mathematics

Article

Generalized Geodesic Convexity on Riemannian Manifolds

Izhar Ahmad [1], Meraj Ali Khan [2,*] and Amira A. Ishan [3]

1 Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia; drizhar@kfupm.edu.sa
2 Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia
3 Department of Mathematics, Taif University, Taif 21944, Saudi Arabia; amiraishan@hotmail.com
* Correspondence: m_khan@ut.edu.sa or meraj79@gmail.com; Tel.: +966-582-903-874

Received: 22 April 2019; Accepted: 11 June 2019; Published: 16 June 2019

Abstract: We introduce log-preinvex and log-invex functions on a Riemannian manifold. Some properties and relationships of these functions are discussed. A characterization for the existence of a global minimum point of a mathematical programming problem is presented. Moreover, a mean value inequality under geodesic log-preinvexity is extended to Cartan-Hadamard manifolds.

Keywords: geodesic log-invex function; geodesic log-preinvex function; global minimum; mean value inequality; Riemannian manifolds

1. Introduction

In Mathematical Sciences, convexity plays very crucial role and contributes a fundamental character in optimization theory, engineering, economics, management science, variational inequalities and Riemannian manifolds etc. However, convexity fails to render accurate results in real world mathematical and economic models. For that reason, many authors have presented the various concepts of generalized convexity. In 1981, Hanson [1] gave a generalization of convex function, which was later known as an invex function. Furthermore, Ben-Israel and Mond [2] introduced preinvex function. The characterizations of preinvex functions and their applications in optimization theory have been studied in [3,4]. Noor [5,6] discussed relations between an equilibrium problem and variational inequalities under these functions. Hermite-Hadamard inequality based on log-preinvex function was presented by Noor [7,8]. Many papers have appeared in the literature under the concept of preinvexity (see, [9–15]).

Some results related to nonlinear analysis and optimization theory have been enhanced on Riemannian manifolds from Euclidean space. The geodesic convexity was introduced by Rapcsak [16] and Udriste [17]. Later on, the concept of invexity on a Riemannian manifold was proposed by Pini [18] and its generalizations were explored in [19]. In [12], Barani and Pouryayevali have presented the geodesic invex set, geodesic η-preinvex and geodesic η-invex functions on a Riemannian manifold. The geodesic α-preinvex function was introduced by Agrawal et al. [9]. In [15], Zhou and Huang presented the B-invex function on a Riemannian manifold. The geodesic geodesic E-convexity was proposed on a Riemannian manifold in [20]. These results were further generalized in [21–23]. Recently, Ahmad et al. [24] have discussed the geodesic sub-b-s convex functions and studied characterizations of these functions.

Analyzing the discussion of Pini [18], Barani and Pouryayevali [12], and Noor [7,8], we attempt an effort to introduce the geodesic log-preinvex and geodesic log-invex functions on Riemannian manifolds.

These functions are a generalization of preinvexity defined in [7,8,12]. The contents of this paper are divided as follows: In Section 2, we recall preliminaries and definitions, which will be used to demonstrate the work. In Section 3, we present a new class of functions namely, geodesic log-preinvex and geodesic log-invex functions. Some properties and relations between geodesic log-preinvex and geodesic log-invex functions are studied on a Riemannian manifold in Section 4. In Section 5, proximal subdifferential and lower semi-continous log pre-invex function are used to observe that a local minimum point for a mathematical optimization problem is also a global minimum point on a Riemannian manifold. Finally, we present a mean value inequality on a Cartan-Hadamard manifold in Section 6 and conclude the paper in Section 7.

2. Notations and Preliminaries

We recall some basic definitions and concepts on Riemannian manifolds. For more details, authors may consult [25]. Let $\bar{M}$ be a n-dimensional Riemannian manifold and let $T_u\bar{M}$ be the tangent space of $\bar{M}$. The Riemannian metric is denoted by $\langle .,. \rangle$ on the $T_u\bar{M}$ and the associated norm is denoted by $\|.\|_u$. Suppose $T\bar{M} = \bigcup_{u\in\bar{M}} T_u\bar{M}$ is tangent bundle of $\bar{M}$. If u and v be two points on $\bar{M}$ and $\gamma : [a,b] \to \bar{M}$ is a piecewise smooth curve joining $\gamma(a) = u$ to $\gamma(b) = v$, its length $L(\gamma)$ is given by

$$L(\gamma) = \int_a^b \left\| \gamma'(s) \right\| ds.$$

The Riemannian distance $d(u,v)$ between the points u and v defined as:

$$d(u,v) = \inf \{L(\gamma) : \gamma \text{ is a piecewise smooth curve connecting the points } u \text{ and } v\}.$$

