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Article

The Modified Inertial Iterative Algorithm for Solving Split Variational Inclusion Problem for Multi-Valued Quasi Nonexpansive Mappings with Some Applications

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Abstract: Based on the very recent work by Shehu and Agbebaku in Comput. Appl. Math. 2017, we introduce an extension of their iterative algorithm by combining it with inertial extrapolation for solving split inclusion problems and fixed point problems. Under suitable conditions, we prove that the proposed algorithm converges strongly to common elements of the solution set of the split inclusion problems and fixed point problems.

Keywords: variational inequality problem; split variational inclusion problem; multi-valued quasi-nonexpasive mappings; Hilbert space

MSC: 47H06; 47H09; 47J05; 47J25

1. Introduction

The *split monotone variational inclusion problem* **(SMVIP)** was introduced by Moudafi [1]. This problem is as follows:

$$\text{Find a point } x^* \in H_1 \text{ such that } 0 \in \hat{f}(x^*) + B_1(x^*) \tag{1}$$

and such that

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in \hat{g}(y^*) + B_2(y^*), \tag{2}$$

where 0 is the zero vector, H_1 and H_2 are real Hilbert spaces, $\hat{f}$ and $\hat{g}$ are given single-valued operators defined on H_1 and H_2, respectively, B_1 and B_2 are multi-valued maximal monotone mappings defined on H_1 and H_2, respectively, and A is a bounded linear operator defined on H_1 to H_2.

It is well known (see [1]) that

$$0 \in \hat{f}(x^*) + B_1(x^*) \iff x^* = J_\lambda^{B_1}(x^* - \lambda \hat{f}(x^*)),$$

and that

$$0 \in \hat{g}(y^*) + B_2(y^*) \iff y^* = J_\lambda^{B_2}(y^* - \lambda \hat{g}(y^*)), \;\; y^* = Ax^*,$$

where $J_\lambda^{B_1} := (I + \lambda B_1)^{-1}$ and $J_\lambda^{B_2} := (I + \lambda B_2)^{-1}$ are the resolvent operators of B_1 and B_2, respectively, with $\lambda > 0$. Note that $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are nonexpansive and firmly nonexpansive.

Recently, Shehu and Agbebaku [2] proposed an algorithm involving a step-size selected and proved strong convergence theorem for split inclusion problem and fixed point problem for multi-valued quasi-nonexpansive mappings. In [1], Moudafi pointed out that the problem **(SMVIP)** [3–5] includes, as special cases, the split variational inequality problem [6], the split zero problem, the split common fixed point problem [7–9] and the split feasibility problem [10,11], which have already been studied and used in image processing and recovery [12], sensor networks in computerized tomography and data compression for models of inverse problems [13].

If $\hat{f} \equiv 0$ and $\hat{g} \equiv 0$ in the problem **(SMVIP)**, then the problem reduces to the *split variational inclusion problem* **(SVIP)** as follows:

$$\text{Find a point } \; x^* \in H_1 \; \text{ such that } \; 0 \in B_1(x^*) \tag{3}$$

and such that

$$y^* = Ax^* \in H_2 \; \text{ solves } \; 0 \in B_2(y^*). \tag{4}$$

Note that the problem **(SVIP)** is equivalent to the following problem:

$$\text{Find a point } \; x^* \in H_1 \; \text{ such that } \; x^* = J_\lambda^{B_1}(x^*) \; \text{ and } \; y^* = J_\lambda^{B_2}(y^*), \;\; y^* = Ax^*$$

for some $\lambda > 0$.

We denote the solution set of the problem **(SVIP)** by Ω, i.e.,

$$\Omega = \{x^* \in H_1 : 0 \in B_1(x^*) \text{ and } 0 \in B_2(y^*), \; y^* = Ax^*\}.$$

Many works have been developed to solve the split variational inclusion problem **(SVIP)**. In 2002, Byrne et al. [7] introduced the iterative method $\{x_n\}$ as follows: For any $x_0 \in H_1$,

$$x_{n+1} = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n) \tag{5}$$

for each $n \geq 0$, where A^* is the adjoint of the bounded linear operator A, $\gamma \in (0, 2/L)$, $L = \|A^*A\|$ and $\lambda > 0$. They have shown the weak and strong convergence of the above iterative method for solving the problem **(SVIP)**.

Later, inspired by the above iterative algorithm, many authors have extended the algorithm $\{x_n\}$ generated by (5). In particular, Kazmi and Rizvi [4] proposed an algorithm $\{x_n\}$ for approximating a solution of the problem **(SVIP)** as follows:

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n f_n(x_n) + (1 - \alpha_n)Su_n \end{cases} \tag{6}$$

for each $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\lambda > 0$, $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A, $f : H_1 \to H_1$ is a contraction and $S : H_1 \to H_1$ is a nonexpansive mapping. In 2015, Sitthithakerngkiet et al. [5] proposed an algorithm $\{x_n\}$ for solving the problem **(SVIP)** and the fixed point problem **(FPP)** of a countable family of nonexpansive mappings as follows:

$$\begin{cases} y_n = J_\lambda^{B_1}(x_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n D)S_n y_n \end{cases} \tag{7}$$

for each $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\lambda > 0$, $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A, $f : H_1 \to H_1$ is a contraction, $D : H_1 \to H_2$ is strongly positive bounded linear operator and, for each $n \geq 1$, $S_n : H_1 \to H_1$ is a nonexpansive mapping.

In both their works, they obtained some strong convergence results by using their proposed iterative methods (for some more results on algorithms, see [14,15]).

Recall that a point $x^* \in H_1$ is called a fixed point of a given multi-valued mapping $S : H_1 \to 2^{H_1}$ if

$$x^* \in Sx^* \tag{8}$$

and the *fixed point problem* **(FPP)** for a multi-valued mapping $S : H_1 \to 2^{H_1}$ is as follows:

$$\text{Find a point } \ x^* \in H_1 \ \text{ such that } \ x^* \in Sx^*.$$

The set of fixed points of the multi-valued mapping S is denoted by $F(S)$.

As applications, the fixed point theory for multi-valued mappings was applied to various fields, especially mathematical economics and game theory (see [16–18]).

Recently, motivated by the results of Byrne et al. [7], Kazmi and Rizvi [4] and Sitthithakerngkiet [5], Shehu and Agbebaku [2] introduced the *split fixed point inclusion problem* **(SFPIP)** from the problems **(SVIP)** and **(FPP)** for a multi-valued quasi-nonexpansive mapping $S : H_1 \to 2^{H_1}$ as follows:

$$\text{Find a point } \ x^* \in H_1 \ \text{ such that } \ 0 \in B_1(x^*), \ \ x^* \in Sx^* \tag{9}$$

and such that

$$y^* = Ax^* \in H_2 \ \text{ solves } \ 0 \in B_2(y^*), \tag{10}$$

where H_1 and H_2 are real Hilbert spaces, B_1 and B_2 are multi-valued maximal monotone mappings defined on H_1 and H_2, respectively, and A is a bounded linear operator defined on H_1 to H_2.

Note that the problem **(SFPIP)** is equivalent to the following problem: for some $\lambda > 0$,

$$\text{Find a point } \ x^* \in H_1 \ \text{ such that } \ x^* = J_\lambda^{B_1}(x^*), \ \ x^* \in Sx^* \ \text{ and } \ Ax^* = J_\lambda^{B_2}(Ax^*).$$

The solution set of the problem **(SFPIP)** is denoted by $F(S) \bigcap \Omega$, i.e.,

$$F(S) \bigcap \Omega = \{x^* \in H_1 : 0 \in B_1(x^*), \ x^* \in Sx^* \text{ and } 0 \in B_2(Ax^*)\}.$$

Notice that, if S is the identity operator, then the problem **(SFPIP)** reduces to the problem **(SVIP)**. Moreover, if $J_\lambda^{B_1} = J_\lambda^{B_2} = A = I$, then the problem **(SFPIP)** reduces to the problem **(FPP)** for a multi-valued quasi-nonexpansive mapping.

