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Article

Asymmetric Orlicz Radial Bodies

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Abstract: Based on the L_p-harmonic radial combination, Li and Wang researched the asymmetric L_p-harmonic radial bodies, which belong to the asymmetric L_p-Brunn-Minkowski theory initiated by Ludwig, Haberl and Schuster. In this paper, combined with Orlicz radial combination, we introduce the asymmetric Orlicz radial bodies and research their properties. Further, we also establish some inequalities for this concept.

Keywords: star body; Orlicz radial combination; asymmetric Orlicz radial body

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1. Introduction

The classical Brunn-Minkowski theory, also known as the mixed volume theory, has been thought to be the core of modern (convex) geometry. Many significant results such as the Brunn-Minkowski inequality and the Minkowski inequality play a significant role in attacking problems in geometry, random matrices and many other fields. In the mid-1990s, Lutwak [1,2] generalized this theory and brought L_p-Brunn-Minkowski theory and its dual version to fruition. We refer the reader to the excellent treatises by Gardner [3] and Schneider [4] for more details.

For star bodies K, L, real number $p \geq 1$ and $\alpha, \beta \geq 0$ (not both zero), Lutwak [2] proposed the L_p-harmonic radial combination $\alpha \cdot K \tilde{+}_{-p} \beta \cdot L$ as follows:

$$\rho(\alpha \cdot K \tilde{+}_{-p} \beta \cdot L, u)^{-p} = \alpha \rho(K, u)^{-p} + \beta \rho(L, u)^{-p},\ u \in S^{n-1},$$

where "$\tilde{+}_{-p}$" denotes L_p-harmonic radial addition, and "$\cdot$" denotes L_p-harmonic radial multiplication. In addition, S^{n-1} and ρ denote the unit sphere in Euclidean space $\mathbb{R}^n$ and radial function, respectively.

In 2017, Li and Wang [5] researched asymmetric L_p-harmonic radial bodies by introducing parameter τ, their work belongs to the new and rapidly evolving asymmetric L_p-Brunn-Minkowski theory. In fact, as an important content in convex geometry, this theory has its origins in the works of Ludwig, Haberl and Schuster [6–11], and was further developed in many articles (see [12–21]).

As a further extension of classical Brunn-Minkowski theory, Orlicz-Brunn-Minkowski theory is the latest development, which originated from the groundbreaking works of Lutwak, Yang and Zhang [22,23]. The lack of homogeneity in this theory makes the corresponding Orlicz addition of convex bodies hard to present (note that previous additions in the L_p or classical case are homogeneous). It was not until 2014 that this obstacle was overcome by Gardner et al. in [24], where a general framework is introduced for Orlicz-Brunn-Minkowski theory that contains both the new additions and previously proposed concepts, and makes clear for the first time the relation to Orlicz spaces and norms. Meanwhile, the dual Orlicz-Brunn-Minkowski theory for star bodies has received considerable attention. Ye [25] developed the basic setting for the dual Orlicz-Brunn-Minkowski theory and thereafter, Gardner, Hug and Weil [26] and independently Zhu, Zhou and Xu [27], introduced dual theory. More literature advancing the Orlicz-Brunn-Minkowski theory, can be found in, e.g., [28–41].

Let Φ denote the set of all convex and strictly decreasing functions $\phi : (0, \infty) \to (0, \infty)$ such that $\lim_{t\to 0} \phi(t) = \infty$ and $\lim_{t\to+\infty} \phi(t) = 0$. Zhu, Zhou and Xu ([27]) have proposed the Orlicz radial combination as follows:

Definition 1. *For $K, L \in \mathcal{S}_o^n$, $\phi \in \Phi$ and $\alpha, \beta \geq 0$ (not both zero), the Orlicz radial combination, $\alpha \cdot K \tilde{+}_\phi \beta \cdot L$, of K and L is defined by*

$$\rho_{\alpha\cdot K\tilde{+}_\phi\beta\cdot L}(u) = \sup\left\{t > 0 : \alpha\phi\left(\frac{\rho_K(u)}{t}\right) + \beta\phi\left(\frac{\rho_L(u)}{t}\right) \leq \phi(1)\right\}, \tag{1}$$

for any $u \in S^{n-1}$.