A metric d on $\bar{M}$, which is similar topology as the one $\bar{M}$ naturally has as a manifold. We define the open ball for this metric centered at the point v with radius $r > 0$ as

$$B(v,r) = \{u \in \bar{M} : d(u,v) < r\}.$$

On a Riemannian manifold $\bar{M}$ there exists a unique affine connection which is without torsion and metric. This affine connection is called the Levi-Civita connection which is symbolized by $\nabla_X Y$ for any vector fields X, Y on $\bar{M}$. A geodesic is a smooth path γ, that is γ satisfies the equation $\nabla_\gamma \gamma' = 0$ when tangent of geodesic is parallel with the path γ. Any path γ connecting u and v in $\bar{M}$ such that $L(\gamma) = d(u,v)$ is a geodesic and it is known as minimizing geodesic. Moreover, the exponential map at u, $exp_u : T_u\bar{M} \to \bar{M}$ is well defined on $T_u\bar{M}$. A simply connected complete Riemannian manifold with non-positive sectional curvature is called Hadamard manifold.

Barani and Pouryayevali [12] have presented the following definitions on a Riemannian manifold $\bar{M}$.

Definition 1. *A nonempty subset U of $\bar{M}$ is said to be a geodesic invex set with respect to $\eta(u,v) : \bar{M} \times \bar{M} \to T\bar{M}$, if for every $u, v \in U$, there exists a unique geodesic $\gamma_{u,v} : [0,1] \to \bar{M}$ such that*

$$\gamma_{u,v}(0) = v, \ \ \gamma'_{u,v}(0) = \eta(u,v), \ \ \gamma_{u,v}(s) \in U, \ \ \textit{for all } s \in [0,1].$$

Let $h : U \to \mathbb{R}$ be a real valued differentiable function.

Definition 2. *A function $h : U \to \mathbb{R}$ is said to be geodesic η-invex on U with respect to η, if*

$$h(u) - h(v) \geq dh_v(\eta(u,v)), \ \ \textit{for all } u, \ v \in U.$$

Definition 3. *A function $h : U \to \mathbb{R}$ is said to be geodesic preinvex on U, if for any $u, v \in U$*

$$h(\gamma_{u,v}(s)) \leq sh(u) + (1-s)h(v), \text{ for all } s \in [0,1],$$

where $\gamma_{u,v}$ is the particular geodesic defined in Definition 1. If the above inequality is strict, then h is said to be a strictly geodesic preinvex function.

Definition 4. *The function $\eta(u,v) : \bar{M} \times \bar{M} \to T\bar{M}$ satisfies the condition (C), if for every $u, v \in \bar{M}$ and for geodesic $\gamma : [0,1] \to \bar{M}$ satisfying $\gamma_{u,v}(0) = v$ and $\gamma'_{u,v}(0) = \eta(u,v)$, we have*

$$\begin{aligned} (C_1) \quad & P^0_{t,\gamma_{u,v}}[\eta(v, \gamma_{u,v(t)}] = -\,t\eta(u,v), \\ (C_2) \quad & P^0_{t,\gamma_{u,v}}[\eta(u, \gamma_{u,v(t)}] = (1-t)\eta(u,v), \end{aligned}$$

for each $t \in [0,1]$.

Throughout the subsequent sections, h is a positive real valued function.

3. Geodesic Log-Invex and Geodesic Log-Preinvex Functions

Now we introduce the following geodesic log-invex and geodesic log-preinvex functions on a Riemannian manifold $\bar{M}$.

Definition 5. *A function $h : U \to \mathbb{R}$ is said to be a geodesic log-invex function with respect to η, if*

$$dh_v(\eta(u,v)) \leq h(v)(\ln(h(u)) - \ln(h(v))), \text{ for all } u,\ v \in U.$$

The following definition of geodesic log-preinvex function on a Riemannian manifold is the generalization of log-preinvex function defined in [7,8].

Definition 6. *The function $h : U \to \mathbb{R}$ is said to be a geodesic log-preinvex with respect to η on U, if for any $u, v \in U$,*

$$h(\gamma_{u,v}(s)) \leq (h(v))^{1-s}(h(u))^s, \text{ for all } u,\ v \in U,\ s \in [0,1].$$

If the above inequality is a strict inequality, then h is called a strictly geodesic log-preinvex function.