Furthermore, Shehu and Agbebaku [2] introduced an algorithm $\{x_n\}$ for solving the problem **(SFPIP)** for a multi-valued quai-nonexpasive mapping S as follows: For any $x_1 \in H_1$,

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma_n A^*(J_\lambda^{B_2} - 1)Ax_n), \\ x_{n+1} = \alpha_n f_n(x_n) + \beta_n x_n + \delta_n(\sigma w_n + (1 - \sigma)u_n), \ \ w_n \in Sx_n, \end{cases} \tag{11}$$

for each $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are the real sequences in $(0,1)$ such that

$$\alpha_n + \beta_n + \delta_n = 1, \quad \sigma \in (0,1), \quad \gamma_n := \tau_n \frac{\|(J_\lambda^{B_2} - I)Ax_n\|^2}{\|A^*(J_\lambda^{B_2} - I)\|^2},$$

where $0 < a \leq \tau_n \leq b < 1$, and $\{f_n(x)\}$ is the uniform convergence sequence for any x in a bounded subset D of H_1, and proved that the sequences $\{u_n\}$ and $\{x_n\}$ generated by (11) both converge strongly to $p \in F(S) \cap \Omega$, where $p = P_{F(S)\cap\Omega} f(p)$.

In optimization theory, the second-order dynamical system, which is called the heavy ball method, is used to accelerate the convergence rate of algorithms. This method is a two-step iterative method for minimizing a smooth convex function which was firstly introduced by Polyak [19].

The following is a modified heavy ball method for the improvement of the convergence rate, which was introduced by Nesterov [20]:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = y_n - \lambda_n \nabla f(y_n) \end{cases}$$

for each $n \geq 1$, where $\lambda_n > 0$, $\theta_n \in [0,1)$ is an extrapolation factor. Here, the term $\theta_n(x_n - x_{n-1})$ is the inertia (for more recent results on the inertial algorithms, see [21,22]).

The following method is called the *inertial proximal point algorithm*, which was introduced by Alvarez and Attouch [23]. This method combined the proximal point algorithm [24] with the inertial extrapolation [25,26]:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + \lambda_n \widehat{T})^{-1}(y_n) \end{cases} \tag{12}$$

for each $n \geq 1$, where I is identity operator and $\widehat{T}$ is a maximal monotone operator. It was proven that, if a positive sequence λ_n is non-decreasing, $\theta_n \in [0,1)$ and the following summability condition holds:

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty, \tag{13}$$

then $\{x_n\}$ generated by (12) converges to a zero point of T.

In fact, recently, some authors have pointed out some problems in this summability condition (13) given in [27], that is, to satisfy this summability condition (13) of the sequence $\{x_n\}$, one needs to calculate $\{\theta_n\}$ at each step. Recently, Bot et al. [28] improved this condition, that is, they got rid of the summability condition (13) and replaced the other conditions.

In this paper, inspired by the results of Shehu and Agbebaku [2], Nesterov [20] and Alvarez and Attouch [23], we proposed a new algorithm by combining the iterative algorithm (11) with the inertial extrapolation for solving the problem **(SFPIP)** and prove some strong convergence theorems of the proposed algorithm to show the existence of a solution of the problem **(SFPIP)**. Furthermore, as applications, we consider our proposed algorithm for solving the variational inequality problem and give some applications in game theory.

2. Preliminaries

In this section, we recall some definitions and results which will be used in the proof of the main results.

Let H_1 and H_2 be two real Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C be a nonempty closed and convex subset of H_1 and D be a nonempty bounded subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of A.

Let $\{x_n\}$ be a sequence in H, we denote the strong and weak convergence of a sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Recall that a mapping $T : C \to C$ is said to be:

(1) *Lipschitz* if there exists a positive constant α such that, for all $x, y \in C$,

$$\|Tx - Ty\| \leq \alpha \|x - y\|.$$

If $\alpha \in (0, 1)$ and $\alpha = 1$, then the mapping T is contractive and nonexpansive, respectively.

(2) *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle$$

for all $x, y \in C$.

A mapping P_C is said to be the *metric projection* of H_1 onto C if, for all point $x \in H_1$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$.

It is well known that P_C is nonexpansive mapping and satisfies

$$\langle x - y, P_C x - P_C y \rangle \leq \|P_C x - P_C y\|^2$$

for all $x, y \in H_1$. Moreover, $P_C x$ is characterized by the fact $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \leq 0$$

for all $y \in C$ and $x \in H_1$ (see [6,22]).

A multi-valued mapping $B_1 : H_1 \to 2^{H_1}$ is said to be *monotone* if, for all $x, y \in H_1$, $u \in B_1(x)$ and $v \in B_1(y)$,

$$\langle x - y, u - v \rangle \geq 0.$$

A monotone mapping $B_1 : H_1 \to 2^{H_1}$ is said to be *maximal* if the graph $G(B_1)$ of B_1 is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping B_1 is maximal if and only if, for all $(x, u) \in H_1 \times H_1$,

$$\langle x - y, u - v \rangle \geq 0$$

for all $(y, v) \in G(B_1)$ implies that $u \in B_1(x)$.

Let $B_1 : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping. Then the *resolvent mapping* $J_\lambda^{B_1} : H_1 \to H_1$ associated with B_1 is defined by

$$J_\lambda^{B_1}(x) := (I + \lambda B_1)^{-1}(x)$$

for all $x \in H_1$ and for some $\lambda > 0$, where I is the identity operator on H_1. It is well known that, for any $\lambda > 0$, the resolvent operator $J_\lambda^{B_1}$ is single-valued firmly nonexpansive (see [2,5,6,14]).

Definition 1. *Suppose that $\{f_n(x)\}$ is a sequence of functions defined on a bounded set D. Then $f_n(x)$ converges uniformly to the function $f(x)$ on D if, for all $x \in D$,*

$$f_n(x) \to f(x) \ \ as \ \ n \to \infty.$$

Let $f_n : D \to H_1$ be a uniformly convergent sequence of contraction mappings on D, i.e., there exists $\mu_n \in (0,1)$ such that

$$f_n(x) - f_n(y)\| \leq \mu_n \|x - y\|$$

for all $x, y \in D$.

Let $CB(H_1)$ denote the family of nonempty closed and bounded subsets of H_1. The *Hausdorff metric* on $CB(H_1)$ is defined by

$$\widehat{H}(x,y) = \max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\}$$

for all $A, B \in CB(H_1)$ (see [18]).

Definition 2. [2] *Let $S : H_1 \to CB(H_1)$ be a multi-valued mapping. Assume that $p \in H_1$ is a fixed point of S, that is, $p \in Sp$. The mapping S is said to be:*

(1) nonexpansive if, for all $x, y \in H_1$,

$$\widehat{H}(Sx, Sy) \leq \|x - y\|.$$

(2) quasi-nonexpansive if $F(S) \neq \emptyset$ and, for all $x \in H_1$ and $p \in F(S)$,

$$\widehat{H}(Sx, Sp) \leq \|x - p\|$$

Definition 3. [2] A single-valued mapping $S : H \to H$ is said to be *demiclosed at the origin* if, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ and $Sx_n \to 0$, we have $Sx = 0$.

Definition 4. [2] A multi-valued mapping $S : H_1 \to CB(H_1)$ is said to be *demiclosed at the origin* if, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ and $d(x_n, Sx_n) \to 0$, we have $x \in Sx$.

Lemma 1. [29,30] *Let H be a Hilbert space. Then, for any $x, y, z \in X$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Lemma 2. [2,31] *Let H be a real Hilbert space. Then the following results hold:*

(1) $\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$.
(2) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$.
(3) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ *for all $x, y \in H$.*

Lemma 3. [2,32,33] *Let $\{a_n\}$, $\{c_n\} \subset \mathbb{R}_+$, $\{\sigma_n\} \subset (0,1)$ and $\{b_n\} \subset \mathbb{R}$ be sequences such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + b_n + c_n \quad for\ all \quad n \geq 0.$$

Assume $\sum_{n=0}^{\infty} |c_n| < \infty$. Then the following results hold:

(1) *If $b_n \leq \beta\sigma_n$ for some $\beta \geq 0$, then $\{a_n\}$ is a bounded sequence.*

(2) *If we have*

$$\sum_{n=0}^{\infty} \sigma_n = \infty \ \ and \ \ \limsup_{n\to\infty} \frac{b_n}{\sigma_n} \leq 0,$$

then $\lim_{n\to\infty} a_n = 0$.

Lemma 4. [32,33] *Let* $\{s_n\}$ *be a sequence of non-negative real numbers such that*

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n t_n + r_n$$

for each $n \geq 1$*, where*

(a) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$;
(b) $\limsup t_n \leq 0$;
(c) $r_n \geq 0$ *and* $\sum_{n=1}^{\infty} r_n < \infty$.

Then $s_n \to 0$ *as* $n \to \infty$.

3. The Main Results

In this section, we prove some strong convergence theorems of the proposed algorithm for solving the problem **(SFPIP)**.