It is easy to check that $\phi(t) = t^{-p}$ with $p \geq 1$ yields Lutwak's L_p-harmonic radial combination. Let $\alpha = \beta = \frac{1}{2}$ and $L = -K$ in (1). We define the Orlicz radial body $\tilde{\Delta}_\phi K$ of $K \in \mathcal{S}_o^n$ by

$$\tilde{\Delta}_\phi K = \frac{1}{2} \cdot K \tilde{+}_\phi \frac{1}{2} \cdot (-K). \tag{2}$$

Clearly, $\tilde{\Delta}_\phi K$ is an origin-symmetric star body; we use $\tilde{\Delta}^*_\phi K$ to denote its polar body.

In this paper, based on (1), we research the asymmetric geometric bodies of the Orlicz version and define the following asymmetric Orlicz radial body.

Definition 2. *For $K \in \mathcal{S}_o^n$, $\phi \in \Phi$ and $\tau \in [-1, 1]$, the asymmetric Orlicz radial body, $\tilde{\Delta}^\tau_\phi K \in \mathcal{S}_o^n$, of K is defined by*

$$\rho(\tilde{\Delta}^\tau_\phi K, u) = \sup\left\{t > 0 : f_1(\tau)\phi\left(\frac{\rho_K(u)}{t}\right) + f_2(\tau)\phi\left(\frac{\rho_{-K}(u)}{t}\right) \leq \phi(1)\right\}, \tag{3}$$

for any $u \in S^{n-1}$, i.e.,

$$\tilde{\Delta}^\tau_\phi K = f_1(\tau) \cdot K \tilde{+}_\phi f_2(\tau) \cdot (-K). \tag{4}$$

Here, the functions $f_1(\tau)$, $f_2(\tau)$ are given by

$$f_1(\tau) = \frac{(1+\tau)^2}{2(1+\tau^2)}, \quad f_2(\tau) = \frac{(1-\tau)^2}{2(1+\tau^2)}. \tag{5}$$

From this, we can easily see that

$$f_1(\tau) + f_2(\tau) = 1, \tag{6}$$

$$f_1(-\tau) = f_2(\tau), \quad f_2(-\tau) = f_1(\tau). \tag{7}$$

From (2), (4) and (5), we have $\tilde{\Delta}^0_\phi K = \tilde{\Delta}_\phi K$, $\tilde{\Delta}^{+1}_\phi K = K$ and $\tilde{\Delta}^{-1}_\phi K = -K$.

In particular, when $\phi(t) = t^{-p}$ with $p \geq 1$, (3) deduces the asymmetric L_p-harmonic radial bodies (see [5]).

Let $V(K)$ denote the n dimensional volume of a body K. For the asymmetric Orlicz radial bodies, we research some properties and establish several inequalities. First of all, we use the following Orlicz-Brunn-Minkowski inequality to compare the volumes of body K and its asymmetric Orlicz radial body $\tilde{\Delta}^\tau_\phi K$.

Lemma 1. *For $K, L \in S_o^n$ and $\alpha, \beta \geq 0$ ([27]). If $\phi \in \Phi$, then*

$$\alpha\phi\left(\left(\frac{V(K)}{V(\alpha \cdot K \tilde{+}_\phi \beta \cdot L)}\right)^{\frac{1}{n}}\right) + \beta\phi\left(\left(\frac{V(L)}{V(\alpha \cdot K \tilde{+}_\phi \beta \cdot L)}\right)^{\frac{1}{n}}\right) \leq \phi(1), \tag{8}$$

with equality if and only if K and L are dilates.

This, together with (4) and (6), yields the following conclusion:

Corollary 1. *Let $K \in \mathcal{S}_o^n$, $\phi \in \Phi$ and $\tau \in [-1,1]$, we then have*

$$V(\tilde{\Delta}_\phi^\tau K) \leq V(K), \tag{9}$$

when $\tau \neq \pm 1$ and equality holds if and only if K is an origin-symmetric star body.

Next, in order to establish the reverse inequality, we consider the following case. Let Ψ denote the class of strictly increasing, convex functions ψ: $[0,\infty) \to [0,\infty)$. For the function $\psi(t) \in \Psi$, we define function $\varphi(t) = \psi(\frac{1}{t})$ with $t > 0$ and use $\tilde{\Phi}$ to denote the set of all such functions. That is to say,

$$\tilde{\Phi} = \{\varphi : (0,\infty) \to (0,\infty) : \varphi(t) = \psi(\frac{1}{t}), \psi(t) \in \Psi\}. \tag{10}$$

Obviously, $\tilde{\Phi}$ is a set of strictly decreasing, strictly convex functions and the relationship between Φ and $\tilde{\Phi}$ is $\tilde{\Phi} \subset \Phi$. When $\varphi \in \tilde{\Phi}$, the definition of Orlicz radial body $\tilde{\Delta}_\varphi^\tau K$ is the same as (4) where ϕ is replaced by φ. For the polar bodies of $\tilde{\Delta}_\varphi^\tau K$ and $\tilde{\Delta}_\varphi K$, we write $\tilde{\Delta}_\varphi^{\tau,*} K$ and $\tilde{\Delta}_\varphi^* K$, respectively. For simplicity, we always suppose ψ is strictly convex in our discussion.