Remark 1. *Every geodesic preinvex function is also a geodesic log-preinvex function, but the converse is not true as can be seen in the following example.*

Example 1. *Let $\bar{M} = \left\{e^{i\theta} : 0 < \theta < \frac{\pi}{2}\right\}$ and $h : \bar{M} \to \mathbb{R}$ be a function defined as $h(e^{i\theta}) = (30 + \cos\theta + \sin\theta)^2$ with $u, v \in \bar{M}$, $u = e^{i\alpha}$ and $v = e^{i\beta}$. Let $\gamma_{u,v}(s) = e^{i((1-s)\beta + s\alpha)}$.*

$$h(\gamma_{u,v}(s)) = (30 + \cos((1-s)\beta + s\alpha) + \sin((1-s)\beta + s\alpha))^2,$$

and $h(e^{i\alpha}) = (30 + \cos\alpha + \sin\alpha)^2$, $h(e^{i\beta}) = (30 + \cos\beta + \sin\beta)^2$.

The function h is a geodesic log-preinvex for all $u, v \in \bar{M}$ *and* $s \in [0,1]$*, can be seen below:*

$$\begin{aligned}
h(\gamma_{u,v}(s)) - \left((h(v))^{1-s}(h(u))^{s}\right) =& h(e^{i((1-s)\beta+s\alpha)}) - \left((h(e^{i\beta}))^{1-s}\ (h(e^{i\alpha}))^{s}\right) \\
&= (30+\cos((1-s)\beta+\alpha s)+\sin((1-s)\beta+\alpha s))^2 \\
&\quad - \left(((30+\cos\beta+\sin\beta)^2)^{(1-s)}\ ((30+\cos\alpha+\sin\alpha)^2)^{s}\right) \\
&= (30+\cos((1-s)\beta+\alpha s)+\sin((1-s)\beta+\alpha s))^2 \\
&\quad - \left((30+\cos\beta+\sin\beta)^{2(1-s)}\ (30+\cos\alpha+\sin\alpha)^{2s}\right) \leq 0.
\end{aligned}$$

However, the function h is not a geodesic preinvex function at $\alpha = \frac{\pi}{4}$*,* $\beta = \frac{\pi}{6}$ *and* $s = \frac{1}{2}$*, since we have*

$$\begin{aligned}
h(\gamma_{u,v}(s)) - sh(u) - (1-s)h(v) = &\left(30+\cos\left(\left(1-\frac{1}{2}\right)\frac{\pi}{6}+\frac{\pi}{4}\times\frac{1}{2}\right)+\sin\left(\left(1-\frac{1}{2}\right)\frac{\pi}{6}+\frac{\pi}{4}\times\frac{1}{2}\right)\right)^2 \\
&-\frac{1}{2}\left(30+\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\right)^2-\left(1-\frac{1}{2}\right)\left(30+\cos\frac{\pi}{6}+\sin\frac{\pi}{6}\right)^2 \\
=&0.75 \nleq 0.
\end{aligned}$$

4. Geodesic Log-Preinvexity and Differentiability

The following section deals with some properties and relationships between the geodesic log-preinvex function and the geodesic log-invex function. Let U be a geodesic invex set.

Theorem 1. *Let* $h : U \to \mathbb{R}$ *be a geodesic log-preinvex function with respect to* η *on* U*. Then, the level set* $U_\alpha = \{u | u \in U, h(u) \leq \alpha\}$ *is a geodesic invex set for each real number* $\alpha \in \mathbb{R}$*.*

Proof of Theorem 1. Assume that $u, v \in U_\alpha$ and $0 \leq s \leq 1$. It follows that $h(u) \leq \alpha$ and $h(v) \leq \alpha$. By the definition of log-preinvexity of h, we have

$$\begin{aligned}
&h(\gamma_{u,v}(s)) \leq (h(v))^{1-s}(h(u))^{s}, \\
&h(\gamma_{u,v}(s)) \leq \alpha^{1-s}\alpha^{s}, \\
&\qquad\qquad \text{or} \\
&h(\gamma_{u,v}(s)) \leq \alpha.
\end{aligned}$$

Therefore, $\gamma_{u,v}(s) \in U_\alpha$ for all $s \in [0,1]$. This completes the proof of the theorem. □

Theorem 2. *Let* $h : U \to \mathbb{R}$ *be a geodesic log-preinvex function with respect to* η *on* U*. Let* $\psi : I \to \mathbb{R}$ *be an increasing log-preinvex function such that range* $h \subset I$*. Then the composite function* $\psi\ o\ h$ *is a geodesic log-preinvex on* U*.*

Proof of Theorem 2. From the definition of geodesic log-preinvexity of h, we have

$$h(\gamma_{u,v}(s)) \leq (h(v))^{1-s}(h(u))^{s}, \quad \text{for all } u,\ v \in U \text{ and } s \in [0,1].$$