Theorem 1. *Let* H_1, H_2 *be two real Hilbert spaces,* $A : H_1 \to H_2$ *be bounded operator with adjoint operator* A^* *and* $B_1 : H_1 \to 2^{H_1}$*,* $B_2 : H_2 \to 2^{H_2}$ *be maximal monotone mappings. Let* $S : H_1 \to CB(H_1)$ *be a multi-valued quasi-nonexpansive mapping and* S *be demiclosed at the origin. Let* $\{f_n\}$ *be a sequence of* μ_n*-contractions* $f_n : H_1 \to H_1$ *with* $0 < \mu_* \leq \mu_n \leq \mu^* < 1$ *and* $\{f_n(x)\}$ *be uniformly convergent for any* x *in a bounded subset* D *of* H_1*. Suppose that* $F(S) \cap \Omega \neq \emptyset$*. For any* $x_0, x_1 \in H_1$*, let the sequences* $\{y_n\}$*,* $\{u_n\}$*,* $\{z_n\}$ *and* $\{x_n\}$ *be generated by*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = J_\lambda^{B_1}(y_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ay_n), \\ z_n = \xi v_n + (1-\xi)u_n, \ v_n \in Sx_n, \\ x_{n+1} = \alpha_n f_n(x_n) + \beta_n x_n + \delta_n z_n \end{cases} \tag{14}$$

for each $n \geq 1$*, where* $\xi \in (0,1)$*,* $\gamma_n := \tau_n \frac{\|(J_\lambda^{B_2}-I)Ay_n\|^2}{\|A^*(J_\lambda^{B_2}-I)Ay_n\|^2}$ *with* $0 < \tau_* \leq \tau_n \leq \tau^* < 1$*,* $\{\theta_n\} \subset [0, \bar{\omega})$ *for some* $\bar{\omega} > 0$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\} \in (0,1)$ *with* $\alpha_n + \beta_n + \delta_n = 1$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < \epsilon_1 \leq \beta_n$ *and* $0 < \epsilon_2 \leq \delta_n$;
(C4) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Then $\{x_n\}$ *generated by* (14) *converges strongly to* $p \in F(S) \cap \Omega$*, where* $p = P_{F(S)\cap\Omega} f(p)$.

Proof. First, we show that $\{x_n\}$ is bounded. Let $p = P_{F(S)\cap\Omega} f(p)$. Then $p \in F(S) \cap \Omega$ and so $J_\lambda^{B_1} p = p$ and $J_\lambda^{B_2} Ap = Ap$. By the triangle inequality, we get

$$\begin{aligned} \|y_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\ &\leq \|x_n - p\| + \theta_n \|x_n - x_{n-1}\|. \end{aligned} \tag{15}$$

By the Cauchy-Schwarz inequality and Lemma 2 (1) and (2), we get

$$\begin{aligned}\|y_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\ &= \|x_n - p\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle \\ &\leq \|x_n - p\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|x_n - p\|. \end{aligned} \tag{16}$$

By using (15) and the fact that S is quasi-nonexpansive S, we get

$$\begin{aligned}\|z_n - p\| &= \|\xi v_n + (1-\xi)u_n - p\| \\ &= \|\xi(v_n - p) + (1-\xi)(u_n - p)\| \\ &\leq \xi\|v_n - p\| + (1-\xi)\|u_n - p\| \\ &\leq \xi d(v_n, Sp) + (1-\xi)\|y_n - p\| \\ &\leq \xi\widehat{H}(Sx_n, Sp) + (1-\xi)[\|x_n - p\| + \theta_n\|x_n - x_{n-1}\|] \\ &\leq \xi\|x_n - p\| + (1-\xi)\|x_n - p\| + (1-\xi)\theta_n\|x_n - x_{n-1}\| \\ &\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\|, \end{aligned} \tag{17}$$

which implies that

$$\begin{aligned}\|z_n - p\|^2 &\leq (\|x_n - p\| + \theta_n\|x_n - x_{n-1}\|)^2 \\ &= \|x_n - p\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|x_n - p\| + \theta_n^2\|x_n - x_{n-1}\|^2. \end{aligned} \tag{18}$$

Since $J_\lambda^{B_1}$ is nonexpansive, by Lemma 2 (2), we get

$$\begin{aligned}\|u_n - p\|^2 &= \|J_\lambda^{B_I}(y_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ay_n) - p\|^2 \\ &= \|J_\lambda^{B_1}(y_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ay_n) - J_\lambda^{B_1}p\|^2 \\ &\leq \|y_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ay_n - p\|^2 \\ &= \|y_n - p\|^2 + \gamma_n^2\|A^*(J_\lambda^{B_2} - I)Ay_n\|^2 + 2\gamma_n\langle y_n - p, A^*(J_\lambda^{B_2} - I)Ay_n\rangle. \end{aligned} \tag{19}$$

Again, by Lemma 2 (2), we get

$$\begin{aligned}&\langle y_n - p, A^*(J_\lambda^{B_2} - I)Ay_n\rangle \\ &= \langle A(y_n - p), (J_\lambda^{B_2} - I)Ay_n\rangle \\ &= \langle J_\lambda^{B_2}Ay_n - Ap - (J_\lambda^{B_2} - I)Ay_n, (J_\lambda^{B_2} - I)Ay_n\rangle \\ &= \langle J_\lambda^{B_2}Ay_n - Ap, (J_\lambda^{B_2} - I)Ay_n\rangle - \langle (J_\lambda^{B_2} - I)Ay_n, (J_\lambda^{B_2} - I)Ay_n\rangle \\ &= \langle J_\lambda^{B_2}Ay_n - Ap, (J_\lambda^{B_2} - I)Ay_n\rangle - \|(J_\lambda^{B_2} - I)Ay_n\|^2 \\ &= \frac{1}{2}\big(\|J_\lambda^{B_2}Ay_n - Ap\|^2 + \|(J_\lambda^{B_2} - I)Ay_n\|^2 \\ &\quad - \|J_\lambda^{B_2}Ay_n - Ap - (J_\lambda^{B_2} - I)Ay_n\|^2\big) - \|(J_\lambda^{B_2} - I)Ay_n\|^2 \\ &= \frac{1}{2}\big(\|J_\lambda^{B_2}Ay_n - Ap\|^2 + \|(J_\lambda^{B_2} - I)Ay_n\|^2 - \|J_\lambda^{B_2}Ay_n - Ap - J_\lambda^{B_2}Ay_n + Ay_n\|^2\big) \\ &\quad - \|(J_\lambda^{B_2} - I)Ay_n\|^2 \\ &= \frac{1}{2}\big(\|J_\lambda^{B_2}Ay_n - Ap\|^2 + \|(J_\lambda^{B_2} - I)Ay_n\|^2 - \|Ay_n - Ap\|^2\big) - \|(J_\lambda^{B_2} - I)Ay_n\|^2 \\ &= \frac{1}{2}\big(\|J_\lambda^{B_2}Ay_n - Ap\|^2 - \|Ay_n - Ap\|^2 - \|(J_\lambda^{B_2} - I)Ay_n\|^2\big) \\ &\leq \frac{1}{2}\big(\|Ay_n - Ap\|^2 - \|Ay_n - Ap\|^2 - \|(J_\lambda^{B_2} - I)Ay_n\|^2\big) \\ &= -\frac{1}{2}\|(J_\lambda^{B_2} - I)Ay_n\|^2. \end{aligned} \tag{20}$$

Using (20) into (19), we get

$$\begin{aligned}\|u_n - p\|^2 &\leq \|y_n - p\|^2 + \gamma_n^2 \|A^*(J_\lambda^{B_2} - I)Ay_n\|^2 - \gamma_n \|(J_\lambda^{B_2} - I)Ay_n\|^2 \\ &= \|y_n - p\|^2 - \gamma_n \big(\|(J_\lambda^{B_2} - I)Ay_n\|^2 - \gamma_n \|A^*(J_\lambda^{B_2} - I)Ay_n\|^2\big).\end{aligned} \tag{21}$$

By the definition of γ_n, (21) can then be written as follows:

$$\|u_n - p\|^2 \leq \|y_n - p\|^2 - \gamma_n(1 - \tau_n)\|(J_\lambda^{B_2} - I)Ay_n\|^2 \leq \|y_n - p\|^2.$$