Corollary 2. *If $K \in \mathcal{K}_o^n$, $\varphi \in \tilde{\Phi}$ and $\varphi(1) = 1$, then for $\tau \in [-1,1]$,*

$$V(\tilde{\Delta}_\varphi^{\tau,*} K) \geq V(K^*), \tag{11}$$

when $\tau \neq \pm 1$ and equality holds if and only if K is an origin-symmetric convex body.

This shows the volume relationship of the polar bodies for K and $\tilde{\Delta}_\varphi^\tau K$. Finally, with the help of the famous Blaschke-Santaló inequality established by Schneider, we give a new Santaló-type inequality.

Lemma 2. *If $L \in \mathcal{K}_c^n$, then*

$$V(L)V(L^*) \leq \omega_n^2, \tag{12}$$

with equality if and only if L is an ellipsoid. Here, ω_n denotes the volume of the standard unit ball ([4]).

Corollary 3. *If $K \in \mathcal{K}_o^n$, $\varphi \in \tilde{\Phi}$ and $\varphi(1) = 1$, then*

$$V(\tilde{\Delta}_\varphi K)V(\tilde{\Delta}_\varphi K^*) \leq \omega_n^2, \tag{13}$$

and equality holds if and only if K is an ellipsoid centered at the origin.

We would like to mention that research on various geometric bodies plays a central role in the Orlicz-Brunn-Minkowski theory. The more remarkable contributions related to geometric bodies can be found in articles ([42–50]). Inspired by the works of the asymmetric L_p-Brunn-Minkowski theory (e.g., [16–19]), we can do research on the extremum problems and Busemann-Petty problems for asymmetric Orlicz radial bodies, which will enrich and further develop the asymmetric theory. In this paper, we focus first on the basic properties and extremal inequalities.

2. Preliminaries

For a compact, star-shaped (about the origin) K in Euclidean space $\mathbb{R}^n$, its radial function $\rho(K,\cdot)$ is defined by [3]

$$\rho(K,u) = \rho_K(u) = \max\{c \geq 0 : cu \in K\},\ u \in S^{n-1}.$$

If $\rho_K(u)$ is positive and continuous, K will be called a star body (about the origin). Two star bodies K and L are dilates if $\rho(K,u)/\rho(L,u)$ is independent of $u \in S^{n-1}$. For the set of star bodies with respected to the origin in $\mathbb{R}^n$ and the set of origin-symmetric star bodies, we write $\mathcal{S}_o^n$ and $\mathcal{S}_{os}^n$, respectively. For $\Lambda \in GL(n)$, we have [3]

$$\rho(\Lambda K, u) = \rho(K, \Lambda^{-1}u),\ u \in S^{n-1}, \tag{14}$$

where $GL(n)$ denotes the group of general linear transformations.

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors in $\mathbb{R}^n$). For the set of convex bodies containing the origin in their interiors and the set of convex bodies whose centroids lie at the origin, we write $\mathcal{K}_o^n$ and $\mathcal{K}_c^n$, respectively.

For the convex body $K \in \mathcal{K}^n$, its support function $h(K,\cdot)$ is defined by [3]

$$h_K(x) = h(K,x) = \max\{x \cdot y : y \in K\},\ x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y. From this, one has

$$h(-K,x) = h(K,-x). \tag{15}$$

If $K \subseteq \mathbb{R}^n$ is nonempty, the polar set K^* of K is defined by [3]