As ψ is an increasing log-preinvex function, then we obtain

$$\psi(h(\gamma_{u,v}(s))) \leq \psi((h(v))^{1-s}(h(u))^{s})$$
or
$$\psi(h(\gamma_{u,v}(s))) \leq h^{1-s}(\psi(v))^{1-s}\psi^{s}(h(u))^{s},$$

by using the definition of composite function, we have

$$(\psi \; o \; h)(\gamma_{u,v}(s)) \leq ((\psi \; o \; h)(v))^{1-s}((\psi \; o \; h)(u))^{s},$$

which implies that $\psi \; o \; h$ is a geodesic log-preinvex function on U. □

Theorem 3. *Let $h_j : U \to \mathbb{R}$, $j = 1, 2, \ldots, r$ be geodesic log-preinvex functions with respect to η on U. Then*

$$h = \sum_{j=1}^{r} \alpha_j h_j, \;\; \textit{for all } \alpha_j \in \mathbb{R}, \; \alpha_j \geq 0, \; j = 1, 2, \ldots, r$$

is a geodesic log-preinvex function on U.

Proof of Theorem 3. From the Definition 6, the proof is obvious. □

Theorem 4. *Let $h : U \to \mathbb{R}$ be a geodesic log-preinvex function on U. Then the function h is a geodesic log-invex on U.*

Proof of Theorem 4. Since U is a geodesic invex set with respect to η, then for all $u, v \in U$, there exists a particular geodesic such that $\gamma_{u,v}(0) = v$, $\gamma'_{u,v}(0) = \eta(u, v)$, $\gamma_{u,v}(s) \in U$, for all $s \in [0, 1]$. Using the differentiability of h at $v \in \bar{M}$, we get

$$dh_v(\eta(u, v)) = \lim_{s \to 0} \frac{1}{s}[h(\gamma_{u,v}(s)) - h(v)],$$

and so

$$h(v) + dh_v(\eta(u, v))s + O^2(s) = h(\gamma_{u,v}(s)).$$

where $O^2(s)$ denotes the higher order terms of the variable s. By the geodesic log-preinvexity of h for $s \in (0, 1)$, we have

$$h(v) + dh_v(\eta(u, v))s + O^2(s) \leq (h(v))^{1-s}(h(u))^{s},$$
or
$$ln\Big(h(v) + dh_v(\eta(u, v))s + O^2(s)\Big) \leq s \; \ln(h(u)) + (1 - s)\ln(h(v)).$$

On further simplification, we have

$$\ln\Big(h(v) + dh_v(\eta(u, v))s + O^2(s)\Big) - \ln(h(v)) \leq s \; \ln(h(u)) - s \; \ln(h(v)).$$

Using the properties of lograthimic function, it is easy to see that

$$\frac{\ln\left(1+\frac{dh_v(\eta(u,v))s}{h(v)}+\frac{O^2(s)}{h(v)}\right)}{s} \leq \ln(h(u)) - \ln(h(v)),$$

taking the limit $s \to 0$, we get

$$dh_v(\eta(u,v)) \leq h(v)(ln(h(u)) - \ln(h(v))).$$

Therefore, h is a geodesic log-invex function on U. □

Theorem 5. *Let $U \subseteq \bar{M}$ be a geodesic invex set with respect to $\eta : \bar{M} \times \bar{M} \to T\bar{M}$ and η satisfies the condition (C). Let $h : U \to R$ be a geodesic log-invex function on U. Then h is a geodesic log-preinvex function on U.*

Proof of Theorem 5. It is clear that for a geodesic invex set with respect to η for each $u, v \in U$, there exists a particular geodesic $\gamma_{u,v} : [0,1] \to \bar{M}$ such that $\gamma_{u,v}(0) = v$, $\gamma'_{u,v}(0) = \eta(u,v)$, $\gamma_{u,v}(s) \in U$, for all $s \in [0,1]$.

Fix $s \in [0,1]$ and set $\bar{u} = \gamma_{u,v}(s)$. By geodesic log-invexity of h on U, we have

$$\ln(h(u)) - \ln(h(\bar{u})) \geq \frac{1}{h(\bar{u})} dh_{\bar{u}}(\eta(u,\bar{u})), \tag{1}$$

$$\ln(h(v)) - \ln(h(\bar{u})) \geq \frac{1}{h(\bar{u})} dh_{\bar{u}}(\eta(v,\bar{u})). \tag{2}$$

Inequalities (1) and (2) are multiplied by s and $(1-s)$, respectively and adding them yields

$$s(\ln(h(u)) - \ln(h(\bar{u}))) + (1-s)(\ln(h(v)) - \ln(h(\bar{u}))) \geq \frac{1}{h(\bar{u})} dh_{\bar{u}}[s\eta(u,\bar{u}) + (1-s)\eta(v,\bar{u})]. \tag{3}$$

