mathematics

Article

Generalizations of Several Inequalities Related to Multivariate Geometric Means

Bo-Yan Xi [1], **Ying Wu** [1], **Huan-Nan Shi** [2] **and Feng Qi** [3,4,*]

1. College of Mathematics, Inner Mongolia University for Nationalities, Tongliao 028043, Inner Mongolia, China; baoyintu78@imun.edu.cn (B.-Y. Xi); nmwuying@163.com (Y. Wu)
2. Department of Electronic Information, Teacher's College, Beijing Union University, Beijing 100011, China; sfthuannan@buu.edu.cn
3. Institute of Mathematics, Henan Polytechnic University, Jiaozuo 454010, Henan, China
4. School of Mathematical Sciences, Tianjin Polytechnic University, Tianjin 300387, China

* Correspondence: qifeng618@gmail.com

Received: 16 May 2019; Accepted: 11 June 2019; Published: 17 June 2019

Abstract: In the paper, by some methods in the theory of majorization, the authors generalize several inequalities related to multivariate geometric means.

Keywords: Schur convex function; Schur geometrically-convex function; elementary symmetric function; inequality; multivariate geometric mean; majorization

MSC: Primary 26E60; Secondary 26D15; 26A51

1. Introduction

Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n = (0, \infty)^n$ and $n \geq 2$. Then, the arithmetic and geometric means of n positive numbers $x_1, x_2, \dots, x_n$ are defined by

$$A(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad G(\boldsymbol{x}) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}.$$

In [1] (p. 208, 3.2.34), it was stated that

$$\prod_{i=1}^{n}(x_i + 1) \geq [G(\boldsymbol{x}) + 1]^n. \tag{1}$$

In the paper [2], Wang and Chen established that the inequality

$$\prod_{i=1}^{n}[(x_i + 1)^p - 1] \geq [(G(\boldsymbol{x}) + 1)^p - 1]^n \tag{2}$$

is valid for $p > 1$. In the paper [3], Wang extended Inequality (2) as follows:

1. if $p \geq 1$, then Inequality (2) is valid;
2. if $0 < p \leq 1$, then Inequality (2) is reversed;
3. if $p > 0$, then
$$\prod_{i=1}^{n}[(x_i + 1)^p + 1] \geq [(G(\boldsymbol{x}) + 1)^p + 1]^n; \tag{3}$$
4. if $p < 0$ and $0 < x_i < \frac{1}{|p|}$ for $1 \leq i \leq n$, then Inequality (3) is reversed.

In [4] (p. 11), it was proven that

$$\sqrt[n]{\prod_{k=1}^{n}(a_k+b_k)} \ge \sqrt[n]{\prod_{k=1}^{n}a_k} + \sqrt[n]{\prod_{k=1}^{n}b_k}, \tag{4}$$

where $a_k, b_k \ge 0$. When $a_k = x_k$ and $b_k = 1$, Inequality (4) becomes (1). Inequality (4) is the Minkowski inequality of the product form.

We observe that the inequalities in (1)–(4) can be rearranged as

$$G(\boldsymbol{x}+\mathbf{1}) \ge G(\boldsymbol{x})+1, \quad G((\boldsymbol{x}+\mathbf{1})^p \pm \mathbf{1}) \ge [G(\boldsymbol{x})+1]^p \pm 1 \tag{5}$$

and

$$G(\boldsymbol{a}+\boldsymbol{b}) > G(\boldsymbol{a})+G(\boldsymbol{b}), \tag{6}$$

where

$$\boldsymbol{x}+\mathbf{1} = (x_1+1, x_2+1, \dots, x_n+1)$$

and

$$(\boldsymbol{x}+\mathbf{1})^p \pm \mathbf{1} = ((x_1+1)^p \pm 1, (x_2+1)^p \pm 1, \dots, (x_n+1)^p \pm 1).$$

Inequality (6) reveals that the geometric mean $G(\boldsymbol{x})$ is sub-additive. For information about the sub-additivity, please refer to [5–12] and the closely related references therein. The sub-additive property of the geometric mean $G(\boldsymbol{x})$ can also be derived from the property that the geometric mean $G(\boldsymbol{x})$ is a Bernstein function; see [13–19] and the closely related references therein.

In this paper, by some methods in the theory of majorization, we will generalize the above inequalities in (1)–(3), (5), and (6).

2. Definitions and Lemmas

Now, we recall some definitions and lemmas.

It is well known that a function $f(x_1, x_2, \dots, x_n)$ of n variables is said to be symmetric if its value is unchanged for any permutation of its n variables $x_1, x_2, \dots, x_n$.

Definition 1 ([20,21])**.** *Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.*

1. *If*

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]} \quad \text{and} \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}$$

for $1 \le k \le n-1$, then $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbol $\boldsymbol{x} \prec \boldsymbol{y}$), where $x_{[1]} \ge x_{[2]} \ge \cdots \ge x_{[n]}$ and $y_{[1]} \ge y_{[2]} \ge \cdots \ge y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in descending order.

2. *For $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0,1]$, if*

$$\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} = (\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2, \dots, \lambda x_n + (1-\lambda)y_n) \in \Omega,$$

then $\Omega \subseteq \mathbb{R}^n$ is said to be a convex set.

Definition 2 ([20,21])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a symmetric and convex set.*

1. *If $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$, then we say that the function $\varphi : \Omega \to \mathbb{R}$ is Schur convex on Ω.*
2. *If $-\varphi$ is a Schur convex function on Ω, then we say that φ is Schur concave on Ω.*

Lemma 1 ([20,21])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a symmetric and convex set with nonempty interior Ω°, and let $\varphi : \Omega \to \mathbb{R}$ be continuous on Ω and continuously differentiable on Ω°. If and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \geq 0, \quad \boldsymbol{x} \in \Omega^\circ,$$

the function φ is Schur convex on Ω.

Definition 3 ([22] (pp. 64 and 107))**.** *Let $\boldsymbol{x}, \boldsymbol{y} \in \Omega \subseteq \mathbb{R}^n_+$.*

1. *If $\left(x_1^\lambda y_1^{1-\lambda}, x_2^\lambda y_2^{1-\lambda}, \dots, x_n^\lambda y_n^{1-\lambda}\right) \in \Omega$ for $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0,1]$, then Ω is called a geometrically-convex set.*
2. *If Ω is a geometrically-convex set and*

$$(\ln x_1, \ln x_2, \dots, \ln x_n) \prec (\ln y_1, \ln y_2, \dots, \ln y_n)$$

implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$ for any $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, then $\varphi : \Omega \to \mathbb{R}_+$ is said to be a Schur geometrically-convex function on Ω.

3. *If $-\varphi$ is a Schur geometrically-convex function on Ω, then φ is said to be a Schur geometrically-concave function on Ω.*

Lemma 2 ([22] (p. 108))**.** *Let $\Omega \subseteq \mathbb{R}^n_+$ be a symmetric and geometrically-convex set with a nonempty interior Ω°, and let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω°. If and only if φ is symmetric on Ω and*

$$(\ln x_1 - \ln x_2)\left(x_1\frac{\partial\varphi}{\partial x_1} - x_2\frac{\partial\varphi}{\partial x_2}\right) \geq 0, \quad \boldsymbol{x} \in \Omega^\circ,$$

the function φ is Schur geometrically-convex on Ω.

For more information on the Schur convexity and the Schur geometric convexity, please refer to the papers [23–26] and the monographs [20,22].

Lemma 3 (Bernoulli's inequality [27,28])**.** *For $0 \neq x > -1$, if $\alpha \geq 1$ or $\alpha < 0$, then*

$$(1+x)^\alpha \geq 1 + \alpha x; \tag{7}$$

if $0 < \alpha < 1$, then Inequality (7) is reversed.