Thus we have

$$\|u_n - p\| \leq \|y_n - p\|. \tag{22}$$

Using the condition (C3) and (17), we get

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n f_n(x_n) + \beta_n x_n + \delta_n z_n - p\| \\ &= \|\alpha_n (f_n(x_n) - f_n(p)) + \alpha_n (f_n(p) - p) + \beta_n (x_n - p) + \delta_n (z_n - p)\| \\ &\leq \alpha_n \|f_n(x_n) - f_n(p)\| + \alpha_n \|f_n(p) - p\| + \beta_n \|x_n - p\| + \delta_n \|z_n - p\| \\ &\leq \alpha_n \mu_n \|x_n - p\| + \alpha_n \|f_n(p) - p\| + \beta_n \|x_n - p\| + \delta_n (\|x_n - p\| \\ &\quad + (1 - \xi)\theta_n \|x_n - x_{n-1}\|) \\ &\leq (\alpha_n \mu^* + (\beta_n + \delta_n))\|x_n - p\| + (1 - \xi)\delta_n \theta_n \|x_n - x_{n-1}\| + \alpha_n \|f_n(p) - p\| \\ &= (1 - \alpha_n (1 - \mu^*)\|x_n - p\| + (1 - \xi)\delta_n \alpha_n \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \alpha_n \|f_n(p) - p\|.\end{aligned}$$

Since $\{f_n\}$ is the uniform convergence on D, there exists a constant $M > 0$ such that

$$\|f_n(p) - p\| \leq M$$

for each $n \geq 1$. So we can choose $\beta := \dfrac{M}{1 - \mu^*}$ and set

$$a_n := \|x_n - p\|, \quad b_n := \alpha_n \|f_n(p) - p\|,$$

$$c_n := (1 - \xi)\delta_n \alpha_n \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|, \quad \sigma_n := \alpha_n (1 - \mu^*).$$

By Lemma 3 (1) and our assumptions, it follows that $\{x_n\}$ is bounded. Moreover, $\{u_n\}$ and $\{y_n\}$ are also bounded.

Now, by Lemma 2, we get

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&=\|\alpha_n(f_n(x_n)-f_n(p))+\alpha_n(f_n(p)-p)+\beta_n(x_n-p)+\delta_n(z_n-p)\|^2\\
&\le \|\alpha_n(f_n(x_n)-f_n(p))+\beta_n(x_n-p)+\delta_n(z_n-p)\|^2+2\alpha_n\langle f_n(p)-p,x_{n+1}-p\rangle\\
&=\|\beta_n(x_n-p)+\delta_n(z_n-p)\|^2+\alpha_n^2\|f_n(x_n)-f_n(p)\|^2\\
&\quad+2\alpha_n\langle f_n(x_n)-f_n(p),\beta_n(x_n-p)+\delta_n(z_n-p)\rangle+2\alpha_n\langle f_n(p)-p,x_{n+1}-p\rangle\\
&\le \beta_n^2\|x_n-p\|^2+\delta_n^2\|z_n-p\|^2+2\beta_n\delta_n\langle x_n-p,z_n-p\rangle+\alpha_n^2\mu_n^2\|x_n-p\|^2\\
&\quad+2\alpha_n\langle f_n(p)-p,x_{n+1}-p\rangle+2\alpha_n\|f_n(x_n)-f_n(p)\|\|\beta_n(x_n-p)+\delta_n(z_n-p)\|\\
&\le \beta_n^2\|x_n-p\|^2+\delta_n^2\|z_n-p\|^2+\beta_n\delta_n\left(\|x_n-p\|^2+\|z_n-p\|^2-\|x_n-z_n\|^2\right)\\
&\quad+\alpha_n^2\mu^{*2}\|x_n-p\|^2+2\alpha_n\mu_n\|x_n-p\|\left(\beta_n\|x_n-p\|+\delta_n\|z_n-p\|\right)\\
&\quad+2\alpha_n\langle f_n(p)-p,x_{n+1}-p\rangle\\
&\le \beta_n(\beta_n+\delta_n)\|x_n-p\|^2+\delta_n(\beta_n+\delta_n)\|z_n-p\|^2-\beta_n\delta_n\|x_n-z_n\|^2+\alpha_n^2\mu^{*2}\|x_n-p\|^2\\
&\quad+2\mu^*\alpha_n(\beta_n+\delta_n)\|x_n-p\|^2+2\mu^*\alpha_n(1-\xi)\delta_n\theta_n\|x_n-x_{n-1}\|\|x_n-p\|\\
&\quad+2\alpha_n\langle f_n(p)-p,x_{n+1}-p\rangle\\
&\le \beta_n(\beta_n+\delta_n)\|x_n-p\|^2+\delta_n(\beta_n+\delta_n)\big(\|x_n-p\|^2+\theta_n^2\|x_n-x_{n-1}\|^2\\
&\quad+2\theta_n\|x_n-x_{n-1}\|\|x_n-p\|\big)-\beta_n\delta_n\|x_n-z_n\|^2+\alpha_n^2\mu^{*2}\|x_n-p\|^2\\
&\quad+2\mu^*\alpha_n(\beta_n+\delta_n)\|x_n-p\|^2+2\mu^*\alpha_n(1-\xi)\delta_n\theta_n\|x_n-x_{n-1}\|\|x_n-p\|\\
&\quad+2\alpha_n\langle f_n(p)-p,x_{n+1}-p\rangle\\
&=\big((1-\alpha_n)^2+\alpha_n^2\mu^{*2}+2\mu^*\alpha_n(1-\alpha_n)\big)\|x_n-p\|^2-\beta_n\delta_n\|x_n-z_n\|^2\\
&\quad+2\big(1-\alpha_n(1-\mu^*(1-\xi))\big)\delta_n\theta_n\|x_n-x_{n-1}\|\|x_n-p\|+(1-\alpha_n)\delta_n\theta_n^2\|x_n-x_{n-1}\|^2\\
&\quad+2\alpha_n\langle f_n(p)-p,x_{n+1}-p\rangle.
\end{aligned}
\tag{23}
$$

Now, we consider two steps for the proof as follows:

Case 1. Suppose that there exists $n_0\in\mathbb{N}$ such that $\{\|x_n-p\|\}_{n=n_0}^{\infty}$ is non-increasing and then $\{\|x_n-p\|\}$ converges. By Lemma 1, we get

$$
\begin{aligned}
\|x_{n+1}-p\|^2&=\|\alpha_nf_n(x_n)+\beta_nx_n+\delta_nz_n-p\|^2\\
&=\alpha_n\|f_n(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\delta_n\|z_n-p\|^2-\alpha_n\beta_n\|f_n(x_n)-x_n\|^2\\
&\quad-\alpha_n\gamma_n\|f_n(x_n)-z_n\|^2-\beta_n\gamma_n\|x_n-z_n\|^2\\
&\le\alpha_n\|f_n(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\delta_n\|z_n-p\|^2\\
&\le\alpha_n\|f_n(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\delta_n\big(\xi\|x_n-p\|^2+(1-\xi)\|u_n-p\|^2\big)\\
&\le\alpha_n\|f_n(x_n)-p\|^2+(\beta_n+\xi\delta_n)\|x_n-p\|^2+(1-\xi)\delta_n\|u_n-p\|^2,
\end{aligned}
$$

which implies that

$$
-\|u_n-p\|^2\le\frac{1}{(1-\xi)\delta_n}\big(\alpha_n\|f_n(x_n)-p\|^2+(\beta_n+\xi\delta_n)\|x_n-p\|^2-\|x_{n+1}-p\|^2\big). \tag{24}
$$

Applying (16) and (24) to (21), we get

$$
\begin{aligned}
&\gamma_n(\|(J_\lambda^{B_2}-I)Ay_n\|^2-\gamma_n\|A^*(J_\lambda^{B_2}-I)Ay_n\|^2)\\
&\le \|y_n-p\|^2-\|u_n-p\|^2\\
&\le \|x_n-p\|^2+2\theta_n\|x_{n-1}-p\|\|x_n-p\|+\theta_n^2\|x_n-x_{n-1}\|^2\\
&\quad+\frac{1}{(1-\xi)\delta_n}(\alpha_n\|f_n(x_n)-p\|^2+(\beta_n+\xi\delta_n)\|x_n-p\|^2-\|x_{n+1}-p\|^2)\\
&=\frac{\beta_n+\delta_n}{(1-\xi)\delta_n}\|x_n-p\|^2+\frac{\alpha_n}{(1-\xi)\delta_n}\|f_n(x_n)-p\|^2-\frac{1}{(1-\xi)\delta_n}\|x_{n+1}-p\|^2\\
&\quad+\theta_n\|x_n-x_{n-1}\|\,(2\|x_n-p\|+\theta_n\|x_n-x_{n-1}\|)\\
&\le \frac{1}{(1-\xi)\epsilon_2}(\|x_n-p\|^2-\|x_{n+1}-p\|^2)+\frac{\alpha_n}{(1-\xi)\epsilon_2}\Big(\|f_n(x_n)-p\|^2-\|x_n-p\|^2\\
&\quad+\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\Big(2\|x_n-p\|+\alpha_n\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\Big)\Big).
\end{aligned}
$$

