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Satisfying Bank Capital Requirements: A Robustness Approach in a Modified Roy Safety-First Framework

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Abstract: This study considers an asset-liability optimization model based on constraint robustness with the chance constraint of capital to risk assets ratio in a safety-first framework under the condition that only moment information is known. This paper aims to extend the proposed single-objective capital to risk assets ratio chance constrained optimization model in the literature by considering the multi-objective constraint robustness approach in a modified safety-first framework. To solve the optimization model, we develop a deterministic convex counterpart of the capital to risk assets ratio robust probability constraint. In a consolidated risk measure of variance and safety-first framework, the proposed distributionally-robust capital to risk asset ratio chance-constrained optimization model guarantees banks will meet the capital requirements of Basel III with a likelihood of 95% irrespective of changes in the future market value of assets. Even under the worst-case scenario, i.e., when loans default, our proposed capital to risk asset ratio chance-constrained optimization model meets the minimum total requirements of Basel III. The practical implications of the findings of this study are that the model, when applied, will provide safety against extreme losses while maximizing returns and minimizing risk, which is prudent in this post-financial crisis regime.

Keywords: robust optimization; capital to risk asset ratio; chance constraint; safety-first principle; Basel III; capital requirements

1. Introduction

Undoubtedly, after the Global Financial Crisis (GFC) of 2007–2008, the spotlight on capital requirement heightened. The Global Financial Crisis was caused by “sub-prime” housing loans in the form of mortgage-backed securities. Numerous determinants for the Global Financial Crisis have been proposed, with various weights assigned by researchers [1]. The Financial Crisis Inquiry Commission stated that the Global Financial Crisis was avertable and its root cause was “widespread failures in financial regulation and supervision...”. A persistent deficit of bank capital and a 25% dip in private investment on average comprise some of the aftermaths of GFC [2]. In tackling the problems and loopholes in financial regulations unveiled by the GFC, Basel III was proposed. Its main objective is to bolster bank capital requirement by expanding bank liquidity and lessening bank leverage. The changes in Basel III include a meaningful surge in the Capital to Risk (weighted) Assets Ratio (CRAR) [3].

The utilization of CRAR safeguards depositors and improves the efficiency and stability of financial frameworks. In [4], a CRAR chance-constrained optimization model was proposed to guarantee that a bank can cope with the capital requirements of Basel III with a probability of 95%, irrespective of the changes in the future market value of assets. The proposed model considered loans

having truncated Gaussian distributed returns, which allowed reformulating the chance constraint related to capital requirements in a second-order cone condition. For the purpose of completeness, the CRAR chance constraint is re-introduced:

$$\mathbb{P} \left\{ y^0(x) + y(x)^T \zeta \leq 0 \right\} \geq \alpha, \tag{1}$$

where $y^0(x) = TL - DR - Rr - LP - M \sum_{k=u+1}^{u+v} (1 + \mathbf{R}_k)x_k$ and $y(x) = M \sum_{k=1}^u (\lambda \omega_k - 1)x_k$. TL is the bank's total liability, and M is the bank's total asset amount. For the purpose of this study, Tier 1 capital (core capital) consists of shareholders' equity and disclosed reserves, DR . Tier 2 (supplementary capital) consists of revaluation reserves, Rr , and general loan loss provisions, LP . Denote $\mathbf{R} = [\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_u, \mathbf{R}_{u+1}, \dots, \mathbf{R}_{u+v}]^T$ as the vector of the annual interest rate of loans and the treasury bill, fixed assets, and non-interest earning assets (riskless). $\Omega = [\zeta^T, \xi^T]^T$ is the vector of assets with $\zeta = [\zeta_1, \zeta_2, \dots, \zeta_u]^T$ and $\xi = [\xi_1, \xi_2, \dots, \xi_v]^T$ corresponding to loans and riskless assets (treasury bill, fixed assets, and non-interest earning assets), respectively. ζ constitutes uncertain parameters that can be estimated, and ξ is a deterministic vector of $[1 + \mathbf{R}_{u+1}, 1 + \mathbf{R}_{u+2}, \dots, 1 + \mathbf{R}_{u+v}]^T$. The Basel III total capital requirement ratio is denoted as λ ; ω_k is the k^{th} asset's weight factor; and α is the safety factor. Denote $x = [x_1, x_2, \dots, x_u, x_{u+1}, \dots, x_{u+v}]^T$ as the vector of asset allocation or investment proportion, which is the decision variable.