Lemma 4. *For $p \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}_+$, define*

$$h_1(x) = \frac{p(x+1)^{p-1}}{(x+1)^p + 1}, \quad g_1(x) = \frac{px(x+1)^{p-1}}{(x+1)^p + 1}, \quad h_2(x) = \frac{p(x+1)^{p-1}}{(x+1)^p - 1}, \quad g_2(x) = \frac{px(x+1)^{p-1}}{(x+1)^p - 1}.$$

1. *If $p < 0$, the function $h_1(x)$ is increasing on $\mathbb{R}_+$; if $0 < p \leq 2$, the function $h_1(x)$ is decreasing on $\mathbb{R}_+$; if $p > 2$, the function $h_1(x)$ is increasing on $\left(0, (p-1)^{1/p} - 1\right)$ and decreasing on $\left((p-1)^{1/p} - 1, \infty\right)$.*
2. *If $p > 0$, the function $g_1(x)$ is increasing on $\mathbb{R}$; if $p < 0$, the function $g_1(x)$ is decreasing on $\left(0, \frac{1}{|p|}\right]$ and increasing on $\left[\frac{2}{|p|}, \infty\right)$.*
3. *For $p \neq 0$, the function $h_2(x)$ is decreasing on $\mathbb{R}_+$.*
4. *If $p \geq 1$, the function $g_2(x)$ is increasing on $\mathbb{R}_+$; if $p < 1$ and $p \neq 0$, the function $g_2(x)$ is decreasing on $\mathbb{R}_+$.*

Proof. Straightforward computation gives

$$[h_1(x)]' = \frac{p[p-1-(x+1)^p]}{[(x+1)^p+1]^2}(x+1)^{p-2}, \qquad [g_1(x)]' = \frac{p[1+px+(x+1)^p]}{[(x+1)^p+1]^2}(x+1)^{p-2},$$
$$[h_2(x)]' = \frac{p[1-p-(x+1)^p]}{[(x+1)^p-1]^2}(x+1)^{p-2}, \qquad [g_2(x)]' = \frac{p[(x+1)^p-1-px]}{[(x+1)^p-1]^2}(x+1)^{p-2}.$$

If $p < 0$ and $x \in \mathbb{R}_+$ or $p > 2$ and $0 < x \leq (p-1)^{1/p} - 1$, we have $[h_1(x)]' \geq 0$; if $0 < p \leq 2$ and $x \in \mathbb{R}_+$ or $p > 2$ and $(p-1)^{1/p} - 1 < x$, we obtain $[h_1(x)]' \leq 0$.

If $p > 0$ and $x \in \mathbb{R}_+$, we acquire $[g_1(x)]' \geq 0$; if $p < 0$ and $0 < x \leq \frac{1}{|p|}$, we have $[g_1(x)]' \leq 0$; if $p < 0$ and $x \geq \frac{2}{|p|}$, since $(x+1)^p < 1$ and $1 + px \leq -1$, we acquire $[g_1(x)]' \geq 0$.

If $p \in \mathbb{R} \setminus \{0\}$, we obtain $[h_2(x)]' \leq 0$ for $x \in \mathbb{R}_+$.

For $x \in \mathbb{R}_+$, by Lemma 3, if $p \geq 1$, we have $(x+1)^p \geq 1 + px$, then $[g_2(x)]' \geq 0$; if $0 < p \leq 1$, we have $(x+1)^p \leq 1 + px$, and so, $[g_2(x)]' \leq 0$; if $p < 0$, we obtain $(x+1)^p \geq 1 + px$, hence $[g_2(x)]' \leq 0$. The proof of Lemma 4 is complete. □

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $i_1, \ldots, i_k$ be positive integers. The elementary symmetric functions are defined by $E_0(\boldsymbol{x}) = 1$,

$$E_k(\boldsymbol{x}) = E_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} x_{i_j}, \quad 1 \leq k \leq n,$$

and $E_k(\boldsymbol{x}) = 0$ for $k < 0$ or $k > n$.

Lemma 5 (Newton's inequality [20] (p. 134))**.** *For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$ and $n \geq 2$, let $\widehat{F}_k(\boldsymbol{x}) = \frac{1}{\binom{n}{k}} E_k(\boldsymbol{x})$ for $1 \leq k \leq n-1$. Then*

$$\left[\widehat{F}_k(\boldsymbol{x})\right]^2 \geq \widehat{F}_{k-1}(\boldsymbol{x}) \widehat{F}_{k+1}(\boldsymbol{x}).$$

3. Main Results

In this section, we will make use of the Schur convexity of the symmetric function

$$E_k((\boldsymbol{x}+\boldsymbol{1})^p + \boldsymbol{\xi}) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} [(x_{i_j}+1)^p + \xi], \quad 2 \leq k \leq n, \quad n \geq 2$$

to generalize the inequalities in (1)–(3), (5) and (6), where $\boldsymbol{x} \in \mathbb{R}^n_+$, the quantities $i_1, i_2, \ldots, i_k$ are positive integers, $\xi = 0, \pm 1$, and $\boldsymbol{\xi} = \boldsymbol{0}, \pm\boldsymbol{1}$.

Our main results are Theorems 1–3 below.

Theorem 1. *Let $2 \leq k \leq n$, $p \in \mathbb{R} \setminus \{0\}$, $i_1, \ldots, i_k \in \mathbb{N}$, and $a = \frac{(n-k)(p-1)}{k-1}$ for $p > 1$.*

1. *If $2 \leq k \leq n-1$, $0 < p \leq 1$, and $\boldsymbol{x} \in \mathbb{R}^n_+$, or if $2 \leq k \leq n-1$, $p > 1$ with $a < 1$, and $\boldsymbol{x} \in \left(0, a^{-1/p} - 1\right]^n$, or if $k = n$, $p > 0$, and $\boldsymbol{x} \in \mathbb{R}^n_+$, then*

$$\binom{n}{k} [G(\boldsymbol{x}) + 1]^{kp} \leq \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} (x_{i_j}+1)^p \leq \binom{n}{k} [A(\boldsymbol{x}) + 1]^{kp}; \tag{8}$$

   if $p < 0$ and $\boldsymbol{x} \in \left(0, \frac{1}{|p|}\right]^n$ or if $k = n$, $p < 0$, and $\boldsymbol{x} \in \mathbb{R}^n_+$, then the double Inequality (8) is reversed.

2. *If $2 \leq k \leq n-1$, $p > 1$ with $a > 1$, and $\boldsymbol{x} \in \left(0, a^{1/p} - 1\right]^n$, then*

$$\binom{n}{k} [A(\boldsymbol{x}) + 1]^{kp} \leq \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} (x_{i_j}+1)^p.$$

3. *If* $2 \le k \le n-1$, $p > 0$, *and* $\boldsymbol{x} \in \mathbb{R}^n_+$, *then*

$$\binom{n}{k}[G(\boldsymbol{x})+1]^{kp} \le \sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k}(x_{i_j}+1)^p; \tag{9}$$

if $p < 0$ *and* $\boldsymbol{x} \in \mathbb{R}^n_+$, *then Inequality* (9) *is reversed.*

Proof. When $k = n$, we have

$$\frac{\partial E_n((\boldsymbol{x}+\mathbf{1})^p)}{\partial x_1} = \frac{p}{(x_1+1)}E_n((\boldsymbol{x}+\mathbf{1})^p).$$