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \le 1, y \in K\}.$$

For $K \in \mathcal{K}_o^n$, we have $(K^*)^* = K$ and

$$\rho_{K^*} = 1/h_K,\quad h_{K^*} = 1/\rho_K. \tag{16}$$

For $K, L \in \mathcal{K}_o^n$, $\psi \in \Psi$ and real $\alpha, \beta \ge 0$ (not both zero), Xi et al. [37] have defined the Orlicz combination $M_\psi(\alpha,\beta;K,L)$ by

$$h_{M_\psi(\alpha,\beta;K,L)}(u) = \inf\left\{t > 0 : \alpha\psi(\frac{h_K(u)}{t}) + \beta\psi(\frac{h_L(u)}{t}) \le 1\right\},\ u \in S^{n-1}. \tag{17}$$

They also deduced the Orlicz-Brunn-Minkowski inequality. For $K, L \in \mathcal{K}_o^n$ and $\alpha, \beta > 0$. If $\psi \in \Psi$, then

$$\alpha\psi\left(\frac{V(K)^{\frac{1}{n}}}{V(M_\psi(\alpha,\beta;K,L))^{\frac{1}{n}}}\right) + \beta\psi\left(\frac{V(L)^{\frac{1}{n}}}{V(M_\psi(\alpha,\beta;K,L))^{\frac{1}{n}}}\right) \le 1, \tag{18}$$

and equality holds if K and L are dilates. When ψ is strictly convex, equality holds if and only if K and L are dilates.

The following result actually reflects the relationship between Orlicz combination and Orlicz radial combination (also called polar dual relationship and which plays a crucial role for our work).

Lemma 3. *For $K, L \in \mathcal{K}_o^n$, $\alpha, \beta > 0$, $\varphi \in \tilde{\Phi}$ and $\varphi(1) = 1$, one has*

$$(\alpha \cdot K \tilde{+}_\varphi \beta \cdot L)^* = M_\psi(\alpha,\beta;K^*,L^*). \tag{19}$$

Proof. It follows from (10) that $\psi(t) = \varphi(\frac{1}{t})$. Since $\varphi \in \tilde{\Phi} \subset \Phi$, this together with (1), (16) and (17) yields that

$$\begin{aligned}\rho(\alpha \cdot K \tilde{+}_\varphi \beta \cdot L, u)^{-1} &= \left[\sup\left\{t > 0 : \alpha\varphi\left(\frac{\rho_K(u)}{t}\right) + \beta\varphi\left(\frac{\rho_L(u)}{t}\right) \le \varphi(1)\right\}\right]^{-1} \\ &= \inf\left\{\frac{1}{t} > 0 : \alpha\psi\left(\frac{t}{\rho_K(u)}\right) + \beta\psi\left(\frac{t}{\rho_L(u)}\right) \le 1\right\}\end{aligned}$$

$$= \inf\left\{t > 0 : \alpha\psi\left(\frac{h_{K^*}(u)}{t}\right) + \beta\psi\left(\frac{h_{L^*}(u)}{t}\right) \leq 1\right\}$$
$$= h(M_\psi(\alpha, \beta; K^*, L^*), u).$$

By (16), one gets (19). □

3. Properties of Asymmetric Orlicz Radial Bodies

In this section, we research some basic properties of asymmetric Orlicz radial bodies. First of all, we demonstrate the transformation property as follows.

Theorem 1. *For $K \in \mathcal{S}_o^n$, $\phi \in \Phi$ and $\tau \in [-1, 1]$, if $\Lambda \in GL(n)$,then*

$$\tilde{\Delta}_\phi^\tau \Lambda K = \Lambda \tilde{\Delta}_\phi^\tau K. \tag{20}$$

Proof. From (14) and (3), it follows that for all $u \in S^{n-1}$,

$$\rho(\tilde{\Delta}_\phi^\tau \Lambda K, u) = \sup\left\{t > 0 : f_1(\tau)\phi\left(\frac{\rho_{\Lambda K}(u)}{t}\right) + f_2(\tau)\phi\left(\frac{\rho_{-\Lambda K}(u)}{t}\right) \leq \phi(1)\right\}$$
$$= \sup\left\{t > 0 : f_1(\tau)\phi\left(\frac{\rho_K(\Lambda^{-1}u)}{t}\right) + f_2(\tau)\phi\left(\frac{\rho_{-K}(\Lambda^{-1}u)}{t}\right) \leq \phi(1)\right\}$$
$$= \rho(\tilde{\Delta}_\phi^\tau K, \Lambda^{-1}u)$$
$$= \rho(\Lambda \tilde{\Delta}_\phi^\tau K, u).$$

This gives (20). □

Next, we give two important conclusions for the parameter τ.