The work in [4] assumed that the full and accurate probability distribution of the random vector ζ is known, given estimations from historical data and information from the literature. However, one might have only partial information about the probability distribution: its moment information. Therefore, replacing an unknown distribution with a particular distribution might lead to an over-optimistic solution, resulting in an unsatisfactory chance constraint under the true or actual distribution of random vector ζ . The work in [5] stated that a more difficult challenge that arises is the need to commit to a particular distribution of random vector ζ given only restricted information about the stochastic parameter. To avoid the difficulty of selecting a proper distribution and uncertainty surrounding it, the work in [6] explored the distributionally-robust optimization approach. In this approach, after defining a set \mathcal{P} of possible probability distributions that are assumed to include the true probability distribution \mathcal{D} of random vector ζ , the optimization problem is reconstructed with respect to the worst case expected function over the selection of the probability distribution in this set. Uncertainty in parameter ζ is described through uncertainty sets that contain many possible values realized for random vector ζ . When the uncertainty set is characterized by statistical estimates of the mean and covariance, the work in [7] provided a sufficient condition to guarantee the satisfaction of the constraint with distribution uncertainty at a specified confidence level.

A natural way to tackle a chance constraint against parameter uncertainty is to use the constraint robustness approach. In particular, the distributionally-robust CRAR chance constraint can be expressed as:

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{P} \{ y^0(x) + y(x)^T \zeta \leq 0 \} \geq \alpha, \tag{2}$$

where \mathcal{P} denotes the set of all probability distributions that are consistent with the first and second moments of the probability distribution of the random vector, ζ . Whenever x satisfies (2) and $\mathcal{D} \in \mathcal{P}$ is the true distribution, x satisfies the chance constraint (1) under true probability distribution \mathcal{D} . The work in [8] revealed that contrary to the stochastic programming approach, the distributionally robust chance constraint reflects investors' risk and aversion towards exposure to uncertainty about the probability distribution of the outcomes via consideration of the worst probability distribution within \mathcal{P} . Thus, this study aims to close the research gap by employing the distributionally-robust approach as a way to avoid the difficulty of selecting a proper distribution and uncertainty surrounding the random vector in the framework of meeting capital requirements.

Empirical evidence indicates that the failures of several banks during the GFC kindled concern about maintaining the excessive risk-taking behaviour of banks. Thus, employing variance-Roy's safety-first risk measure as a way of minimizing risk while providing a safety net against extreme

losses is reasonable and worth investigating. Therefore, we employed the modified and improved Roy's safety-first principle investigated by [9]. It is important to note that appropriately modelling risk and meeting capital requirements among other objectives are important for financial stability and that an economy with an efficient financial market structure develops faster.

This paper aims to extend the proposed single-objective CRAR chance constrained optimization model in [4] by considering the multi-objective constraint robustness approach in a modified safety-first framework. This paper also considers credit risk and the expected value of the portfolio. In solving the model, Section 3 introduces steps in constructing a deterministic convex counterpart of robust the probability constraint (2).

In summary, this paper considers a multi-objective distributionally-robust chance-constrained model for capital adequacy. A deterministic equivalent of the robust chance constraint is developed, and computational results are provided to suggest that the model is effective at generating capital allocation decisions even under the worst case realizations (i.e., default) of the debt instruments considered.

The structure of this study is organized as follows: in the next sections, literature review on optimization under uncertainty is discussed, and problem definition and assumption are presented. Section 4 provides the model formulation and approach, and the next section presents the development of the model. Numerical examples and computational results of our method are shown in Section 6. The last section concludes the paper.

2. Literature Review: Optimization under Uncertainty

Dependent on objectives, constraints, and decision variables, the literature on deterministic programming models categorizes problems as linear programming [10], non-linear programming [11], and integer programming [12], among others. However, real-life data are usually not certain, and some methods have been proposed for treating such parameter uncertainty. One conventional approach is sensitivity analysis [13], which deals with uncertainty after finding the optimal solution.

Other frameworks that explicitly incorporate uncertainty into the computation of the optimal solution are stochastic programming, dynamic programming, and robust optimization [14]. Although the above-mentioned methods overlap, they have unfolded independently of each other. Stochastic programming incorporates stochastic components into the programming framework. The method represents uncertain data by scenarios via for example, Monte Carlo sampling, and simple average approximation. Dynamic programming deals with stochastic uncertain systems in a multi-stage framework. It is a technique more widely utilized in derivative pricing as it tackles problems with uncertain coefficients over multiple horizons. In recent times, robust optimization method is a widely acceptable approach in tackling uncertainty. Robust optimization models uncertainty by using a certain membership (uncertainty sets that are based on statistical estimates and probabilistic guarantees on the solution) and optimizes the worst possible case of the problem. When the uncertain parameters are known within certain bounds, robust optimization is best suited [14].