By condition (C), we obtain

$$s\eta(u,\bar{u}) + (1-s)\eta(v,\bar{u}) = s(1-s)P^s_{0,\gamma}[\eta(u,v)] - (1-s)sP^s_{0,\gamma}[\eta(u,v)] = 0.$$

The above equation together with inequality (3), yields

$$s\ln(h(u)) + (1-s)\ln(h(v)) \geq \ln(h(\bar{u})),$$

equivalently,

$$h(\bar{u}) \leq (h(v))^{1-s}(h(u))^s.$$

This implies that h is a geodesic log-preinvex function on U. □

5. Semi-Continuous Geodesic Log-Preinvexity

The following definition of a proximal sub-differential of a function defined on a Riemannian manifold will be used in the sequel.

Definition 7. *[26,27] Let $h : \bar{M} \to (-\infty, \infty]$ be a lower semi-continuous function. A vector $\zeta \in T_v\bar{M}$ is said to be a proximal sub-gradient of h at $v \in dom(h)$, if there exist positive numbers δ and σ such that*

$$h(u) \geq h(v) + < \zeta, exp_v^{-1} u >_v - \sigma d^2(u, v), \text{ for all } u \in B(v, \delta),$$

where $dom(h) = \{u \in \bar{M} : 0 < h(u) < \infty\}$. The set of all proximal sub-gradient of $v \in \bar{M}$ is denoted by $\partial_p h(v)$ and is called the proximal sub-differential of h at v.

To study the semicontinous geodesic log-preinvex function, we first show that any local minimum point for a mathematical programming problem (P) is also a global minimum point under geodesic log-preinvexity.

Theorem 6. *Let $U \subseteq \bar{M}$ be a geodesic invex set with respect to $\eta : \bar{M} \times \bar{M} \to T\bar{M}$ and $h : U \to \mathbb{R}$ be a geodesic log-preinvex function at $\bar{u} \in U$. If $\bar{u} \in U$ is a local minimum point of the problem*

$$\text{(P)} \quad \begin{array}{l} \text{Minimize } h(u) \\ \text{subject to } u \in U, \end{array}$$

then $\bar{u}$ is also a global minimum point of problem (P).

Proof of Theorem 6. Suppose that $\bar{u} \in U$ is a local minimum point of the problem (P). Then there exists a neighbourhood $N_\epsilon(\bar{u})$ such as

$$h(\bar{u}) \leq h(u), \text{ for all } u \in U \cap N_\epsilon(\bar{u}). \tag{4}$$

If $\bar{u}$ is not a global minimum point of the problem (P), then there exists a point $u^* \in U$ such that

$$h(u^*) < h(\bar{u}),$$

or

$$\ln(h(u^*)) < \ln(h(\bar{u})). \tag{5}$$

As U is a geodesic invex set with respect to η, then there exists a particular geodesic γ so that $\gamma(0) = \bar{u}, \quad \gamma'(0) = \eta(u^*, \bar{u}), \quad \gamma(s) \in U$ and for all $s \in [0, 1]$.

Although, we set $\epsilon > 0$ such as $d(\gamma(s), \bar{u})\epsilon$, then $\gamma(s) \in N_\epsilon(\bar{u})$. The geodesic log-preinvexity of h gives

$$h(\gamma(s)) \leq (h(u^*))^s (h(\bar{u}))^{1-s},$$

equivalently, we get

$$\ln(h(\gamma(s))) \leq s \ln(h(u^*)) + (1 - s) \ln(h(\bar{u})).$$

The above inequality with inequality (5), implies

$$\ln(h(\gamma(s))) < \ln(h(\bar{u})),$$

or

$$h(\gamma(s)) < h(\bar{u}), \text{ for all } s \in (0, 1).$$

Therefore, for each $\gamma(s) \in U \cap N_\epsilon(\bar{u})$, $h(\gamma(s)) < h(\bar{u})$, which contradicts inequality (4). Hence $\bar{u}$ is a global minimum point of the problem (P). □

Theorem 7. *Let $U \subseteq \bar{M}$ be a geodesic invex set with respect to $\eta : \bar{M} \times \bar{M} \to T\bar{M}$ with $\eta(u, v) \neq 0$, for $u \neq v$. Suppose that $h : U \to (-\infty, \infty]$ is lower semi-continuous geodesic log-preinvex function and $v \in dom(h)$, $\zeta \in \partial_p h(v)$. Then there exists $\delta > 0$ such that*

$$\ln(h(u)) - \ln(h(v)) \geq \frac{\langle \zeta, \eta(u, v) \rangle_v}{h(v)}, \text{ for all } u \in U \cap B(v, \delta).$$