Since $\{\|x_n-p\|\}$ is convergent, we have $\|x_n-p\|-\|x_{n+1}-p\|\to 0$ as $n\to\infty$. By the conditions (C2) and (C4), we get

$$\gamma_n(\|(J_\lambda^{B_2}-I)Ay_n\|^2-\gamma_n\|A^*(J_\lambda^{B_2}-I)Ay_n\|^2)\to 0 \;\text{ as }\; n\to\infty.$$

From the definition of γ_n, we get

$$\frac{\tau_n(1-\tau_n)\|(J_\lambda^{B_2}-I)Ay_n\|^4}{\|A^*(J_\lambda^{B_2}-I)Ay_n\|^2}\to 0 \;\text{ as }\; n\to\infty$$

or

$$\frac{\|(J_\lambda^{B_2}-I)Ay_n\|^2}{\|A^*(J_\lambda^{B_2}-I)Ay_n\|}\to 0 \;\text{ as }\; n\to\infty.$$

Since

$$\|A^*(J_\lambda^{B_2}-I)Ay_n\|\le\|A^*\|\|(J_\lambda^{B_2}-I)Ay_n\|=\|A\|\|(J_\lambda^{B_2}-I)Ay_n\|,$$

it is easy to see that

$$\|(J_\lambda^{B_2}-I)Ay_n\|\le\|A\|\frac{\|(J_\lambda^{B_2}-I)Ay_n\|^2}{\|A^*(J_\lambda^{B_2}-I)Ay_n\|}.$$

Consequently, we get

$$\|(J_\lambda^{B_2}-I)Ay_n\|\to 0 \;\text{ as }\; n\to\infty \tag{25}$$

and also

$$\|A^*(J_\lambda^{B_2}-I)Ay_n\|\to 0 \;\text{ as }\; n\to\infty. \tag{26}$$

Similarly, from (23) and our assumptions, we get

$$
\begin{aligned}
&\|x_n - z_n\|^2 \\
&= \frac{1}{\beta_n\delta_n}\big\{\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (1-\alpha_n)\delta_n\theta_n^2\|x_n - x_{n-1}\|^2 \\
&\quad + 2\big(1-\alpha_n(1-\mu^*(1-\xi))\big)\delta_n\theta_n\|x_n - x_{n-1}\|\|x_n - p\| \\
&\quad + \alpha_n\big[\big(\alpha_n(1+\mu^{*2}) - 2(1-\mu^*(1-\alpha_n))\big)\|x_n - p\|^2 + 2\langle f_n(p) - p, x_{n+1} - p\rangle\big]\big\} \\
&\leq \frac{1}{\epsilon_1\epsilon_2}\big\{\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\big[\delta_n(1-\alpha_n)\alpha_n^2\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\
&\quad + 2\delta_n\big(1-\alpha_n(1-\mu^*(1-\xi))\big)\theta_n\|x_n - p\|\big] + \alpha_n\big[2\langle f_n(p) - p, x_{n+1} - p\rangle \\
&\quad + \big(\alpha_n(1+\mu^{*2}) - 2(1-\mu^*(1-\alpha_n))\big)\|x_n - p\|^2\big]\big\} \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Therefore, we have

$$\|x_n - z_n\| \to 0 \text{ as } n \to \infty. \tag{27}$$

By the condition (C2) and (27), we get

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n f_n(x_n) + \beta_n x_n + \delta_n z_n - x_n\| \\
&\leq \alpha_n\|f_n(x_n) - x_n\| + \delta_n\|x_n - z_n\| \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Thus we have

$$\|x_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + \|x_n - z_n\| \to 0 \text{ as } n \to \infty.$$

Since $J_\lambda^{B_1}$ is firmly nonexpansive, we have

$$
\begin{aligned}
&\|u_n - p\|^2 \\
&= \|J_\lambda^{B_1}(y_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ay_n) - J_\lambda^{B_1}p\|^2 \\
&\leq \langle u_n - p, y_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ay_n - p\rangle \\
&= \frac{1}{2}\big(\|u_n - p\|^2 + \|y_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ay_n - p\|^2 - \|u_n - y_n - \gamma_n A^*(J_\lambda^{B_2} - I)Ay_n\|^2\big) \\
&= \frac{1}{2}\big(\|u_n - p\|^2 + \|y_n - p\|^2 + \gamma_n^2\|A^*(J_\lambda^{B_2} - I)Ay_n\|^2 + 2\langle y_n - p, \gamma_n A^*(J_\lambda^{B_2} - I)Ay_n\rangle \\
&\quad - \|u_n - y_n\|^2 - \gamma_n^2\|A^*(J_\lambda^{B_2} - I)Ay_n\|^2 + 2\langle u_n - y_n, \gamma_n A^*(J_\lambda^{B_2} - I)Ay_n\rangle\big) \\
&\leq \frac{1}{2}\big(\|y_n - p\|^2 + \|y_n - p\|^2 - \|u_n - y_n\|^2 + 2\langle u_n - p, \gamma_n A^*(J_\lambda^{B_2} - I)Ay_n\rangle\big) \\
&\leq \frac{1}{2}\big(2\|y_n - p\|^2 - \|u_n - y_n\|^2 + 2\gamma_n\|u_n - p\|\|A^*(J_\lambda^{B_2} - I)Ay_n\|\big) \\
&\leq \|y_n - p\|^2 - \frac{1}{2}\|u_n - y_n\|^2 + \gamma_n\|u_n - p\|\|A^*(J_\lambda^{B_2} - I)Ay_n\|
\end{aligned}
$$

or

$$\|u_n - y_n\|^2 \leq 2\big(\|y_n - p\|^2 - \|u_n - p\|^2 + \gamma_n\|u_n - p\|\|A^*(J_\lambda^{B_2} - 1)Ay_n\|\big). \tag{28}$$

From (28), (16), (24) and (26) and our assumptions, it follows that

$$\begin{aligned}\|u_n - y_n\|^2 \leq & 2\big[\|x_n - p\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|x_n - p\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\ & + \frac{1}{(1-\xi)\delta_n}\big(\alpha_n\|f_n(x_n) - p\|^2 + (\beta_n + \xi\delta_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2\big) \\ & + \gamma_n\|u_n - p\|\|A^*(J_\lambda^{B_2} - 1)Ay_n\|\big] \\ = & 2\big[\frac{1}{(1-\xi)\epsilon_2}\big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\big) + \gamma_n\|u_n - p\|\|A^*(J_\lambda^{B_2} - 1)Ay_n\| \\ & + \frac{\alpha_n}{(1-\xi)\epsilon_2}\big(\|f_n(x_n) - p\|^2 - \|x_n - p\|^2 \\ & + \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\big(2\|x_n - p\| + \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\big)\big)\big] \to 0 \text{ as } n \to \infty,\end{aligned}$$

that is, we have

$$\|u_n - y_n\| \to 0 \text{ as } n \to \infty. \tag{29}$$

From $y_n := x_n + \theta_n(x_n - x_{n-1})$, we get

$$\|y_n - x_n\| = \|x_n + \theta_n(x_n - x_{n-1}) - x_n\| = \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|,$$

which, with the condition (C4), implies that

$$\|y_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{30}$$

In addition, using (27), (29) and (30), we obtain

$$\begin{aligned}\|z_n - u_n\| &\leq \|u_n - y_n\| + \|y_n - z_n\| \\ &\leq \|u_n - y_n\| + \|y_n - x_n\| + \|x_n - z_n\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

From $z_n := \xi v_n + (1-\xi)u_n$, we get

$$\|v_n - u_n\| = \frac{1}{\xi}\|z_n - u_n\| \to 0 \text{ as } n \to \infty. \tag{31}$$

Thus, by (29)–(31), we also get

$$\begin{aligned}\|x_n - v_n\| &\leq \|x_n - u_n\| + \|u_n - v_n\| \\ &\leq \|x_n - y_n\| + \|y_n - u_n\| + \|u_n - v_n\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Therefore, we have

$$d(x_n, Sx_n) \leq \|x_n - v_n\| \to 0 \text{ as } n \to \infty. \tag{32}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in H_1$ and, consequently, $\{u_{n_k}\}$ and $\{y_{n_k}\}$ converge weakly to the point x^*.

