tilde{A}))(\omega)} P(\omega)\left(\mathbb{E}(\tilde{B})\right)(\omega) \\
&= \textstyle\sum_{(\mathbb{E}(\tilde{B}))(\omega)\neq(\mathbb{E}(\tilde{A}))(\omega)} P(\omega)\left(\mathbb{E}(\tilde{B}\setminus\tilde{A})\right)(\omega) + \sum_{\omega\in \mathrm{supp}\tilde{A}_{\mathbb{E}(\tilde{B})}} P(\omega)\left(\mathbb{E}(\tilde{B})\right)(\omega) \\
&= \textstyle\sum_{(\mathbb{E}(\tilde{B}))(\omega)\neq(\mathbb{E}(\tilde{A}))(\omega)} P(\omega)\left(\mathbb{E}(\tilde{B}\setminus\tilde{A})\right)(\omega) + \sum_{(\mathbb{E}(\tilde{B}))(\omega)=(\mathbb{E}(\tilde{A}))(\omega)} P(\omega)\left(\mathbb{E}(\tilde{B}\setminus\tilde{A})\right)(\omega) \\
&\quad + \textstyle\sum_{\omega\in \mathrm{supp}\tilde{A}_{\mathbb{E}(\tilde{B})}} P(\omega)\left(\mathbb{E}(\tilde{A}_{\mathbb{E}(\tilde{B})})\right)(\omega) \\
&= \textstyle\sum_{\omega\in \mathrm{supp}(\tilde{B}\setminus\tilde{A})} P(\omega)\left(\mathbb{E}(\tilde{B}\setminus\tilde{A})\right)(\omega) + \tilde{P}_G(\tilde{A}_{\mathbb{E}(\tilde{B})}) \\
&= \tilde{P}_G(\tilde{B}\setminus\tilde{A}) + \tilde{P}_G(\tilde{A}_{\mathbb{E}(\tilde{B})})
\end{aligned}$$

so $\tilde{P}_G(\tilde{B}\setminus\tilde{A}) = \tilde{P}_G(\tilde{B}) - \tilde{P}_G(\tilde{A}_{\mathbb{E}(\tilde{B})})$;

(2) From $\tilde{A}_{\mathbb{E}(\tilde{\Omega})} = \{(\omega,r) \in \tilde{A} : (\mathbb{E}(\tilde{\Omega}))(\omega) = (\mathbb{E}(\tilde{A}))(\omega),\ r \in (0,1]\} = \{(\omega,r) \in \tilde{A} : (\mathbb{E}(\tilde{A}))(\omega) = 1,\ r \in (0,1]\} = \tilde{A}_{[1]}$, by Conclusion (1), we have that $\tilde{P}_G(\tilde{A}^C) = \tilde{P}_G(\tilde{\Omega}\setminus\tilde{A}) = \tilde{P}_G(\tilde{\Omega}) - \tilde{P}_G(\tilde{A}_{\mathbb{E}(\tilde{\Omega})}) = 1 - \tilde{P}_G(\tilde{A}_{[1]}) = 1 - \sum_{(\mathbb{E}(\tilde{A}))(\omega)=1} P(\omega)$. □

Proposition 11. *Let* $(\Omega, \sigma(\Omega), P)$ *be a probability space, and* $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P}_{[r]})$ *be the generated finitely additive r-probability fuzzy space (*$r \in (0,1]$*). Then*

(1) $\tilde{P}_{[r]}(\tilde{B}\setminus\tilde{A}) = \tilde{P}_{[r]}(\tilde{B}) - \tilde{P}_{[r]}(\tilde{A}) + P([\tilde{B}\setminus\tilde{A}]^r \cap [\tilde{A}]^r)$ *for any* $\tilde{A}, \tilde{B} \in 2^{\tilde{\Omega}}$ *with* $\tilde{A} \subset \tilde{B}$;

(2) $\tilde{P}_{[r]}(\tilde{A}^C) = 1 - \tilde{P}_{[r]}(\tilde{A}) + P([\tilde{A}^C]^r \cap [\tilde{A}]^r)$ *for any* $\tilde{A} \in 2^{\tilde{\Omega}}$, *where* $\tilde{A}^C = \tilde{\Omega}\setminus\tilde{A}$.