Proof of Theorem 7. From the definition of $\partial_p h(v)$, there are positive numbers δ and σ such that

$$h(u) \geq h(v) + < \zeta, exp_v^{-1} u >_v - \sigma d^2(u, v), \text{ for all } u \in B(v, \delta). \tag{6}$$

Now, fix $u \in U \cap B(v, \delta)$. Since U is a geodesic invex set with respect to η, then there exists a particular geodesic $\gamma_{u,v} : [0, 1] \to \bar{M}$ such as $\gamma_{u,v}(0) = v$, $\gamma'_{u,v}(0) = \eta(u, v)$, $\gamma_{u,v}(s) \in U$, for all $s \in [0, 1]$.

Since $\bar{M}$ is a Hadamard manifold, then $\gamma_{u,v}(s) = exp_v(s\eta(u, v))$ for every $s \in [0, 1]$ (see [25], p. 253). If we take $s_0 = \frac{\delta}{\|\eta(u,v)\|_v}$, then $exp_v(s\eta(u, v)) \in U \cap B(v, \delta)$, for all $s \in (0, s_0)$.

By the geodesic log-preinvexity of h, we have

$$h(exp_v(s(\eta(u, v)))) \leq (h(u))^s (h(v))^{1-s},$$

or

$$\ln(h(exp_v(s(\eta(u, v))))) \leq (1 - s) \ln(h(v)) + sln(h(u)). \tag{7}$$

Inequality (6) for each $s \in (0, s_0)$ can be written as

$$h(exp_v(s(\eta(u, v)))) \geq h(v) + < \zeta, exp_v^{-1} exp_v(s\eta(u, v)) >_v - \sigma d^2(exp_v(s\eta(u, v), v))$$

$$= h(v) + \langle \zeta, s\eta(u, v) \rangle_v - \sigma d^2(exp_v(s\eta(u, v), v)).$$

By the intrinsic property of Cartan-Hadamard manifold $\bar{M}$, we have

$$d^2(exp_v(s\eta(u, v), v)) = \|s\eta(u, v)\|_v^2 = s^2 \|\eta(u, v)\|_v^2.$$

for every $s \in (0, s_0)$.

Thus, we have

$$h(exp_v(s(\eta(u, v)))) \geq h(v) + \langle \zeta, s\eta(u, v) \rangle_v - \sigma s^2 \|\eta(u, v)\|_v^2,$$

or

$$\ln(h(exp_v(s(\eta(u, v))))) \geq \ln[h(v) + \langle \zeta, s\eta(u, v) \rangle_v - \sigma s^2 \|\eta(u, v)\|_v^2]. \tag{8}$$

Inequalities (7) and (8), yield

$$(1 - s) \ln(h(v)) + s \ln(h(u)) \geq \ln[h(v) + \langle \zeta, s\eta(u, v) \rangle_v - \sigma s^2 \|\eta(u, v)\|_v^2],$$

or,

$$\ln(h(u)) - \ln(h(v)) \geq \frac{1}{s} \ln \left[1 + \frac{\langle \zeta, s\eta(u,v)\rangle_v - \sigma s^2 \|\eta(u,v)\|_v^2}{h(v)}\right].$$

Now, taking limit $s \to 0$, we obtain

$$\ln(h(u)) - \ln(h(v)) \geq \frac{\langle \zeta, \eta(u,v)\rangle_v}{h(v)}.$$

Hence the proof of the theorem is complete. □

6. Mean Value Inequality

Antczak [28] proved the mean value inequality under invexity. Barani and Pouryayevali [12] extended this inequality under geodesic invexity to a Cartan-Hadamard manifold. We now generalize mean value inequality under geodesic log-preinvexity on a Cartan-Hadamard manifold.

Definition 8. *[12] Let $U \subseteq \bar{M}$ be a geodesic invex set with respect to $\eta : \bar{M} \times \bar{M} \to T\bar{M}$, and u and x be two arbitrary points of U. Let $\gamma : [0,1] \to \bar{M}$ be the particular geodesic such as $\gamma(0) = x,\ \gamma'(0) = \eta(u,x),\ \ \gamma(s) \in U$, for all $s \in [0,1]$.*

We denote by P_{xy} the closed η-path joining the points x and $y = \gamma(1)$, defined as

$$P_{xy} = \{v : v = \gamma(s), s \in [0,1]\}.$$

Let define P^0_{xy} an open η-path joining the points x and y as

$$P^0_{xy} = \{v : v = \gamma(s), s \in (0,1)\}.$$

If $x = y$ we set $P^0_{xy} = \phi$.