Let us consider a general stochastic programming problem:

$$\text{maximize } f(x) = \text{maximize } \mathbb{E}[F(x, \zeta)], \quad (3)$$

where the expectation is taken over ζ . Here, the objective function $F(x, \zeta)$ is dependent on decision variable x and uncertain parameter ζ . The objective function is well defined, as it is optimized on the average. An important question often asked is what if the uncertainty resides in the constraints [15]. One approach is to formulate such problems similarly by incorporating penalties for constraint violations [9]. An alternative approach also employed in this study is to require that the constraints are satisfied for all possible values of the uncertain parameters with a high probability. Contemporary work in robust optimization has resulted in defining and specifying uncertainty sets to guarantee that chance constraints are satisfied with a targeted probability, thus providing a connection between stochastic

programming and robust optimization. For the purpose of this study, the theory of chance-constrained models is explored further.

The chance-constrained stochastic optimization method is one of the major approaches to solving optimization problems under uncertainty. It ensures that an individual constraint is satisfied with a target probability. Mainly, it restricts the feasible region so that a solution is obtained at a high probability. Chance-constrained programming was first investigated by [16] to ensure that the optimal solution satisfied constraints at a certain probability or confidence level. Many research works have now delved into more ways of tackling chance-constrained problems and increasing the efficiencies of such optimization problems.

A general chance-constrained programming problem takes the form:

$$\text{maximize} \quad f(x) \tag{4}$$

$$\text{subject to} \quad \mathbb{P}\{g(x, \zeta) \leq 0\} \geq \alpha, \tag{5}$$

$$x \in \Lambda, \tag{6}$$

where x denotes decision variables, Λ denotes a set of all feasible solutions, ζ represents uncertain parameters, and $\alpha \in (0, 1)$ is a desired safety factor chosen by the modeler. The chance constraint ensures that the constraint $g(x, \zeta) \leq 0$ is satisfied with a probability α at least.

Chance-constrained optimization problems are challenging computationally. Even checking the feasibility of a chance constraint is NP-hard, and the feasible region is usually non-convex. It is also difficult to obtain samples to estimate the uncertain parameter’s probability distribution accurately. In practice, assumptions about the probability distribution of the uncertain parameters in a chance-constrained problem need to be made to express the probabilistic constraint (5) in closed form. It is, however, difficult to obtain an equivalent deterministic constraint for most probability constraints.

A more difficult challenge that arises is the need to commit to a particular distribution of the uncertain parameter ζ given only restricted information about the stochastic parameters [5]. To avoid the above difficulties such as a selection of the proper probability distribution of the uncertain parameter, NP-hard feasibility checking, and nonconvexity, approximation methods have been proposed. In general, there exist two kinds of approximation approaches for a chance constraint: the analytical approximation method and the sampling-based method. Given the disadvantages of the sampling-based method such as the use of an empirical distribution of the random samples to model the actual distribution [17] among others, we pursue the analytical approximation approach. The analytical approximation method formulates the chance constraint into an equivalent deterministic counterpart. Robust optimization presents a way to approximate analytically a chance constraint. This technique requires a mild assumption on the probability distribution of the uncertain parameters and provides a tractable and feasible solution to the chance-constrained problem. Research contributions using the framework of robust counterpart optimization were explored by [18,19].

A natural way to tackle a chance constraint against parameter uncertainty, which is itself characterized by an uncertain probability distribution, is to use the Distributionally-Robust Chance-Constrained (DRCC) approach, a variant of distributionally-robust optimization. Distributionally-robust optimization is an approach that bridges the gap between robust optimization and stochastic programming [20]. In particular, the distributionally-robust chance constraint can be expressed as:

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{P}\{g(x, \zeta) \leq 0\} \geq \alpha, \tag{7}$$

where \mathcal{P} represents a set of all probability distributions that is in line with the characteristic properties of the true probability \mathcal{D} such as moment information or its support [21]. Whenever x satisfies (7) and $\mathcal{D} \in \mathcal{P}$ is the true distribution, x satisfies the chance constraint (5) under the true probability distribution \mathcal{D} .