From this, we obtain

$$(x_1-x_2)\left[\frac{\partial E_n((\boldsymbol{x}+\mathbf{1})^p)}{\partial x_1} - \frac{\partial E_n((\boldsymbol{x}+\mathbf{1})^p)}{\partial x_2}\right] = -\frac{p(x_1-x_2)^2}{(x_1+1)(x_2+1)}E_n((\boldsymbol{x}+\mathbf{1})^p)\begin{cases}\le 0, & p>0;\\ \ge 0, & p<0\end{cases}$$

and

$$(x_1-x_2)\left[x_1\frac{\partial E_n((\boldsymbol{x}+\mathbf{1})^p)}{\partial x_1} - x_2\frac{\partial E_n((\boldsymbol{x}+\mathbf{1})^p)}{\partial x_2}\right] = \frac{p(x_1-x_2)^2}{(x_1+1)(x_2+1)}E_n((\boldsymbol{x}+\mathbf{1})^p)\begin{cases}\ge 0, & p>0;\\ \le 0, & p<0.\end{cases}$$

Using Lemmas 1 and 2, we arrive at

1. if $p > 0$, then $E_n((\boldsymbol{x}+\mathbf{1})^p)$ is Schur concave on $\mathbb{R}^n_+$;
2. if $p < 0$, then $E_n((\boldsymbol{x}+\mathbf{1})^p)$ is Schur convex on $\mathbb{R}^n_+$;
3. if $p > 0$, then $E_n((\boldsymbol{x}+\mathbf{1})^p)$ is Schur geometrically convex on $\mathbb{R}^n$;
4. if $p < 0$, the $E_n((\boldsymbol{x}+\mathbf{1})^p)$ is Schur geometrically concave on $\mathbb{R}^n$.

Since

$$(A_n(\boldsymbol{x}), A_n(\boldsymbol{x}), \dots, A_n(\boldsymbol{x})) \prec \boldsymbol{x} \tag{10}$$

and

$$(\ln G_n(\boldsymbol{x}), \ln G_n(\boldsymbol{x}), \dots, \ln G_n(\boldsymbol{x})) \prec \ln \boldsymbol{x} \tag{11}$$

for $\boldsymbol{x} \in \mathbb{R}^n_+$, applying Definitions 2 and 3, we obtain the double Inequality (8) for $k = n$.

When $2 \le k \le n-1$, a direct differentiation yields

$$\frac{\partial E_k((\boldsymbol{x}+\mathbf{1})^p)}{\partial x_1} = p(x_1+1)^{p-1}(x_2+1)^p E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p) + p(x_1+1)^{p-1}E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p),$$

where $\widehat{\boldsymbol{x}} = (x_3, \dots, x_n)$. We clearly see that

$$\begin{aligned}\Delta_A(E_k((\boldsymbol{x}+\mathbf{1})^p)) &\triangleq (x_1-x_2)\left[\frac{\partial E_k((\boldsymbol{x}+\mathbf{1})^p)}{\partial x_1} - \frac{\partial E_k((\boldsymbol{x}+\mathbf{1})^p)}{\partial x_2}\right]\\ &= -p(x_1-x_2)^2(x_1+1)^{p-1}(x_2+1)^{p-1}E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p)\\ &\quad + p(x_1-x_2)\left[(x_1+1)^{p-1}-(x_2+1)^{p-1}\right]E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p)\end{aligned} \tag{12}$$

and

$$\begin{aligned}&(\ln x_1 - \ln x_2)\left[x_1\frac{\partial E_k((\boldsymbol{x}+\mathbf{1})^p)}{\partial x_1} - x_2\frac{\partial E_k((\boldsymbol{x}+\mathbf{1})^p)}{\partial x_2}\right]\\ &= p(\ln x_1-\ln x_2)(x_1-x_2)(x_1+1)^{p-1}(x_2+1)^{p-1}E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p)\\ &\quad + p(\ln x_1-\ln x_2)\left[x_1(x_1+1)^{p-1}-x_2(x_2+1)^{p-1}\right]E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p).\end{aligned} \tag{13}$$

Using Equation (13) and Lemma 2, we obtain that

1. if $p > 0$, then $E_k((\boldsymbol{x}+\mathbf{1})^p)$ is Schur geometrically convex on $\mathbb{R}_+^n$;
2. if $p < 0$, then $E_k((\boldsymbol{x}+\mathbf{1})^p)$ is Schur geometrically concave on $\left(0, \frac{1}{|p|}\right]^n$.

By Equation (12) and Lemma 1, we reveal that

1. if $p < 0$, then $E_k((\boldsymbol{x}+\mathbf{1})^p)$ is Schur convex on $\mathbb{R}_+^n$;
2. if $0 < p \le 1$, then $E_k((\boldsymbol{x}+\mathbf{1})^p)$ is Schur concave on $\mathbb{R}_+^n$;
3. if $p > 1$ and $\boldsymbol{x} \in \mathbb{R}_+^n$,
   (a) when $x_1 = x_2$, we have $\Delta_A(E_k((\boldsymbol{x}+\mathbf{1})^p)) = 0$;
   (b) when $x_1 \ne x_2$,
      i. by Lagrange's mean value theorem, we have
$$\frac{1}{(x_2+1)^{p-1}} - \frac{1}{(x_1+1)^{p-1}} = \frac{(p-1)(x_1-x_2)}{(\xi+1)^p} \tag{14}$$
      for at least one interior point $\xi \in (\min\{x_1,x_2\}, \max\{x_1,x_2\})$;
      ii. from Newton's inequality, we obtain
$$\begin{aligned}\frac{(n-k)(n-2)}{(k-1)\sum_{i=3}^n (x_i+1)^{-p}} &= \frac{(n-k)\widehat{F}_{n-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p)}{(k-1)\widehat{F}_{n-3}((\widehat{\boldsymbol{x}}+\mathbf{1})^p)} \le \frac{E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p)}{E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p)} \\ &\le \frac{(n-k)\widehat{F}_1((\widehat{\boldsymbol{x}}+\mathbf{1})^p)}{(k-1)\widehat{F}_0((\widehat{\boldsymbol{x}}+\mathbf{1})^p)} = \frac{n-k}{(k-1)(n-2)} \sum_{i=3}^n (x_i+1)^p.\end{aligned} \tag{15}$$

Substituting (14) and (15) into Inequality (12) yields

$$\begin{aligned}\Delta_A(E_k((\boldsymbol{x}+\mathbf{1})^p)) &= -p(x_1-x_2)^2(x_1+1)^{p-1}(x_2+1)^{p-1}E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p) \\ &\quad + p(x_1-x_2)\left[(x_1+1)^{p-1}-(x_2+1)^{p-1}\right]E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p) \\ &= p(x_1-x_2)(x_1+1)^{p-1}(x_2+1)^{p-1}E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p) \\ &\quad \times\left[-(x_1-x_2)+\left(\frac{1}{(x_2+1)^{p-1}}-\frac{1}{(x_1+1)^{p-1}}\right)\frac{E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p)}{E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p)}\right] \\ &= p(x_1-x_2)^2(x_1+1)^{p-1}(x_2+1)^{p-1}E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p)\left[\frac{p-1}{(\xi+1)^p}\frac{E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p)}{E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p)}-1\right].\end{aligned} \tag{16}$$

If $a > 1$ and $\boldsymbol{x} \in (0, a^{1/p}-1]^n$, using $\min\{x_1,x_2\} < \xi < \max\{x_1,x_2\}$, we have

$$\frac{n-2}{\sum_{i=3}^n (x_i+1)^{-p}} > \frac{n-2}{\sum_{i=3}^n (0+1)^{-p}} = 1$$

and

$$\frac{1}{(\xi+1)^p} \ge \frac{k-1}{(n-k)(p-1)}. \tag{17}$$

Consequently, the inequalities from (16)–(17) imply $\Delta(E_k((\boldsymbol{x}+\mathbf{1})^p)) \ge 0$ for $\boldsymbol{x} \in (0, a^{1/p}-1]^n$.