Theorem 8. (Mean Value Inequality) *Let $\bar{M}$ be a Cartan-Hadamard manifold and $U \subseteq \bar{M}$ be a geodesic invex set with respect to $\eta : \bar{M} \times \bar{M} \to T\bar{M}$ such that $\eta(a,b) \neq 0$ for each $a, b \in U$, $a \neq b$. If $\gamma_{b,a}(s) = exp_a(s\eta(b,a))$ for all $a, b \in U,\ s \in [0,1]$ and $c = \gamma_{b,a}(1)$, then a necessary and sufficient condition for a function $h : U \to \mathbb{R}$ to be a geodesic log-preinvex is that the inequality*

$$\ln(h(u)) \leq \ln(h(a)) + \frac{\ln(h(b)) - \ln(h(a))}{\langle \eta(b,a), \eta(b,a)\rangle_a} \langle exp_a^{-1} u, \eta(b,a)\rangle_a, \tag{9}$$

holds, for all $u \in P_{ca}$.

Proof of Theorem 8. **Necessity:** Let $h : U \to \mathbb{R}$ be a geodesic log-preinvex function and let $a, b \in U$ and $u \in P_{ca}$. If $u = a$ or $u = c$, then (9) is true trivially. If $u \in P^0_{ca}$, then $u = exp_a(s\eta(b,a))$, for some $s \in (0,1)$. From the geodesic invexity of U, we have $u \in U$ and

$$s = \frac{\langle exp_a^{-1} u, \eta(b,a)\rangle_a}{\langle \eta(b,a), \eta(b,a)\rangle_a}.$$

From the geodesic log-preinvexity of h on U, we get

$$h(u) = h(exp_a(s\eta(b,a)) \leq (h(a))^{1-s}(h(b))^s,$$

or

$$\ln(h(u)) \leq s\ \ln(h(b)) + (1-s)\ln(h(a)) = \ln(\mathrm{h}(\mathrm{a})) + \mathrm{s}(\ln(\mathrm{h}(\mathrm{b})) - \ln(\mathrm{h}(\mathrm{a}))).$$

Using the value of s, we obtain

$$\ln(h(u)) \leq \ln(h(a)) + \frac{\ln(h(b)) - \ln(h(a))}{\langle \eta(b,a), \eta(b,a)\rangle_a} \langle exp_a^{-1}u, \eta(b,a)\rangle_a.$$

Sufficiency: Suppose that inequality (9) is true. Let $a, b \in U$ and $u = exp_a(s\eta(b,a))$, for some $s \in [0,1]$. Then $u \in U$, we have

$$h(u) = h(exp_a(s\eta(b,a)).$$

Now from inequality (9), we get

$$\ln(h(u)) \leq \ln(h(a)) + \frac{\ln(h(b)) - \ln(h(a))}{\langle \eta(b,a), \eta(b,a)\rangle_a} \langle exp_a^{-1}u, \eta(b,a)\rangle_a,$$

$$\ln(h(u)) \leq s\ \ln(h(b)) + (1-s)ln(h(a)),$$

that is,

$$h(exp_a(s\eta(b,a)) \leq (h(b))^s(h(a))^{1-s}.$$

Hence, h is a geodesic log-preinvex function on U. □

7. Conclusions

The present paper is based on the concept of generalized geodesic convexity on a Riemannian manifold. We have introduced the geodesic log-invex function and geodesic log-preinvex function. Some properties and relationships between these functions are studied. Using the smoothness condition on a geodesic log-preinvex function with lower semi-continuity, we obtained a result of the existence condition for a global minimum of a mathematical programming problem. Moreover, we prove the mean value inequality on Cartan-Hadamard manifold. As a future work, we can use the introduced functions to establish the relations between variational inequalities and a vector optimization problem on the lines of Jayswal et al. [29].

Author Contributions: All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Funding: This research was funded by the Deanship of Scientific Research, University of Tabuk, KSA.