Theorem 2. *If $K \in \mathcal{S}_o^n$, $\phi \in \Phi$, and $\tau \in [-1, 1]$, then*

$$\tilde{\Delta}_\phi^{-\tau} K = \tilde{\Delta}_\phi^\tau(-K) = -\tilde{\Delta}_\phi^\tau K. \tag{21}$$

Proof. From (4) and (7), we have

$$\tilde{\Delta}_\phi^{-\tau} K = f_1(-\tau) \cdot K \tilde{+}_\phi f_2(-\tau) \cdot (-K)$$
$$= f_1(\tau) \cdot (-K) \tilde{+}_\phi f_2(\tau) \cdot K$$
$$= \tilde{\Delta}_\phi^\tau(-K).$$

This is just one side of (21). Furthermore, together with (14) and (3), we have

$$\rho(-\tilde{\Delta}_\phi^\tau K, u) = \rho(\tilde{\Delta}_\phi^\tau K, -u)$$
$$= \sup\left\{t > 0 : f_1(\tau)\phi\left(\frac{\rho_K(-u)}{t}\right) + f_2(\tau)\phi\left(\frac{\rho_{-K}(-u)}{t}\right) \leq \phi(1)\right\}$$
$$= \sup\left\{t > 0 : f_1(\tau)\phi\left(\frac{\rho_{-K}(u)}{t}\right) + f_2(\tau)\phi\left(\frac{\rho_K(u)}{t}\right) \leq \phi(1)\right\}$$
$$= \rho(\tilde{\Delta}_\phi^\tau(-K), u).$$

This yields the other side of (21). □

Theorem 3. *If $K \in \mathcal{S}_o^n$, $\phi \in \Phi$, $\tau \in [-1,1]$ and $\tau \neq 0$, then $\tilde{\Delta}_\phi^\tau K = \tilde{\Delta}_\phi^{-\tau} K$ if and only if K is an origin-symmetric star body.*

Proof. If K is an origin-symmetric star body, i.e., $K = -K$, the left-hand equality of (21) implies $\tilde{\Delta}_\phi^\tau K = \tilde{\Delta}_\phi^{-\tau} K$.

On the other hand, (3) can be equivalently transformed into

$$\rho(\tilde{\Delta}_\phi^\tau K, u) = t$$

if and only if

$$f_1(\tau)\phi\left(\frac{\rho_K(u)}{t}\right) + f_2(\tau)\phi\left(\frac{\rho_{-K}(u)}{t}\right) = \phi(1). \tag{22}$$

Thus, when $\tilde{\Delta}_\phi^\tau K = \tilde{\Delta}_\phi^{-\tau} K$, (22) and (7) mean

$$[f_1(\tau) - f_2(\tau)]\phi\left(\frac{\rho_K(u)}{t}\right) = [f_1(\tau) - f_2(\tau)]\phi\left(\frac{\rho_{-K}(u)}{t}\right).$$

It follows from (5) that $f_1(\tau) - f_2(\tau) \neq 0$ when $\tau \neq 0$. This together with the monotonicity of function ϕ indicates that K is an origin-symmetric star body. □

A direct result of Theorem 3 can be stated as follows.

Corollary 4. *For $K \in \mathcal{S}_o^n$, $\phi \in \Phi$ and $\tau \in [-1,1]$, if K is not an origin-symmetric star body, then $\tilde{\Delta}_\phi^\tau K = \tilde{\Delta}_\phi^{-\tau} K$ if and only if $\tau = 0$.*

Specifically, for the case of $K \in \mathcal{S}_{os}^n$, we have following facts.

Theorem 4. *If $K \in \mathcal{S}_{os}^n$, $\phi \in \Phi$ and $\tau \in [-1,1]$, then $\tilde{\Delta}_\phi^\tau K = K$.*

Proof. Since $K \in \mathcal{S}_{os}^n$, i.e., $K = -K$, by (3) and (6) we obtain

$$\rho(\tilde{\Delta}_\phi^\tau K, u) = \sup\left\{t > 0 : \phi\left(\frac{\rho_K(u)}{t}\right) \leq \phi(1)\right\} = \rho(K, u),$$

for all $u \in S^{n-1}$. This gets the desired result. □

Corollary 5. *For $K, L \in \mathcal{S}_{os}^n$, $\phi \in \Phi$ and $\tau \in [-1,1]$, then $\tilde{\Delta}_\phi^\tau K = \tilde{\Delta}_\phi^\tau L$ if and only if $K = L$.*

4. Proofs of Corollaries

In this part, we complete the proofs of extremum inequalities separately. The first one is easy to demonstrate, but others are not.