If $a < 1$ and $\boldsymbol{x} \in \left(0, a^{-1/p}-1\right]^n$, we obtain

$$\begin{aligned}\Delta_A(E_k((\boldsymbol{x}+\mathbf{1})^p)) &= p(x_1-x_2)^2(x_1+1)^{p-1}(x_2+1)^{p-1}E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p) \\ &\quad \times\left[-1+\frac{p-1}{(\xi+1)^p}\frac{E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p)}{E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p)}\right] \\ &\le p(x_1-x_2)^2(x_1+1)^{p-1}(x_2+1)^{p-1}E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p)\left[-1+\frac{(n-k)(p-1)}{(k-1)(n-2)(\xi+1)^p}\sum_{i=3}^n (x_i+1)^p\right] \\ &\le p(x_1-x_2)^2(x_1+1)^{p-1}(x_2+1)^{p-1}E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p)\end{aligned}$$

$$\times\left\{-1+\frac{(n-k)(p-1)}{(k-1)(\xi+1)^p}\left[\left(\left(\frac{k-1}{(n-k)(p-1)}\right)^{1/p}-1\right)+1\right]^p\right\}$$
$$=p(x_1-x_2)^2(x_1+1)^{p-1}(x_2+1)^{p-1}E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p)\left[\frac{1}{(\xi+1)^p}-1\right]<0.$$

It is easy to obtain that, if $p>1$ and $a>1$, then $E_k((\boldsymbol{x}+\mathbf{1})^p)$ is Schur convex on $\left(0,a^{1/p}-1\right]^n$; if $p>1$ and $a<1$, then $E_k((\boldsymbol{x}+\mathbf{1})^p)$ is Schur concave on $\left(0,a^{-1/p}-1\right]^n$.

Using (10) and by Definitions 2 and 3, the inequalities in (8) and (9) hold. The proof of Theorem 1 is complete. □

Theorem 2. *Let $2\le k\le n$ and $p\in\mathbb{R}\setminus\{0\}$, $i_1,\dots,i_k\in\mathbb{N}$, and*

$$b=\left[\frac{1}{2}\left(\sqrt{\frac{4(4n-3k-1)(p-1)}{k-1}+1}-1\right)\right]^{1/p},\quad 1<p\le 2.$$

1. *If $k=n$, $0<p\le 2$, and $\boldsymbol{x}\in\mathbb{R}^n_+$, or if $k=n$, $p>2$, and $\boldsymbol{x}\in\left((p-1)^{1/p}-1,\infty\right)^n$, or if $2\le k\le n-1$, $0<p\le 1$, and $\boldsymbol{x}\in\mathbb{R}^n_+$, then*

$$\binom{n}{k}[(G(\boldsymbol{x})+1)^p+1]^k\le\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k}[(x_{i_j}+1)^p+1]\le\binom{n}{k}[(A(\boldsymbol{x})+1)^p+1]^k.\tag{18}$$

If $2\le k\le n$, $p<0$, and $\boldsymbol{x}\in\left(0,\frac{1}{|p|}\right]^n$, then the double Inequality (18) is reversed.

2. *If $2\le k\le n$, $p>2$, and $\boldsymbol{x}\in\left(0,(p-1)^{1/p}-1\right]^n$, or if $2\le k\le n-1$, $1<p\le 2$, and $\boldsymbol{x}\in(0,b]^n$, or if $2\le k\le n$, $p<0$, and $\boldsymbol{x}\in\mathbb{R}^n_+$, then*

$$\binom{n}{k}[(A(\boldsymbol{x})+1)^p+1]^k\le\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k}[(x_{i_j}+1)^p+1].\tag{19}$$

3. *If $2\le k\le n-1$, $p>0$, and $\boldsymbol{x}\in\mathbb{R}^n_+$ or if $2\le k\le n$, $p<0$, and $\boldsymbol{x}\in\left[\frac{2}{|p|},\infty\right)^n$, then*

$$\binom{n}{k}[(G(\boldsymbol{x})+1)^p+1]^k\le\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k}[(x_{i_j}+1)^p+1].\tag{20}$$

4. *If $k=n$, $p>2$, and $\boldsymbol{x}\in\mathbb{R}^n_+$, then*

$$[(G(\boldsymbol{x})+1)^p+1]^n\le\prod_{i=1}^{n}\left[(x_i+1)^p+1\right].\tag{21}$$

Proof. When $k=n$, a direct differentiation yields

$$\frac{\partial E_n((\boldsymbol{x}+\mathbf{1})^p+\mathbf{1})}{\partial x_1}=\frac{p(x_1+1)^{p-1}}{(x_1+1)^p+1}E_n((\boldsymbol{x}+\mathbf{1})^p+\mathbf{1}).\tag{22}$$

From Lemma 4, it follows that

$$(x_1-x_2)\left[\frac{\partial E_n((\boldsymbol{x}+\mathbf{1})^p+\mathbf{1})}{\partial x_1}-\frac{\partial E_n((\boldsymbol{x}+\mathbf{1})^p+\mathbf{1})}{\partial x_2}\right]$$
$$=p(x_1-x_2)\left[\frac{(x_1+1)^{p-1}}{(x_1+1)^p+1}-\frac{(x_2+1)^{p-1}}{(x_2+1)^p+1}\right]E_n((\boldsymbol{x}+\mathbf{1})^p+\mathbf{1})$$

$$\begin{cases} \leq 0, & 0 < p \leq 2, \quad \boldsymbol{x} \in \mathbb{R}_+^n; \\ \geq 0, & p > 2, \quad \boldsymbol{x} \in \left(0, (p-1)^{1/p} - 1\right]^n; \\ \leq 0, & p > 2, \quad \boldsymbol{x} \in \left((p-1)^{1/p} - 1, \infty\right)^n; \\ \geq 0, & p < 0, \quad \boldsymbol{x} \in \mathbb{R}_+^n \end{cases}$$

and

$$\begin{aligned} &(\ln x_1 - \ln x_2)\left[x_1 \frac{\partial E_n((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1})}{\partial x_1} - x_2 \frac{\partial E_n((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1})}{\partial x_2}\right] \\ &= p(\ln x_1 - \ln x_2)\left[\frac{x_1(x_1+1)^{p-1}}{(x_1+1)^p + 1} - \frac{x_2(x_2+1)^{p-1}}{(x_2+1)^p + 1}\right] E_n((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1}) \\ &\begin{cases} \geq 0, & p > 0, \quad \boldsymbol{x} \in \mathbb{R}_+^n, \\ \leq 0, & p < 0, \quad \boldsymbol{x} \in \left(0, \frac{1}{|p|}\right]^n, \\ \geq 0, & p < 0, \quad \boldsymbol{x} \in \left[\frac{2}{|p|}, \infty\right)^n. \end{cases} \end{aligned}$$

Therefore, from Lemmas 1 and 2, we acquire

1. if $0 < p \leq 2$, then $E_n((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1})$ is Schur concave on $\mathbb{R}_+^n$;
2. if $p > 2$, then $E_n((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1})$ is Schur convex on $\left(0, (p-1)^{1/p} - 1\right]^n$;
3. if $p > 2$, then $E_n((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1})$ is Schur concave on $\left((p-1)^{1/p} - 1, \infty\right)^n$;
4. if $p < 0$, then $E_n((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1})$ is Schur convex on $\mathbb{R}_+^n$;
5. if $p > 0$, then $E_n((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1})$ is Schur geometrically convex on $\mathbb{R}_+^n$;
6. if $p < 0$, then $E_n((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1})$ is Schur geometrically concave on $\left(0, \frac{1}{|p|}\right]^n$;
7. if $p < 0$, then $E_n((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1})$ is Schur geometrically convex on $\left[\frac{2}{|p|}, \infty\right)^n$.

From Relations (10) and (11), employing Definitions 2 and 3, we conclude that inequalities in (18)–(21) for $k = n$.