Acknowledgments: We thank three anonymous referees very much for many valuable remarks and suggestions that definitely improved the final version of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hanson, M.A. On sufficiency of the Kuhn-Tucker conditions. *J. Math. Anal. Appl.* **1981**, *80*, 545–550. [CrossRef]
2. Ben-Israel, A.; Mond, B. What is the invexity? *J. Aust. Math. Soc.* **1986**, *8*, 1–9. [CrossRef]
3. Jeyakumar, V. Strong and weak invexity in mathematical programming. *Math. Oper. Res.* **1985**, *55*, 109–125.
4. Weir, T.; Mond, B. Preinvex functions in multiobjective optimization. *J. Math. Anal. Appl.* **1988**, *136*, 29–38. [CrossRef]
5. Noor, M.A. Variational-like inequality. *Optimization* **1994**, *30*, 323–330. [CrossRef]
6. Noor, M.A. Invex equilibrium problems. *J. Math. Anal. Appl.* **2005**, *302*, 463–475. [CrossRef]
7. Noor, M.A. On Hadamard integral inequalities involving two log-preinvex functions. *J. Inequalities Pure Appl. Math.* **2007**, *8*, 1–6.
8. Noor, M.A. Hadamard integral inequalities for product of two preinvex functions. *Nonlinear Anal. Forum* **2009**, *14*, 167–173.
9. Agarwal, R.P.; Ahmad, I.; Iqbal, A.; Ali, S. Generalized invex sets and preinvex functions on Riemannian manifolds. *Taiwanese J. Math.* **2012**, *16*, 1719–1732. [CrossRef]
10. Ahmad, I.; Iqbal, A.; Ali, S. On properties of geodesic $\eta-$preinvex functions. *Adv. Oper. Res.* **2009**, *2009*, 381831. [CrossRef]
11. Antczak, T. r-preinvexity and r-invexity in mathematical programming. *Comput. Math. Appl.* **2005**, *50*, 551–566. [CrossRef]
12. Barani, A.; Pouryayevali, M.R. Invex sets and preinvex functions on Riemannian manifolds. *J. Math. Anal. Appl.* **2007**, *328*, 767–779. [CrossRef]
13. Kilicman, A.; Saleh, W. Generalized preinvex functions and their applications. *Symmetry* **2018**, *10*, 493. [CrossRef]
14. Noor, M.A. On generalized preinvex functions and monotonicities. *J. Inequalities Pure Appl. Math.* **2004**, *110*, 1–9.
15. Zhou, L.W.; Huang, N.J. Roughly geodesic B-invex and optimization problem on Hadamard manifolds. *Taiwan. J. Math.* **2013**, *17*, 833–855. [CrossRef]
16. Rapcsak, T. *Smooth Nonlinear Optimization in R^n*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1997.
17. Udriste, C. *Convex Functions and Optimization Methods on Riemannian Manifolds*; Mathematics and Its Applications; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1994.
18. Pini, R. Convexity along curves and invexity. *Optimization* **1994**, *29*, 301–309. [CrossRef]
19. Mititelu, S. Generalized invexity and vector optimization on differential manifolds. *Differ. Geom. Dyn. Syst.* **2001**, *3*, 21–31.
20. Iqbal, A.; Ali, S.; Ahmad, I. On geodesic E-convex sets, geodesic E-convex functions and E-epigraphs. *J. Optim. Theory Appl.* **2012**, *55*, 239–251. [CrossRef]
21. Iqbal, A.; Ahmad, I.; Ali, S. Some properties of geodesic semi-E-convex functions. *Nonlinear Anal. Theory Methods Appl.* **2011**, *74*, 6805–6813. [CrossRef]
22. Kilicman, A.; Saleh, W. On geodesic strongly E-convex sets and geodesic strongly E-convex functions. *J. Inequalities Appl.* **2015**, *2015*, 297. [CrossRef]
23. Kilicman, A.; Saleh, W. On properties of geodesic semilocal E-preinvex functions. *J. Inequalities Appl.* **2018**, *2018*, 353. [CrossRef] [PubMed]
24. Ahmad, I.; Jayswal, A.; Kumari, B. Characterizations of geodesic sub-b-s-convex functions on Riemannian manifolds. *J. Nonlinear Sci. Appl.* **2018**, *11*, 189–197. [CrossRef]
25. Lang, S. *Fundamental of Differential Geometry Graduate Texts in Mathematics*; Springer: New York, NY, USA, 1999.
26. Azagra, D.; Ferrara, J. Proximal calculus on Riemannian manifolds with applications to fixed theory. *Mediterr. J. Math.* **2005**, *2*, 437–450. [CrossRef]
27. Ferreira, O.P. Proximal subgradient and characterization of Lipschitz function on Riemannian manifolds. *J. Math. Anal. Appl.* **2006**, *313*, 587–597. [CrossRef]
28. Antczak, T. Mean value in invexity analysis. *Nonlinear Anal.* **2005**, *60*, 471–484. [CrossRef]

29. Jayswal, A.; Ahmad, I.; Kumari, B. Vector variational inequalities on Hadamard manifolds involving strongly geodesic convex functions. *Oper. Res. Lett.* **2019**, *47*, 110–114 [CrossRef]