In the DRCC paradigm, the distribution of ζ is not exactly known, but rather assumed to belong to a given set \mathcal{P} . In other words, Equation (7) requires that for all probability distributions of ζ , the chance constraint holds. In the DRCC framework, the work in [22] investigated safe tractable approximations of chance-constrained affinely-perturbed linear matrix inequalities. The work in [7] showed that in some cases of a linear chance constraint problem, the worst case moment expression could be analytically expressed. Based on S-lemma, the work in [23] showed that a distributionally-robust chance constraint is tractable when $f(x, \zeta)$ is linear in the decision variable x and piecewise linear or quadratic in the uncertainty parameter ζ . In this study, to obtain a well-posed optimization problem without assuming full knowledge of the probability measure, in moment-based optimization, a distributionally-robust counterpart to a defined chance constraint of capital requirement is considered to guarantee satisfying bank capital requirements.

3. Problem Definition and Assumption: Chance Constraint with an Unknown Distribution

The proposed asset-liability optimization model is based on constraint robustness with the chance constraint of capital to risk assets ratio in a safety-first framework under the condition that only moment information is known. This paper aims to extend the proposed single-objective capital to the risk asset ratio chance-constrained optimization model in [4] by considering the multi-objective constraint robustness approach in a modified safety-first framework.

The following assumptions were made to develop the model: First, the set of all probability distributions have known first and second moments. Second, the set of probability distributions were assumed to include the true probability distribution of the random vector.

Motivated by [7], this study examines an aspect of a chance-constrained robust problem with known first and second moments. For the purpose of completeness, a reintroduction of the description of the notations and parameters is needed.

Notation and Parameter Description

Without loss of generality, define the single generic constraint as:

$$\mathbb{P} \left\{ y^0(x) + y(x)^T \zeta \leq 0 \right\} \geq \alpha, \alpha \in (0, 1) \tag{8}$$

and define the random vector:

$$r = [1 \quad \zeta^T]^T \in \mathbb{R}^{h+1}$$

and:

$$\begin{aligned} \hat{r} &= E\{r\} = E\{[1 \quad \zeta^T]^T\} = [1 \quad \hat{\zeta}^T]^T \\ \Gamma &= var\{r\} = var\{[1 \quad \zeta^T]\} \end{aligned}$$

Consider $v \leq h + 1$ as the rank of Γ and $\Gamma_{f,r} \in \mathbb{R}^{h+1}$ as a full rank factor such that $\Gamma = \Gamma_{f,r} \Gamma_{f,r}^T$. Let:

$$\tilde{z} = [y^0(x) \quad y(x)^T]^T \in \mathbb{R}^{h+1}$$

Let us define the quantity:

$$\varphi(z) = r^T \tilde{z}$$

and:

$$\hat{\varphi}(z) = E\{\varphi(z)\} = \hat{r}^T \tilde{z} \tag{9}$$

$$\sigma^2(z) = var\{\varphi(z)\} = \tilde{z}^T \Gamma \tilde{z} \tag{10}$$

The normalized random variable is defined as:

$$\tilde{\varphi}(z) = \frac{\varphi(z) - \hat{\varphi}(z)}{\sigma(z)}$$

Therefore, one can rewrite Constraint (1) as:

$$\mathbb{P}\{r^T \tilde{z} \leq 0\} = \mathbb{P}\{\varphi(z) \leq 0\} = \mathbb{P}\{\tilde{\varphi}(z) \leq -\hat{\varphi}(z)/\sigma(z)\} \geq \alpha.$$

By distributional constraint robustness, the chance constraint $\mathbb{P}\{r^T \tilde{z} \leq 0\} \geq \alpha$ should be robustly enforced by considering the problem:

$$\inf_{r \sim \mathcal{P}} \mathbb{P}\{r^T \tilde{z} \leq 0\} \geq \alpha = \inf_{r \sim (\hat{r}, \Gamma)} \mathbb{P}\{r^T \tilde{z} \leq 0\} \geq \alpha \tag{11}$$

where $r \sim \mathcal{P}$ means that the distribution of r belongs to the family \mathcal{P} with \mathcal{P} having known first and second moments.

4. Model Formulation and Approach

4.1. Formulation

Let us consider the random variable \mathbf{Y} such that $\mathbf{Y}^T \tilde{z} = (2\hat{r}^T - r^T)\tilde{z}$. From the result of [24] on the tight bound Chebyshev inequality [25], the following holds:

$$\begin{aligned} \sup_{r \sim (\hat{r}, \Gamma)} \mathbb{P}\{r^T \tilde{z} > 0\} &= \sup_{r \sim (\hat{r}, \Gamma)} \mathbb{P}\{\mathbf{Y}^T \tilde{z} - \hat{r}^T \tilde{z} > \hat{r}^T \tilde{z}\} \\ &\leq \begin{cases} \frac{1}{1 + \frac{(\hat{r}^T \tilde{z})^2}{\tilde{z}^T \Gamma \tilde{z}}}, & \text{if } \hat{r}^T \tilde{z} \geq 0 \\ 1, & \text{Otherwise.} \end{cases} \end{aligned} \tag{12}$$