If $2 \leq k \leq n-1$, since

$$\begin{aligned} \frac{\partial E_k((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1})}{\partial x_1} &= p(x_1+1)^{p-1}[(x_2+1)^p + 1]E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p + 1) \\ &\quad + p(x_1+1)^{p-1}E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p + 1). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\Delta_A(E_k((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1})) \triangleq (x_1 - x_2)\left[\frac{\partial E_k((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1})}{\partial x_1} - \frac{\partial E_k((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1})}{\partial x_2}\right] \\ &= p(x_1 - x_2)\left[\frac{(x_1+1)^{p-1}}{(x_1+1)^p + 1} - \frac{(x_2+1)^{p-1}}{(x_2+1)^p + 1}\right] \prod_{i=1}^{2}[(x_i+1)^p + 1]E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p + 1) \\ &\quad + p(x_1 - x_2)\left[(x_1+1)^{p-1} - (x_2+1)^{p-1}\right]E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p + 1). \end{aligned} \tag{23}$$

By Equation (23) and Lemma 1, it is easy to obtain that

1. if $p < 0$, then $E_k((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1})$ is Schur convex on $\mathbb{R}_+^n$;
2. if $0 < p \leq 1$, then $E_k((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1})$ is Schur concave on $\mathbb{R}_+^n$;
3. if $p > 1$ and $\boldsymbol{x} \in \mathbb{R}_+^n$,
   (a) when $x_1 = x_2$, we have $\Delta_A(E_k((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1}) = 0$;

(b) when $x_1 \neq x_2$, using Cauchy's mean value theorem, we have

$$\frac{\frac{(x_1+1)^{p-1}}{(x_1+1)^p+1}-\frac{(x_2+1)^{p-1}}{(x_2+1)^p+1}}{(x_1+1)^{p-1}-(x_2+1)^{p-1}}=\frac{p-1-(\xi+1)^p}{(p-1)[(\xi+1)^p+1]^2}$$

for some point $\xi \in (\min\{x_1,x_2\},\max\{x_1,x_2\})$ such that

$$\begin{aligned}\Delta_A(E_k((\boldsymbol{x}+\mathbf{1})^p+\mathbf{1})) &= p(x_1-x_2)\left[(x_1+1)^{p-1}-(x_2+1)^{p-1}\right]E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p+1)\\ &\times\left\{\frac{\frac{(x_1+1)^{p-1}}{(x_1+1)^p+1}-\frac{(x_2+1)^{p-1}}{(x_2+1)^p+1}}{(x_1+1)^{p-1}-(x_2+1)^{p-1}}\prod_{i=1}^{2}[(x_i+1)^p+1]+\frac{E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p+1)}{E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p+1)}\right\}\\ &= p(x_1-x_2)\left[(x_1+1)^{p-1}-(x_2+1)^{p-1}\right]E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p+1)\\ &\times\left\{\frac{p-1-(\xi+1)^p}{(p-1)[(\xi+1)^p+1]^2}\prod_{i=1}^{2}[(x_i+1)^p+1]+\frac{E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p+1)}{E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p+1)}\right\}.\end{aligned}$$

If $p>2$ and $x \in \left(0,(p-1)^{1/p}-1\right]$, then $p-1-(x+1)^p \geq 0$, so $\Delta_A(E_k((\boldsymbol{x}+\mathbf{1})^p+\mathbf{1})) \geq 0$ for $\boldsymbol{x} \in \left(0,(p-1)^{1/p}-1\right]^n$.

If $1<p\leq 2$ and $\boldsymbol{x} \in (0,b]$, we derive $p-1-(x+1)^p \leq 0$. Using

$$(\min\{x_1,x_2\}+1)^p+1<(\xi+1)^p+1<(\max\{x_1,x_2\}+1)^p+1$$

and Newton's inequality leads to

$$\begin{aligned}\Delta_A(E_k((\boldsymbol{x}+\mathbf{1})^p+\mathbf{1})) &= p(x_1-x_2)\left[(x_1+1)^{p-1}-(x_2+1)^{p-1}\right]E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p+1)\\ &\times\left\{\frac{p-1-(\xi+1)^p}{(p-1)[(\xi+1)^p+1]^2}\prod_{i=1}^{2}[(x_i+1)^p+1]+\frac{E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p+1)}{E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p+1)}\right\}\\ &\geq p(x_1-x_2)\left[(x_1+1)^{p-1}-(x_2+1)^{p-1}\right]E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p+1)\\ &\times\left\{\frac{p-1-(\xi+1)^p}{(p-1)[(\xi+1)^p+1]^2}\prod_{i=1}^{2}[(x_i+1)^p+1]+\frac{n-k}{k-1}\frac{n-2}{\sum_{i=3}^{n}[(x_i+1)^p+1]^{-1}}\right\}\\ &\geq p(x_1-x_2)\left[(x_1+1)^{p-1}-(x_2+1)^{p-1}\right]E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p+1)\\ &\times\left\{\frac{p-1-(\xi+1)^p}{(p-1)[(\xi+1)^p+1]}[(\max\{x_1,x_2\}+1)^p+1]+\frac{n-k}{k-1}\frac{n-2}{\sum_{i=3}^{n}[(x_i+1)^p+1]^{-1}}\right\}\\ &\geq p(x_1-x_2)\left[(x_1+1)^{p-1}-(x_2+1)^{p-1}\right]E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p+1)\\ &\times\left\{\frac{p-1-(b+1)^p}{2(p-1)}[(b+1)^p+1]+\frac{2(n-k)}{k-1}\right\}\\ &= \frac{p}{2(p-1)}(x_1-x_2)\left[(x_1+1)^{p-1}-(x_2+1)^{p-1}\right]E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p+1)\\ &\times\left\{-(b+1)^p[(b+1)^p+1]+\frac{(4n-3k-1)(p-1)}{k-1}\right\}\geq 0.\end{aligned}$$

By Lemma 1, if $1<p\leq 2$, then $E_k((\boldsymbol{x}+\mathbf{1})^p+\mathbf{1})$ is Schur convex with respect to $\boldsymbol{x} \in (0,b]^n$; if $p>2$, then $E_k((\boldsymbol{x}+\mathbf{1})^p+\mathbf{1})$ is Schur convex on $\left(0,(p-1)^{1/p}-1\right]^n$.

When $2 \le k \le n-1$, from (22), we obtain

$$(\ln x_1 - \ln x_2)\left[x_1 \frac{\partial E_k((\boldsymbol{x}+\boldsymbol{1})^p+\boldsymbol{1})}{\partial x_1} - x_2 \frac{\partial E_k((\boldsymbol{x}+\boldsymbol{1})^p+\boldsymbol{1})}{\partial x_2}\right] = p(\ln x_1 - \ln x_2)$$
$$\times \left[\frac{x_1(x_1+1)^{p-1}}{(x_1+1)^p+1} - \frac{x_2(x_2+1)^{p-1}}{(x_2+1)^p+1}\right] \prod_{i=1}^{2}[(x_i+1)^p+1] E_{k-2}((\widehat{\boldsymbol{x}}+\boldsymbol{1})^p+1) \tag{24}$$
$$+p(x_1-x_2)\left[x_1(x_1+1)^{p-1} - x_2(x_2+1)^{p-1}\right] E_{k-1}((\widehat{\boldsymbol{x}}+\boldsymbol{1})^p+1).$$

Using Equation (24) and Lemma 2, we obtain that

1. if $p > 0$, then $E_k((\boldsymbol{x}+\boldsymbol{1})^p+\boldsymbol{1})$ is Schur geometrically convex on $\mathbb{R}_+^n$;
2. if $p < 0$, then $E_k((\boldsymbol{x}+\boldsymbol{1})^p+\boldsymbol{1})$ is Schur geometrically concave on $\left(0, \frac{1}{|p|}\right]^n$;
3. if $p < 0$, then $E_k((\boldsymbol{x}+\boldsymbol{1})^p+\boldsymbol{1})$ is Schur geometrically convex on $\left[\frac{2}{|p|}, \infty\right)^n$.