From (32), Lemma 4 and the demiclosedness principle for a multi-valued mapping S at the origin, we get $x^* \in Sx^*$, which implies that

$$x^* \in F(S).$$

Next, we show that $x^* \in \Omega$. Let $(v, z) \in G(B_1)$, that is, $z \in B_1(v)$. On the other hand, $u_{n_k} = J_\lambda^{B_1}(y_{n_k} + \gamma_{n_k}A^*(J_\lambda^{B_2} - I)Ay_{n_k})$ can be written as

$$y_{n_k} + \gamma_{n_k}A^*(J_\lambda^{B_1} - I)Ay_{n_k} \in u_{n_k} + \lambda B_1(u_{n_k}),$$

or, equivalently,

$$\frac{(y_{n_k} - u_{n_K}) + \gamma_{n_k} A^*(J_\lambda^{B_1} - I)Ay_{n_k}}{\lambda} \in B_1(u_{n_k}).$$

Since B_1 is maximal monotone, we get

$$\Big\langle v - u_{n_k}, z - \frac{(y_{n_k} - u_{n_k}) + \gamma_{n_k} A^*(J_\lambda^{B_2} - I)Ay_{n_k}}{\lambda} \Big\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned} \langle v - u_{n_k}, z\rangle &\geq \Big\langle v - u_{n_k}, \frac{(y_{n_k} - u_{n_k}) + \gamma_{n_k} A^*(J_\lambda^{B_2} - I)Ay_{n_k}}{\lambda} \Big\rangle \\ &= \Big\langle v - u_{n_k}, \frac{y_{n_k} - u_{n_k}}{\lambda} \Big\rangle + \Big\langle v - u_{n_k}, \frac{\gamma_{n_k} A^*(J_\lambda^{B_2} - I)Ay_{n_k}}{\lambda} \Big\rangle. \end{aligned} \tag{33}$$

Since $u_{n_k} \rightharpoonup x^*$, we have

$$\lim_{k\to\infty} \langle v - u_{n_k}, z\rangle = \langle v - x^*, z\rangle.$$

By (26) and (29), it follows that (33) becomes $\langle v - x^*, z\rangle \geq 0$, which implies that

$$0 \in B_1(x^*).$$

Moreover, from (29), we know that $\{Ay_{n_k}\}$ converges weakly to Ax^* and, by (25), the fact that $J_\lambda^{B_2}$ is nonexpansive and the demiclosedness principle for a multi-valued mapping, we have

$$0 \in B_2(Ax^*),$$

which implies that $x^* \in \Omega$. Thus $x^* \in F(S) \cap \Omega$. Since $\{f_n(x)\}$ is uniformly convergent on D, we get

$$\begin{aligned} \limsup_{n\to\infty} \langle f_n(p) - p, x_{n+1} - p\rangle &= \limsup_{j\to\infty} \langle f_{n_j}(p) - p, x_{n_j+1} - p\rangle \\ &= \langle f(p) - p, x^* - p\rangle \leq 0. \end{aligned}$$

From (23), we get

$$\begin{aligned} \|x_{n+1} - p\|^2 \leq\ & \big(1 - 2\alpha_n(1 - \mu^*(1 - \alpha_n)) + \alpha_n^2(1 + \mu^{*2})\big)\|x_n - p\|^2 - \beta_n\delta_n\|x_n - z_n\|^2 \\ & + 2\big(1 - \alpha_n(1 - \mu^*(1 - \xi))\big)\delta_n\theta_n\|x_n - x_{n-1}\|\|x_n - p\| \\ & + (1 - \alpha_n)\delta_n\theta_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle f_n(p) - p, x_{n+1} - p\rangle \\ \leq\ & \big(1 - 2\alpha_n(1 - \mu^*)\big)\|x_n - p\|^2 + 2\alpha_n(1 - \mu^*)\frac{\langle f_n(p) - p, x_{n+1} - p\rangle}{1 - \mu^*} \\ & + \alpha_n\Big[\delta_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\big(2\big(1 - \alpha_n(1 - \mu^*(1 - \xi))\big)\|x_n - p\| \\ & + \big((1 - \alpha_n)\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\big) + \alpha_n(1 + \mu^{*2})\|x_n - p\|^2\Big]. \end{aligned}$$

By Lemma 4, we obtain

$$\lim_{n\to\infty} x_n = p.$$

Case 2. Suppose that $\{\|x_n - p\|\}_{n=n_0}^\infty$ is not a monotonically decreasing sequence for some n_0 large enough. Set $\Gamma_n = \|x_n - p\|^2$ and let $\tau : \mathbb{B} \to \mathbb{N}$ be a mapping defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}$$

for all $n \geq n_0$. Obviously, τ is a non-decreasing sequence. Thus we have

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$$

for all $n \geq n_0$. That is, $\|x_{\tau(n)} - p\| \leq \|x_{\tau(n)+1} - p\|$ for all $n \geq n_0$. Thus $\lim_{n\to\infty} \|x_{\tau(n)} - p\|$ exists. As in Case 1, we can show that

$$\lim_{n\to\infty} \|(J_\lambda^{B_2} - I)Ay_{\tau(n)}\| = 0, \quad \lim_{n\to\infty} \|A^*(J_\lambda^{B_2} - I)Ay_{\tau(n)}\| = 0, \tag{34}$$

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0, \quad \lim_{n\to\infty} \|u_{\tau(n)} - x_{\tau(n)}\| = 0, \tag{35}$$

$$\lim_{n\to\infty} \|v_{\tau(n)} - u_{\tau(n)}\| = 0, \quad \lim_{n\to\infty} \|x_{\tau(n)} - v_{\tau(n)}\| = 0. \tag{36}$$

Therefore, we have

$$d(x_{\tau(n)}, Sx_{\tau(n)}) \leq \|x_{\tau(n)} - v_{\tau(n)}\| \to 0 \text{ as } n \to \infty. \tag{37}$$

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence $\{u_{\tau(n)}\}$ of $\{x_{\tau(n)}\}$ that converges weakly to a point $x^* \in H_1$. From $\|u_{\tau(n)} - x_{\tau(n)}\| \to 0$, it follows that $u_{\tau(n)} \rightharpoonup x^* \in H_1$.

Moreover, as in Case 1, we show that $x^* \in F(S) \cap \Omega$. Furthermore, since $\{f_n(x)\}$ is uniformly convergent on $D \subset H_1$, we obtain that

$$\limsup_{n\to\infty} \langle f_{\tau(n)}(p) - p, x_{\tau(n)+1} - p\rangle \leq 0.$$

From (23), we get

$$\begin{aligned}
\|x_{\tau(n)+1} - p\|^2 \leq{} & \big(1 - 2\alpha_{\tau(n)}(1 - \mu^*(1 - \alpha_{\tau(n)})) + \alpha_{\tau(n)}^2(1 + \mu^{*2})\big)\|x_{\tau(n)} - p\|^2 \\
& - \beta_{\tau(n)}\delta_{\tau(n)}\|x_{\tau(n)} - z_{\tau(n)}\|^2 + 2\alpha_{\tau(n)}\langle f_{\tau(n)}(p) - p, x_{\tau(n)+1} - p\rangle \\
& + 2\big(1 - \alpha_{\tau(n)}(1 - \mu^*(1 - \xi))\big)\delta_{\tau(n)}\theta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|\|x_{\tau(n)} - p\| \\
& + (1 - \alpha_{\tau(n)})\delta_{\tau(n)}\theta_{\tau(n)}^2\|x_{\tau(n)} - x_{\tau(n)-1}\|^2 \\
\leq{} & \big(1 - 2\alpha_{\tau(n)}(1 - \mu^*)\big)\|x_{\tau(n)} - p\|^2 + \alpha_{\tau(n)}^2(1 + \mu^{*2})\|x_{\tau(n)} - p\|^2 \\
& + \delta_{\tau(n)}\theta_n\|x_{\tau(n)} - x_{\tau(n)-1}\|\big(2(1 - \alpha_{\tau(n)}(1 - \mu^*))\|x_{\tau(n)} - p\| \\
& + (1 - \alpha_{\tau(n)})\theta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|\big) + 2\alpha_{\tau(n)}\langle f_{\tau(n)}(p) - p, x_{\tau(n)+1} - p\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
2\alpha_{\tau(n)}(1 - \mu^*)\|x_{\tau(n)} - p\|^2 \leq{} & \|x_{\tau(n)} - p\|^2 - \|x_{\tau(n)+1} - p\|^2 + \alpha_{\tau(n)}^2(1 + \mu^{*2})\|x_{\tau(n)} - p\|^2 \\
& + \delta_{\tau(n)}\theta_n\|x_{\tau(n)} - x_{\tau(n)-1}\|\big(2(1 - \alpha_{\tau(n)}(1 - \mu^*))\|x_{\tau(n)} - p\| \\
& + (1 - \alpha_{\tau(n)})\theta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|\big) \\
& + 2\alpha_{\tau(n)}\langle f_{\tau(n)}(p) - p, x_{\tau(n)+1} - p\rangle,
\end{aligned}$$