Proof of Corollary 1. According to (4), (6) and (8), we know that

$$\begin{aligned}\phi(1) &\geq f_1(\tau)\phi\left(\left(\frac{V(K)}{V(\tilde{\Delta}_\phi^\tau K)}\right)^{\frac{1}{n}}\right) + f_2(\tau)\phi\left(\left(\frac{V(-K)}{V(\tilde{\Delta}_\phi^\tau K)}\right)^{\frac{1}{n}}\right) \\ &= \phi\left(\left(\frac{V(K)}{V(\tilde{\Delta}_\phi^\tau K)}\right)^{\frac{1}{n}}\right).\end{aligned}$$

By the monotonicity of function ϕ, one can get the desired result (9). Obviously, when $\tau \neq \pm 1$, equality holds if and only if K is an origin-symmetric star body. □

By the crucial polar dual relationship (19), we prove (11) as follows.

Proof of Corollary 2. Since $\varphi \in \tilde{\Phi} \subset \Phi$, it follows from (4) and (19) that

$$\begin{aligned}\tilde{\Delta}_{\varphi}^{\tau,*}K &= (f_1(\tau)\cdot K \tilde{+}_{\varphi} f_2(\tau)\cdot(-K))^* \\ &= M_{\psi}(f_1(\tau), f_2(\tau); K^*, (-K)^*). \end{aligned} \tag{23}$$

By (14)–(16), one has, for all $u \in S^{n-1}$,

$$h((-K)^*, u) = \frac{1}{\rho(-K, u)} = \frac{1}{\rho(K, -u)} = h(K^*, -u) = h(-K^*, u).$$

This means $-K^* = (-K)^*$. Hence, (23) can be rewritten as

$$\tilde{\Delta}_{\varphi}^{\tau,*}K = M_{\psi}(f_1(\tau), f_2(\tau); K^*, -K^*).$$

This together with (18) and (6) implies

$$\begin{aligned} 1 &\geq f_1(\tau)\psi\left(\left(\frac{V(K^*)}{V(\tilde{\Delta}_{\varphi}^{\tau,*}K)}\right)^{\frac{1}{n}}\right) + f_2(\tau)\psi\left(\left(\frac{V(-K^*)}{V(\tilde{\Delta}_{\varphi}^{\tau,*}K)}\right)^{\frac{1}{n}}\right) \\ &= \psi\left(\left(\frac{V(K^*)}{V(\tilde{\Delta}_{\varphi}^{\tau,*}K)}\right)^{\frac{1}{n}}\right). \end{aligned}$$

Since $\varphi(1) = 1$ if and only if $\psi(1) = 1$, combined with the monotonicity of function ψ, we can get (11). Obviously, when $\tau \neq \pm 1$, the supposition that ψ is strictly convex implies that equality holds if and only if K is an origin-symmetric convex body. □

The Santaló-type inequality (13) can be proved as follows.

Proof of Corollary 3. Since $\varphi \in \tilde{\Phi} \subset \Phi$, we replace K by the polar body K^* in (9) and apply (11). Then

$$V(\tilde{\Delta}_{\varphi}^{\tau}K^*) \leq V(K^*) \leq V(\tilde{\Delta}_{\varphi}^{\tau,*}K).$$

Taking $\tau = 0$, it follows that

$$V(\tilde{\Delta}_{\varphi}K^*) \leq V(\tilde{\Delta}_{\varphi}^{*}K). \tag{24}$$

From (9) and (11), we know that equality holds if and only if K is an origin-symmetric convex body. □

The fact that $\tilde{\Delta}_{\varphi}K$ is an origin-symmetric convex body means $\tilde{\Delta}_{\varphi}K \in \mathcal{K}_c^n$. It follows from the Blaschke-Santaló inequality (12) that

$$V(\tilde{\Delta}_{\varphi}K)V(\tilde{\Delta}_{\varphi}^{*}K) \leq \omega_n^2, \tag{25}$$

with equality if and only if $\tilde{\Delta}_{\varphi}K$ is an ellipsoid centered at the origin.

From (24) and (25), it is easy to obtain the desired result (13). When K is an origin-symmetric convex body, (2) implies $\tilde{\Delta}_{\varphi}K = K$. Hence, equality holds in (13) if and only if K is an ellipsoid centered at the origin.

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