By (10) and Lemmas 1 and 2, we arrive at Inequalities (18)–(21). The proof of Theorem 2 is complete. □

Theorem 3. *Let $2 \le k \le n$, $p \in \mathbb{R} \setminus \{0\}$, $i_1, \ldots, i_k \in \mathbb{N}$,*

$$c = \left[\frac{n-1+\sqrt{(n-1)^2+4(n-k)(k-1)/(p-1)}}{2(n-k)}\right]^{1/p} - 1, \quad p > 1,$$

and

$$d = \left(\frac{2n+k-3}{k-1}\right)^{1/p} - 1, \quad 0 < p < 1.$$

1. *If $k = n$, $p \ge 1$, and $\boldsymbol{x} \in \mathbb{R}_+^n$, or if $2 \le k \le n-1$, $p > 1$, and $\boldsymbol{x} \in (0, c]^n$, or if $2 \le k \le n-1$, $p = 1$, and $\boldsymbol{x} \in \mathbb{R}_+^n$, then*

$$\binom{n}{k}[(G(\boldsymbol{x})+1)^p-1]^k \le \sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k}[(x_{i_j}+1)^p-1] \le \binom{n}{k}[(A(\boldsymbol{x})+1)^p-1]^k. \tag{25}$$

2. *If $k = n$, $0 < p < 1$, and $\boldsymbol{x} \in \mathbb{R}_+^n$ or f $2 \le k \le n-1$, $0 < p < 1$, and $\boldsymbol{x} \in (0, d]^n$, then*

$$\sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k}[(x_{i_j}-1)^p+1] \le \binom{n}{k}[(G(\boldsymbol{x})+1)^p-1]^k.$$

3. *If $p < 0$, $k = n$ is an even integer, and $\boldsymbol{x} \in \mathbb{R}_+^n$, or if $2 \le k \le n-1$, k is an even integer, $p < 0$, and $\boldsymbol{x} \in \left(0, \frac{1}{|p|}\right]^n$, then*

$$\sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k}[(x_{i_j}-1)^p+1] \le \binom{n}{k}[(A(\boldsymbol{x})+1)^p-1]^k. \tag{26}$$

   If $p < 0$, $k = n$ is an odd integer, and $\boldsymbol{x} \in \mathbb{R}_+^n$, or if $2 \le k \le n-1$, k is an odd integer, $p < 0$, and $\boldsymbol{x} \in \left(0, \frac{1}{|p|}\right]^n$, then Inequality (26) is reversed.

Proof. The proof is divided into three cases.

Case 1. If $k = n$, a direct differentiation yields

$$\frac{\partial E_n((\boldsymbol{x}+\boldsymbol{1})^p-\boldsymbol{1})}{\partial x_1} = \frac{p(x_1+1)^{p-1}}{(x_1+1)^p-1} E_n((\boldsymbol{x}+\boldsymbol{1})^p-\boldsymbol{1}).$$

From Lemma 4, it follows that

$$
\begin{aligned}
&(x_1 - x_2)\left[\frac{\partial E_n((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})}{\partial x_1} - \frac{\partial E_n((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})}{\partial x_2}\right] \\
&= p(x_1 - x_2)\left[\frac{(x_1+1)^{p-1}}{(x_1+1)^p - 1} - \frac{(x_2+1)^{p-1}}{(x_2+1)^p - 1}\right] E_n((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1}) \\
&\begin{cases} \leq 0, & \text{if } p > 0; \\ \leq 0, & \text{if } p < 0 \text{ and } n \text{ is an even integer;} \\ \geq 0, & \text{if } p < 0 \text{ and } n \text{ is an odd integer} \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&(\ln x_1 - \ln x_2)\left[x_1 \frac{\partial E_n((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})}{\partial x_1} - x_2 \frac{\partial E_n((\boldsymbol{x}+\mathbf{1})^p + \mathbf{1})}{\partial x_2}\right] \\
&= p(\ln x_1 - \ln x_2)\left[\frac{x_1(x_1+1)^{p-1}}{(x_1+1)^p - 1} - \frac{x_2(x_2+1)^{p-1}}{(x_2+1)^p - 1}\right] E_n((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1}) \\
&\begin{cases} \geq 0, & \text{if } p \geq 1; \\ \leq 0, & \text{if } 0 < p < 1; \\ \leq 0, & \text{if } p < 0 \text{ and } n \text{ is an even integer;} \\ \geq 0, & \text{if } p < 0 \text{ and } n \text{ is an odd integer.} \end{cases}
\end{aligned}
$$

Therefore, from Lemmas 1 and 2, we have

1. if $p > 0$, then $E_n((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})$ is Schur concave on $\mathbb{R}^n_+$; if $p < 0$ and n is an even (or odd, respectively) integer, then $E_n((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})$ is Schur concave (or Schur convex, respectively) on $\mathbb{R}^n_+$;
2. if $p \geq 1$, then $E_n((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})$ is Schur geometrically convex on $\mathbb{R}^n_+$; if $0 < p < 1$, then $E_n((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})$ is Schur geometrically concave on $\mathbb{R}^n_+$; if $p < 0$ and n is an even (or odd, respectively) integer, then $E_n((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})$ is Schur geometrically concave (or convex, respectively) on $\mathbb{R}^n_+$.

For $k = n$, by Relations (10) and (11) and by Definitions 2 and 3, the inequalities in (25) and (26) hold.

Case 2. When $2 \leq k \leq n-1$, since

$$
\begin{aligned}
&\tfrac{\partial E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})}{\partial x_1} = p(x_1+1)^{p-1}[(x_2+1)^p - 1]E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1}) \\
&+ p(x_1+1)^{p-1}E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1}),
\end{aligned} \tag{27}
$$

we have

$$
\begin{aligned}
\Delta(E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})) &\triangleq (x_1 - x_2)\left[\frac{\partial E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})}{\partial x_1} - \frac{\partial E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})}{\partial x_2}\right] \\
&= p(x_1 - x_2)\{(x_1+1)^{p-1}[(x_2+1)^p - 1] - [(x_1+1)^p - 1](x_2+1)^{p-1}\} \\
&\times E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1}) + p(x_1 - x_2)\left[(x_1+1)^{p-1} - (x_2+1)^{p-1}\right] E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1}) \\
&= p(x_1 - x_2)\left[\frac{(x_1+1)^{p-1}}{(x_1+1)^p - 1} - \frac{(x_2+1)^{p-1}}{(x_2+1)^p - 1}\right] \prod_{i=1}^{2}[(x_i+1)^p - 1] E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1}) \\
&+ p(x_1 - x_2)\left[(x_1+1)^{p-1} - (x_2+1)^{p-1}\right] E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1}).
\end{aligned} \tag{28}
$$

Utilizing the monotonicity of $p(x+1)^{p-1}$ and Lemma 4, we obtain that

1. if $0 < p \leq 1$, then $\Delta(E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})) \leq 0$ for $\boldsymbol{x} \in \mathbb{R}^n_+$, so the function $E_n((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})$ is a Schur-concave function on $\mathbb{R}^n_+$;