or

$$\begin{aligned}
2(1 - \mu^*)\|x_{\tau(n)} - p\|^2 \leq{} & \alpha_{\tau(n)}(1 + \mu^{*2})\|x_{\tau(n)} - p\|^2 + 2\langle f_{\tau(n)}(p) - p, x_{\tau(n)+1} - p\rangle \\
& + \delta_{\tau(n)}\frac{\theta_{\tau(n)}}{\alpha_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\|\Big(2(1 - \alpha_{\tau(n)}(1 - \mu^*))\|x_{\tau(n)} - p\| \\
& + (1 - \alpha_{\tau(n)})\alpha_{\tau(n)}\frac{\theta_{\tau(n)}}{\alpha_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\|\Big).
\end{aligned}$$

Thus we have

$$\limsup_{n\to\infty} \|x_{\tau(n)} - p\| \le 0$$

and so

$$\lim_{n\to\infty} \|x_{\tau(n)} - p\| = 0. \tag{38}$$

By (35) and (38), we get

$$\|x_{\tau(n)+1} - p\| \le \|x_{\tau(n)+1} - x_{\tau(n)}\| + \|x_{\tau(n)} - p\| \to 0,\ n \to \infty.$$

Furthermore, for all $n \ge n_0$, it is easy to see that $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ if $n \ne \tau(n)$ (that is, $\tau(n) < n$) because of $\Gamma_j \ge \Gamma_{j+1}$ for $\tau(n) + 1 \le j \le n$. Consequently, it follows that, for all $n \ge n_0$,

$$0 \le \Gamma_n \le \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Therefore, $\lim \Gamma_n = 0$, that is, $\{x_n\}$ converges strongly to the point x^*. This completes the proof. □

Remark 1. [22] *The condition (C4) is easily implemented in numerical results because the value of $\|x_n - x_{n-1}\|$ is known before choosing θ_n. Indeed, we can choose the parameter θ_n such as*

$$\theta_n = \begin{cases} \min\left\{\bar{\omega}, \frac{\omega_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } \|x_n - x_{n-1}\| \ne 0, \\ \bar{\omega}, & \text{otherwise,} \end{cases}$$

where $\{\omega_n\}$ is a positive sequence such that $\omega_n = o(\alpha_n)$. Moreover, in the condition (C4), we can take $\alpha_n = \dfrac{1}{n+1}$, $\bar{\omega} = \dfrac{4}{5}$ and

$$\theta_n = \begin{cases} \min\left\{\bar{\omega}, \frac{\alpha_n^2}{\|x_n - x_{n-1}\|}\right\}, & \text{if } \|x_n - x_{n-1}\| \ne 0, \\ \bar{\omega}, & \text{otherwise,} \end{cases}$$

or

$$\theta_n = \begin{cases} \min\left\{\frac{4}{5}, \frac{1}{(n+1)^2\|x_n - x_{n-1}\|}\right\}, & \text{if } \|x_n - x_{n-1}\| \ne 0, \\ \frac{4}{5}, & \text{otherwise.} \end{cases}$$

If the multi-valued quasi-nonexpansive mapping S in Theorem 1 is a single-valued quasi-nonexpansive mapping, then we obtain the following:

Corollary 1. *Let H_1 and H_2 be two real Hilbert spaces. Suppose that $A : H_1 \to H_2$ is a bounded linear operator with adjoint operator A^*. Let $\{f_n\}$ be a sequence of μ_n-contractions $f_n : H_1 \to H_1$ with $0 < \mu_* \le \mu_n \le \mu^* < 1$ and $\{f_n(x)\}$ be uniformly convergent for any x in a bounded subset D of H_1. Suppose that $S : H_1 \to H_1$ is a single-valued quasi-nonexpansive mapping, $I - S$ is demiclosed at the origin and $F(S) \cap \Omega \ne \emptyset$. For any $x_0, x_1 \in H_1$, let the sequences $\{y_n\}$, $\{u_n\}$, $\{z_n\}$ and $\{x_n\}$ be generated by*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = J_\lambda^{B_1}(y_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ay_n), \\ z_n = \xi S x_n + (1-\xi)u_n, \\ x_{n+1} = \alpha_n f_n(x_n) + \beta_n x_n + \delta_n z_n \end{cases} \tag{39}$$

for each $n \geq 1$, *where* $\xi \in (0,1)$, $\gamma_n := \tau_n \frac{\|(J_\lambda^{B_2}-I)Ay_n\|^2}{\|A^*(J_\lambda^{B_2}-I)Ay_n\|^2}$ *with* $0 < \tau_* \leq \tau_n \leq \tau^* < 1$, $\{\theta_n\} \subset [0,\bar{\omega})$ *for some* $\bar{\omega} > 0$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\} \in (0,1)$ *with* $\alpha_n + \beta_n + \delta_n = 1$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < \epsilon_1 \leq \beta_n$ *and* $0 < \epsilon_2 \leq \delta_n$;
(C4) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Then the sequence $\{x_n\}$ *generated by* (39) *converges strongly to a point* $p \in F(S) \cap \Omega$, *where* $p = P_{F(S)\cap\Omega} f(p)$.

Remark 2. *If* $\theta_n = 0$, *then the iterative scheme (14) in Theorem 1 reduces to the iterative (11).*

4. Applications

In this section, we give some applications of the problem **(SFPIP)** in the variational inequality problem and game theory. First, we introduce variational inequality problem in [34] and game theory (see [35]).

4.1. The Variational Inequality Problem

Let C be a nonempty closed and convex subset of a real Hilbert space H_1. Suppose that an operator $F : H_1 \to H_1$ is monotone.

Now, we consider the following variational inequality problem **(VIP)**:

$$\text{Find a point } \; x^* \in C \; \text{ such that } \; \langle Fx^*, y - x^*\rangle \geq 0 \; \text{ for all } \; y \in C. \tag{40}$$

The solution set of the problem **(VIP)** is denoted by Γ.

Moreover, it is well-known that x^* is a solution of the problem **(VIP)** if and only if x^* is a solution of the problem **(FPP)** [34], that is, for any $\gamma > 0$,

$$x^* = P_C(x^* - \gamma F x^*).$$

The following lemma is extracted from [2,36]. This lemma is used for finding a solution of the split inclusion problem and the variational inequality problem:

Lemma 5. *Let* H_1 *be a real Hilbert space,* $F : H_1 \to H_1$ *be a monotone and L-Lipschitz operator on a nonempty closed and convex subset* C *of* H_1. *For any* $\gamma > 0$, *let* $T = P_C(I - \gamma F(P_C(I - \gamma F)))$. *Then, for any* $y \in \Gamma$ *and* $L\gamma < 1$, *we have*

$$\|Tx - Ty\| \leq \|x - y\|,$$

$I - T$ *is demiclosed at the origin and* $F(T) = \Gamma$.

Now, we apply our Theorem 1, by combining with Lemma 5, to find a solution of the problem **(VIP)**, that is, a point in the set Γ.

let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be maximal monotone mappings defined on H_1 and H_2, respectively, and $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*.