Proof. (1) By the definition of $\tilde{P}_{[r]}$ and Corollary 2, we have that

$$\begin{aligned}
\tilde{P}_{[r]}(\tilde{B}) &= P([\tilde{B}]^r) \\
&= P([(\tilde{B}\setminus\tilde{A}) \cup \tilde{A})]^r) \\
&= P([\mathbb{E}\left((\tilde{B}\setminus\tilde{A}) \cup \tilde{A}\right)]^r) \\
&= P([\mathbb{E}(\tilde{B}\setminus\tilde{A}) \cup \mathbb{E}(\tilde{A})]^r) \\
&= P([\mathbb{E}(\tilde{B}\setminus\tilde{A})]^r \cup [\mathbb{E}(\tilde{A})]^r) \\
&= P([\mathbb{E}(\tilde{B}\setminus\tilde{A})]^r) + P([\mathbb{E}(\tilde{A})]^r) - P([\mathbb{E}(\tilde{B}\setminus\tilde{A})]^r \cap [\mathbb{E}(\tilde{A})]^r) \\
&= \tilde{P}([(\tilde{B}\setminus\tilde{A})]^r) + \tilde{P}([\tilde{A})]^r) - P([\mathbb{E}(\tilde{B}\setminus\tilde{A})]^r \cap [\mathbb{E}(\tilde{A})]^r) \\
&= P_{[r]}((\tilde{B}\setminus\tilde{A})) + P_{[r]}(\tilde{A})) - P([\tilde{B}\setminus\tilde{A}]^r \cap [\tilde{A}]^r)
\end{aligned}$$

so $\tilde{P}_{[r]}(\tilde{B}\setminus\tilde{A}) = \tilde{P}_{[r]}(\tilde{B}) - \tilde{P}_{[r]}(\tilde{A}) + P([\tilde{B}\setminus\tilde{A}]^r \cap [\tilde{A}]^r)$;

(2) By Conclusion (1), we can directly obtain Conclusion (2). □

Proposition 12. *Let* $(\Omega, \sigma(\Omega), P)$ *be a probability space, and* $(\tilde{\Omega}, \sigma(\tilde{\Omega}), \tilde{P}_{(r)})$ *be the generated strong-r-probability fuzzy space (*$r \in (0,1)$*). Then*

(1) $\tilde{P}_{(r)}(\tilde{B}\setminus\tilde{A}) = \tilde{P}_{(r)}(\tilde{B}) - \tilde{P}_{(r)}(\tilde{A}) + P((\tilde{B}\setminus\tilde{A})^r \cap (\tilde{A})^r)$ *for any* $\tilde{A}, \tilde{B} \in 2^{\tilde{\Omega}}$ *with* $\tilde{A} \subset \tilde{B}$;

(2) $\tilde{P}_{(r)}(\tilde{A}^C) = 1 - \tilde{P}_{(r)}(\tilde{A}) + P((\tilde{A}^C)^r \cap (\tilde{A})^r)$ *for any* $\tilde{A} \in 2^{\tilde{\Omega}}$, *where* $\tilde{A}^C = \tilde{\Omega}\setminus\tilde{A}$.

Proof. The proof of the proposition is similar to the proof of Proposition 11, so we omit it. □

5. Conclusions

In this paper, we studied the problems of basic fuzzy event space and of probability fuzzy space, the obtained results provide the basis for the future systematic establishment of the theory of "random fuzzy sets" and "random fuzzy numbers" with strong usability in the future. Firstly, we proposed the concepts of basic fuzzy event (Definition 1), and from the concept, we defined basic fuzzy event space (Definition 1), fuzzy events (Definition 2). In order to establish a reasonable probability distribution of fuzzy events (i.e., give reasonable definition of probability fuzzy space), we also defined the concepts of canonical mapping of fuzzy events (Definition 3) and simple fuzzy events, and obtained some results (Propositions 1–8) that will be used in studying probability distribution of fuzzy events. Then, we introduced the definitions of the probability function about fuzzy events and probability fuzzy space (Definition 6) and obtained some properties (Proposition 8) of the defined probability function. Then, we gave the model $\tilde{P}_{(r)}(\tilde{A}) = P((\tilde{A})^r)$ (Theorem 1), model $\tilde{P}_{[r]}(\tilde{A}) = P([\tilde{A}]^r)$ (Theorem 2) and model $\tilde{P}(\tilde{A}) = \sum_{\omega \in \mathrm{supp}\tilde{A}} P(\omega) \left(\mathbb{E}(\tilde{A})\right)(\omega)$ (Definition 7 and Theorem 4) for probability distribution of probability fuzzy space based on a known probability space, obtained some properties (Properties 2–4) of these probability distribution of probability fuzzy space, and gave some examples (Examples 4 and 5) to show the usability of the proposed models of probability distribution.

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