2. if $p < 0$ and k is an even (or odd, respectively) integer, then $\Delta(E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})) \lesseqgtr 0$ for $\boldsymbol{x} \in \mathbb{R}^n_+$. This shows from Lemma 1 that, if $p < 0$ and n is an even (or odd, respectively) integer, the function $E_n((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})$ is Schur concave (or Schur convex, respectively) on $\mathbb{R}^n_+$;
3. if $p > 1$ and $\boldsymbol{x} \in (0, c]^n$, from (28), it follows that

$$\begin{aligned}\Delta(E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})) &= p(x_1 - x_2)\{-(x_1 - x_2)(x_1+1)^{p-1}(x_2+1)^{p-1} \\ &\quad -[(x_1+1)^{p-1} - (x_2+1)^{p-1}]\}E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1}) \\ &\quad +p(x_1 - x_2)[(x_1+1)^{p-1} - (x_2+1)^{p-1}]E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1}) \\ &= p(x_1 - x_2)E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})\Big\{-(x_1 - x_2)(x_1+1)^{p-1}(x_2+1)^{p-1} \\ &\quad +[(x_1+1)^{p-1} - (x_2+1)^{p-1}]\left[\frac{E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})}{E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})} - 1\right]\Big\}.\end{aligned} \tag{29}$$

Suppose $x_1 \neq x_2$ and $\boldsymbol{x} \in (0, c]^n$. By Lagrange's mean value theorem, we have

$$(x_1+1)^{p-1} - (x_2+1)^{p-1} = (p-1)(x_1 - x_2)(\xi+1)^{p-2} \tag{30}$$

for some ξ in the open interval $(\min\{x_1, x_2\}, \max\{x_1, x_2\})$ and

$$\frac{(x_1+1)^{p-1}(x_2+1)^{p-1}}{(p-1)(\xi+1)^{p-2}} \geq \frac{1}{(p-1)(\xi+1)^p} \geq \frac{1}{(p-1)(c+1)^p}.$$

For $k = 2, \ldots, n$ and $\boldsymbol{x} \in (0, c]^n$, using Lemma 5 yields

$$\frac{E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})}{E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})} \leq \frac{(n-k)\widehat{F}_1((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})}{(k-1)\widehat{F}_0((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})} \leq \frac{n-k}{k-1}(c+1)^p - \frac{n-k}{k-1}. \tag{31}$$

For $\boldsymbol{x} \in (0, c]^n$, by Equation (29) and the inequalities in (30) and (31), we obtain

$$\begin{aligned}&\Delta(E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})) = p(x_1 - x_2)^2 E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1}) \\ &\times\left\{-(x_1+1)^{p-1}(x_2+1)^{p-1} + (p-1)(\xi+1)^{p-2}\left[\frac{E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})}{E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})} - 1\right]\right\} \\ &\leq p(x_1 - x_2)^2(p-1)(\xi+1)^{p-2}E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1}) \\ &\times\left[-\frac{(x_1+1)^{p-1}(x_2+1)^{p-1}}{(p-1)(\xi+1)^{p-2}} + \frac{n-k}{(k-1)(n-2)}\sum_{i=3}^{n}(x_i+1)^p - \frac{n-1}{k-1}\right] \\ &\leq p(x_1 - x_2)^2(p-1)(\xi+1)^{p-2}E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1}) \\ &\times\left[-\frac{1}{(p-1)(c+1)^p} + \frac{n-k}{k-1}(c+1)^p - \frac{n-1}{k-1}\right] = 0.\end{aligned}$$

Therefore, if $p > 1$, then $\Delta(E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})) \leq 0$ for $\boldsymbol{x} \in (0, c]^n$. Therefore, from Lemma 1, it follows that

1. if $0 < p \leq 1$, the function $E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})$ is Schur concave on $\mathbb{R}^n_+$;
2. if $p < 0$ and k is an even (or odd, respectively) integer, the function $E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})$ is Schur concave (or Schur convex, respectively) on $\mathbb{R}^n_+$;
3. if $p > 1$, the function $E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})$ is Schur concave on $(0, c]^n$.

Case 3. If $2 \le k \le n-1$, then from (27), it follows that

$$\Delta_G(E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1}) \triangleq (\ln x_1 - \ln x_2)\left[x_1 \frac{\partial E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})}{\partial x_1} - x_2 \frac{\partial E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1})}{\partial x_2}\right]$$
$$= p(\ln x_1 - \ln x_2)\left[\frac{x_1(x_1+1)^{p-1}}{(x_1+1)^p - 1} - \frac{x_2(x_2+1)^{p-1}}{(x_2+1)^p - 1}\right]\prod_{i=1}^{2}[(x_i+1)^p - 1]E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})$$
$$+p(\ln x_1 - \ln x_2)\left[x_1(x_1+1)^{p-1} - x_2(x_2+1)^{p-1}\right]E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1}).$$

Using the monotonicity of function $px(x+1)^{p-1}$ and Lemma 4 results in

1. if $p \ge 1$, then $\Delta(E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1}) \ge 0$ for $\boldsymbol{x} \in \mathbb{R}^n_+$;
2. if $p < 0$ and k is an even (or odd, respectively) integer, then $\Delta(E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1}) \lesseqgtr 0$ for $\boldsymbol{x} \in \left(0, \frac{1}{|p|}\right]^n$;
3. if $0 < p < 1$, then

$$\begin{aligned}\Delta_G(E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1}) &= p(\ln x_1 - \ln x_2)\{x_1(x_1+1)^{p-1}[(x_2+1)^p - 1] \\ &\quad -[(x_2+1)^p - 1]x_2(x_2+1)^{p-1}\}E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1}) \\ &\quad +p(\ln x_1 - \ln x_2)\left[x_1(x_1+1)^{p-1} - x_2(x_2+1)^{p-1}\right]E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1}) \\ &= p(\ln x_1 - \ln x_2)E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})\Big\{(x_1 - x_2)(x_1+1)^{p-1}(x_2+1)^{p-1} \\ &\quad +\left[x_1(x_1+1)^{p-1} - x_2(x_2+1)^{p-1}\right]\left[\frac{E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})}{E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})} - 1\right]\Big\}.\end{aligned} \tag{32}$$

Suppose $x_1 \ne x_2$, by Lagrange's mean value theorem, we have

$$x_1(x_1+1)^{p-1} - x_2(x_2+1)^{p-1} = (x_1 - x_2)(\xi+1)^{p-2}(1+p\xi) \tag{33}$$

for some $\xi \in (\min\{x_1, x_2\}, \max\{x_1, x_2\})$. Therefore, using Lemma 5 leads to

$$\frac{E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})}{E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})} \le \frac{n-k}{(k-1)(n-2)}\sum_{i=3}^{n}(x_i+1)^p - \frac{n-k}{k-1}, \quad \boldsymbol{x} \in \mathbb{R}^n_+. \tag{34}$$

For $\boldsymbol{x} \in (0, d]^n$, substituting (33) and (34) into (32) yields

$$\begin{aligned}\Delta_G(E_k((\boldsymbol{x}+\mathbf{1})^p - \mathbf{1}) &= p(\ln x_1 - \ln x_2)(x_1 - x_2)E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1}) \\ &\times\left\{(x_1+1)^{p-1}(x_2+1)^{p-1} + (\xi+1)^{p-2}(1+p\xi)\left[\frac{E_{k-1}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})}{E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})} - 1\right]\right\} \\ &\le p(\ln x_1 - \ln x_2)(x_1 - x_2)E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1})\Big\{(x_1+1)^{p-1}(x_2+1)^{p-1} \\ &\quad +(\xi+1)^{p-2}(1+p\xi)\left[\frac{n-k}{(k-1)(n-2)}\sum_{i=3}^{n}(x_i+1)^p - \frac{n-1}{k-1}\right]\Big\} \\ &= p(\ln x_1 - \ln x_2)(x_1 - x_2)(\xi+1)^{p-2}(1+p\xi)E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1}) \\ &\times\left[\frac{(x_1+1)^{p-1}(x_2+1)^{p-1}}{(\xi+1)^{p-2}(1+p\xi)} + \frac{n-k}{(k-1)(n-2)}\sum_{i=3}^{n}(x_i+1)^p - \frac{n-1}{k-1}\right] \\ &\le p(\ln x_1 - \ln x_2)(x_1 - x_2)(\xi+1)^{p-2}(1+p\xi)E_{k-2}((\widehat{\boldsymbol{x}}+\mathbf{1})^p - \mathbf{1}) \\ &\times\left[(d+1)^{2p} + \frac{n-k}{k-1}(d+1)^p - \frac{n-1}{k-1}\right] = 0.\end{aligned}$$