Now, we consider the *split fixed point variational inclusion problem* **(SFPVIP)** as follows:

$$\text{Find a point } \; x^* \in H_1 \; \text{ such that } \; 0 \in B_1(x^*), \;\; x^* \in \Gamma \tag{41}$$

and

$$y^* = Ax^* \in H_2 \quad \text{such that} \quad 0 \in B_2(y^*). \tag{42}$$

Theorem 2. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $\{f_n\}$ be a sequence of μ_n-contractions $f_n : H_1 \to H_1$ with $0 < \mu_* \leq \mu_n \leq \mu^* < 1$ and $\{f_n(x)\}$ be uniformly convergent for any x in a bounded subset D of H_1. For any $\lambda > 0$, let $T = P_C(I - \gamma F(P_C(I - \gamma F)))$ with $L\gamma < 1$, where $F : H_1 \to H_1$ is a L-Lipschitz and monotone operator on $C \subset H_1$ and $F(T) \cap \Omega \neq \emptyset$. For any $x_0, x_1 \in H_1$, let the sequences $\{y_n\}$, $\{u_n\}$, $\{z_n\}$ and $\{x_n\}$ be generated by*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = J_\lambda^{B_1}(y_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ay_n), \\ z_n = \xi T x_n + (1 - \xi)u_n, \\ x_{n+1} = \alpha_n f_n(x_n) + \beta_n x_n + \delta_n z_n \end{cases} \tag{43}$$

for each $n \geq 1$, where $\xi \in (0,1)$, $\gamma_n := \tau_n \frac{\|(J_\lambda^{B_2} - I)Ay_n\|^2}{\|A^*(J_\lambda^{B_2} - I)Ay_n\|^2}$ with $0 < \tau_* \leq \tau_n \leq \tau^* < 1$, $\{\theta_n\} \subset [0, \bar{\omega})$ for some $\bar{\omega} > 0$ and $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\} \in (0,1)$ with $\alpha_n + \beta_n + \delta_n = 1$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < \epsilon_1 \leq \beta_n, 0 < \epsilon_2 \leq \delta_n$;
(C4) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$.

Then the sequence $\{x_n\}$ generated by (43) converges strongly to a point $p \in F(T) \cap \Omega = \Gamma \cap \Omega$, where $p = P_{\Gamma\cap\Omega} f(p)$.

Proof. Since $I - T$ is demiclosed at the origin and $F(T) = \Gamma$, by using Lemma (5) and Corollary (1), the sequence $\{x_n\}$ converges strongly to a point $p \in F(T) \cap \Omega$, that is, the sequence $\{x_n\}$ converges strongly to a point $p \in \Gamma$. □

4.2. Game Theory

Now, we consider a game of N players in strategic form

$$G = (p_i, S_i),$$

where $i = 1, \cdots, N$, $p_i : S = S_1 \times S_2 \times \cdots \times S_N \to \mathbb{R}$ is the pay-off function (continuous) of the ith player and $S_i \in \mathbb{R}^{M_i}$ is the set of strategy of the ith player such that $M_i = |S_i|$.

Let S_i be nonempty compact and convex set, $s_i \in S_i$ be the strategy of the ith player and $s = (s_1, s_2, \cdots, s_N)$ be the collective strategy of all players. For any $s \in S$ and $z_i \in S_i$ of the ith player for each i, the symbols S_{-i}, s_{-i} and (z_i, s_{-i}) are defined by

- $S_{-i} := (S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_N)$ is the set of strategies of the remaining players when s_i was chosen by ith player,
- $s_{-i} := (s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_N)$ is the strategies of the remaining players when ith player has s_i and
- $(z_i, s_{-i}) := (s_1, \cdots, s_{i-1}, z_i, s_{i+1}, \cdots, s_N)$ is the strategies of the situation that z_i was chosen by ith player when the rest of the remaining players have chosen s_{-i}.

Moreover, $\bar{s}_i$ is a special strategy of the ith player, supporting the player to maximize his pay-off, which equivalent to the following:

$$p_i(\bar{s}_i, s_{-i}) = \max_{z_i \in S_i} p_i(z_i, s_{-i}).$$

Definition 5. [37,38] *Given a game of N players in strategic form, the collective strategies $s^* \in S$ is said to be a Nash equilibrium point if*

$$p_i(s^*) = \max_{z_i \in S_i} p_i(z_i, s_i^*)$$

for all $i = 1, \cdots, N$ and $s_i^* \in S_{-i}$.

If no player can change his strategy to bring advantages, then the collective strategies $s^* = (s_i^*, s_{-i}^*)$ is a Nash equilibrium point. Furthermore, a Nash equilibrium point is the collective strategies of all players, i.e., s_i^* (for each $i \geq 1$) is the best response of ith player. There is a multi-valued mapping $T_i : S_{-i} \to 2^{S_i}$ such that

$$\begin{aligned} T_i(s_{-i}) &= \arg\max p_i(z_i, s_{-i}) \\ &= \{s_i \in S_i : p_i(s_i, s_{-i}) = \max_{z_i \in S_i} p_i(z_i, s_{-i})\} \end{aligned}$$

for all $s_{-i} \in S_{-i}$. Therefore, we can define the mapping $T : S \to 2^S$ by

$$T := T_1 \times T_2 \times \cdots \times T_N$$

such that the Nash equilibrium point is the collective strategies s^*, where $s^* \in F(T)$. Note that $s^* \in F(T)$ is equivalent to $s_i^* \in T(s_{-i}^*)$.

Let H_1 and H_2 be two real Hilbert spaces, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued mappings. Suppose S is nonempty compact and convex subset of $H_1 = \mathbb{R}^{M_N}$, $H_2 = \mathbb{R}$ and the rest of the players have made their best responses s_{-i}^*. For each $s \in S$, define a mapping $A : S \to H_2$ by

$$As = p_i(s) - p_i(z_i, s_{-i}^*),$$

where p_i is linear, bounded and convex. Indeed, A is also linear, bounded and convex.

The *Nash equilibrium problem* **(NEP)** is the following:

$$\text{Find a point } \; s^* \in S \; \text{ such that } \; As^* > 0, \;\; 0 \in H_2. \tag{44}$$

However, the solution to the problem **(NEP)** may not be single-valued. Then the problem **(NEP)** reduces to finding the fixed point problem **(FPP)** of a multi-valued mapping, i.e.,

$$\text{Find a point } \; s^* \in S \; \text{ such that } \; s^* \in Ts^*, \tag{45}$$

where T is multi-valued pay-off function.

Now, we apply our Theorem 1 to find a solution to the problem **(FPP)**.

Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be maximal monotone mappings defined on H_1 and H_2, respectively, and $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*.

Now, we consider the following problem:

$$\text{Find a point } \; s^* \in H_1 \; \text{ such that } \; 0 \in B_1(s^*), \;\; s^* \in Ts^* \tag{46}$$

and

$$y^* = As^* \in H_2 \; \text{ such that } \; 0 \in B_2(y^*). \tag{47}$$

Theorem 3. *Assume that B_1 and B_2 are maximal monotone mappings defined on Hilbert spaces H_1 and H_2, respectively. Let $T : S \to CB(S)$ be a multi-valued quasi-nonexpansive mapping such that T is demiclosed at the origin. Let $\{f_n\}$ be a sequence of μ_n-contractions $f_n : H_1 \to H_1$ with $0 < \mu_* \leq \mu_n \leq \mu^* < 1$ and $\{f_n(x)\}$ be uniformly convergent for any x in a bounded subset D of H_1. Suppose that the problem* **(NEP)** *has* *a nonempty solution and* $F(T) \cap \Omega \neq \emptyset$*. For arbitrarily chosen* $x_0, x_1 \in H_1$*, let the sequences* $\{y_n\}$*,* $\{u_n\}$*,* $\{z_n\}$ *and* $\{x_n\}$ *be generated by*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = J_\lambda^{B_1}(y_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ay_n), \\ z_n = \xi v_n + (1-\xi)u_n, \quad v_n \in Tx_n, \\ x_{n+1} = \alpha_n f_n(x_n) + \beta_n x_n + \delta_n z_n \end{cases} \tag{48}$$

for each $n \geq 1$*, where* $\xi \in (0,1)$*,* $\gamma_n := \tau_n \frac{\|(J_\lambda^{B_2} - I)Ay_n\|^2}{\|A^*(J_\lambda^{B_2} - I)Ay_n\|^2}$ *with* $0 < \tau_* \leq \tau_n \leq \tau^* < 1$*,* $\{\theta_n\} \subset [0, \bar{\omega})$ *for some* $\bar{\omega} > 0$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\} \in (0,1)$ *with* $\alpha_n + \beta_n + \delta_n = 1$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $0 < \epsilon_1 \leq \beta_n$ *and* $0 < \epsilon_2 \leq \delta_n$;

(C4) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Then the sequence $\{x_n\}$ *generated by Equation* (48) *converges strongly to Nash equilibrium point.*

Proof. By Theorem 1, the sequence $\{x_n\}$ converges strongly to a point $p \in F(T) \cap \Omega$, then the sequence $\{x_n\}$ converges strongly to a Nash equilibrium point. □

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