Therefore, using Lemma 2, we obtain that

1. if $p \geq 1$, the function $E_k((\boldsymbol{x}+\boldsymbol{1})^p - \boldsymbol{1})$ is Schur geometrically convex on $\mathbb{R}_+^n$;
2. if $p < 0$ and k is an even (or odd, respectively) integer, the function $E_k((\boldsymbol{x}+\boldsymbol{1})^p - \boldsymbol{1})$ is a Schur geometrically-concave (or convex, respectively) function on $\left(0, \frac{1}{|p|}\right]^n$;
3. if $0 < p < 1$, the function $E_k((\boldsymbol{x}+\boldsymbol{1})^p - \boldsymbol{1})$ is Schur geometrically concave on $(0, d]^n$.

For $k = n$, by Relations (10) and (11) and by Definitions 2 and 3, the inequalities in (25) and (26) hold. Theorem 3 is thus proven. □

Remark 1. *This paper is a corrected and revised version of the preprint [29].*

Author Contributions: The authors contributed equally to this work. The authors read and approved the final manuscript.

Funding: The authors Bo-Yan Xi, Ying Wu and Feng Qi were partially supported by the National Natural Science Foundation of China under Grant No. 11361038, by the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region under Grant No. NJZZ18154, and by the Inner Mongolia Autonomous Region Natural Science Foundation Project under Grant No. 2018LH01002, China.

Acknowledgments: The authors are thankful to the anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Mitrinović, D.S. *Analytic Inequalities*; Springer: Berlin, Germany; New York, NY, USA, 1970.
2. Wang, Z.; Chen, J. Simple proof and application of a geometric inequality. *J. Ningbo Univ.* **1995**, *8*, 70–72. (In Chinese)
3. Wang, W.-L. Some inequalities for symmetric functions and applications. *J. Ningbo Univ.* **1995**, *8*, 27–29. (In Chinese)
4. Kuang, J.-C. *Chángyòng Bùděngshì (Applied Inequalities)*, 4th ed.; Shandong Science and Technology Press: Ji'nan, China, 2010. (In Chinese)
5. Chen, C.-P.; Qi, F.; Srivastava, H.M. Some properties of functions related to the gamma and psi functions. *Integral Transform. Spec. Funct.* **2010**, *21*, 153–164. [CrossRef]
6. Guo, B.-N.; Qi, F.; Luo, Q.-M. The additivity of polygamma functions. *Filomat* **2015**, *29*, 1063–1066. [CrossRef]
7. Guo, S.; Qi, F. A class of completely monotonic functions related to the remainder of Binet's formula with applications. *Tamsui Oxf. J. Math. Sci.* **2009**, *25*, 9–14.
8. Li, W.-H.; Qi, F. Some Hermite-Hadamard type inequalities for functions whose n-th derivatives are (α, m)-convex. *Filomat* **2013**, *27*, 1575–1582. [CrossRef]
9. Liu, A.-Q.; Li, G.-F.; Guo, B.-N.; Qi, F. Monotonicity and logarithmic concavity of two functions involving exponential function. *Internat. J. Math. Ed. Sci. Tech.* **2008**, *39*, 686–691. [CrossRef]
10. Qi, F.; Guo, B.-N. Some properties of extended remainder of Binet's first formula for logarithm of gamma function. *Math. Slovaca* **2010**, *60*, 461–470. [CrossRef]
11. Qi, F.; Guo, S.; Guo, B.-N. Complete monotonicity of some functions involving polygamma functions. *J. Comput. Appl. Math.* **2010**, *233*, 2149–2160. [CrossRef]
12. Qi, F.; Niu, D.-W.; Guo, B.-N. Monotonic properties of differences for remainders of psi function. *Int. J. Pure Appl. Math. Sci.* **2007**, *4*, 59–66.
13. Guo, B.-N.; Qi, F. On the degree of the weighted geometric mean as a complete Bernstein function. *Afr. Mat.* **2015**, *26*, 1253–1262. [CrossRef]
14. Qi, F.; Lim, D. Integral representations of bivariate complex geometric mean and their applications. *J. Comput. Appl. Math.* **2018**, *330*, 41–58. [CrossRef]
15. Qi, F.; Zhang, X.-J.; Li, W.-H. An elementary proof of the weighted geometric mean being a Bernstein function. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **2015**, *77*, 35–38.
16. Qi, F.; Zhang, X.-J.; Li, W.-H. An integral representation for the weighted geometric mean and its applications. *Acta Math. Sin.* **2014**, *30*, 61–68. [CrossRef]

17. Qi, F.; Zhang, X.-J.; Li, W.-H. Lévy-Khintchine representation of the geometric mean of many positive numbers and applications. *Math. Inequal. Appl.* **2014**, *17*, 719–729. [CrossRef]
18. Qi, F.; Zhang, X.-J.; Li, W.-H. Lévy-Khintchine representations of the weighted geometric mean and the logarithmic mean. *Mediterr. J. Math.* **2014**, *11*, 315–327. [CrossRef]
19. Qi, F.; Zhang, X.-J.; Li, W.-H. The harmonic and geometric means are Bernstein functions. *Bol. Soc. Mat. Mex.* **2017**, *23*, 713–736. [CrossRef]
20. Marshall, A.W.; Olkin, I.; Arnold, B.C. *Inequalities: Theory of Majorization and its Applications*, 2nd ed.; Springer: New York, NY, USA, 2011. [CrossRef]
21. Wang, B.-Y. *Foundations of Majorization Inequalities*; Beijing Normal University Press: Beijing, China, 1990. (In Chinese)
22. Zhang, X.-M. *Geometrically Convex Functions*; An'hui University Press: Hefei, China, 2004. (In Chinese)
23. Chu, Y.-M.; Zhang, X.-M.; Wang, G.-D. The Schur geometrical convexity of the extended mean values. *J. Convex Anal.* **2008**, *15*, 707–718.
24. Guan, K.-Z. Schur convexity of the complete symmetric function. *Math. Inequal. Appl.* **2006**, *9*, 567–576. [CrossRef]
25. Sun, T.-C.; Lü, Y.-P.; Chu, Y.-M. Schur multiplicative and harmonic convexities of generalized Heronian mean in n variables and their applications. *Int. J. Pure Appl. Math.* **2009**, *55*, 25–33.
26. Xi, B.-Y.; Gao, D.-D.; Zhang, T.; Guo, B.-N.; Qi, F. Shannon type inequalities for Kapur's entropy. *Mathematics* **2019**, *7*, 22. [CrossRef]
27. Bullen, P.S. Handbook of Means and Their Inequalities. In *Mathematics and Its Applications*; Kluwer Academic Publishers Group: Dordrecht, The Netherlands, 2003; Volume 560.
28. Mitrinović, D.S.; Pečarić, J.E.; Fink, A.M. *Classical and New Inequalities in Analysis*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1993; doi:10.1007/978-94-017-1043-5.
29. Xi, B.-Y.; Wu, Y.; Shi, H.-N.; Qi, F. Generalizations of Several Inequalities Related to Multivariate Geometric Means. HAL Archives. 2018. Available online: https://hal.archives-ouvertes.fr/hal-01865896 (accessed on 8 June 2019).