Obviously,

$$\begin{aligned} \mathbb{P}\{\mathbf{Y}^T \tilde{z} - \hat{r}^T \tilde{z} > \hat{r}^T \tilde{z}\} &= \mathbb{P}\{\hat{r}^T \tilde{z} - \mathbf{Y}^T \tilde{z} < -\hat{r}^T \tilde{z}\} \\ &= \mathbb{P}\{r^T \tilde{z} - \hat{r}^T \tilde{z} < -\hat{r}^T \tilde{z}\}. \end{aligned}$$

This, combined with (12), implies that:

$$\sup_{r \sim (\hat{r}, \Gamma)} \mathbb{P}\{r^T \tilde{z} - \hat{r}^T \tilde{z} < -\hat{r}^T \tilde{z}\} \leq \frac{\tilde{z}^T \Gamma \tilde{z}}{\tilde{z}^T \Gamma \tilde{z} + (\hat{r}^T \tilde{z})^2}$$

Hence,

$$\frac{\tilde{z}^T \Gamma \tilde{z}}{\tilde{z}^T \Gamma \tilde{z} + (\hat{r}^T \tilde{z})^2} \leq 1 - \alpha$$

is sufficient for Constraint (11) to hold. The expression can be recast in various forms as:

$$\begin{aligned} \tilde{z}^T \Gamma \tilde{z} &\leq (1 - \alpha)(\tilde{z}^T \Gamma \tilde{z} + (\hat{r}^T \tilde{z})^2), \\ \alpha \tilde{z}^T \Gamma \tilde{z} &\leq (1 - \alpha)(\hat{r}^T \tilde{z})^2, \\ \frac{\alpha}{(1 - \alpha)} \tilde{z}^T \Gamma \tilde{z} &\leq (\hat{r}^T \tilde{z})^2, \\ (\hat{r}^T \tilde{z})^2 &\geq \tilde{z}^T \Gamma \tilde{z} \frac{\alpha}{(1 - \alpha)}, \\ \hat{\varphi}^2(z) &\geq \sigma^2(z) \frac{\alpha}{(1 - \alpha)}. \end{aligned}$$

Theorem 1. The chance constraint, $\mathbb{P}\{\hat{r}^T \tilde{z} \leq 0\} \geq \alpha$, for any $\alpha \in (0, 1)$ expressed as a constraint robust term in the form $\inf_{r \sim (\hat{r}, \Gamma)} \mathbb{P}\{r^T \tilde{z} \leq 0\} \geq \alpha$ is equivalent to the second-order cone constraint $(\hat{r}^T \tilde{z})^2 \geq \tilde{z}^T \Gamma \tilde{z} \frac{\alpha}{(1-\alpha)}$.

4.2. CreditMetrics Approach

Future bank capital depends on future market values of assets and liabilities. This paper employs a modified CreditMetrics approach to estimate the future market value of loans, i.e., the uncertainty parameter in (1). A modified CreditMetrics approach has been proposed by [4]. We adopted this approach for the sole aim of determining the future market value of loans and the associated risk measure.

Migration of Ratings

Credit rating transition is the migration of loans across different ratings over a risk period. Credit risk arises from changes in the loan value as a result of upgrades and downgrades. Therefore, it is prudent to evaluate the probability of default and the possibility of migration to other ratings. Table 1 shows a transition matrix developed from historical data by CRISIL (Credit Rating Information Services of India Limited). The likelihood that an A borrower will remain at A over the next year is 71.23%.

Table 1. CRISIL’s one-year mean transition rates (1993–2014)(%). Source: CRISIL Default Study 2014.

Rating	AAA	AA	A	BBB	BB	B	C	D
AAA	98.23	1.54	0.23	0.00	0.00	0.00	0.00	0.00
AA	17.04	78.52	3.70	0.74	0.00	0.00	0.00	0.00
A	9.59	15.07	71.23	4.11	0.00	0.00	0.00	0.00
BBB	4.02	3.29	15.69	76.28	0.00	0.37	0.37	0.00
BB	11.11	22.22	22.22	22.22	22.22	0.00	0.00	0.00
B	0.00	0.00	0.00	0.00	0.00	50.00	0.00	50.00
C	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100

The recovery rate is a measure of the extent to which a creditor recovers the principal and accrued interest due on a defaulted debt [26]. According to Moody’s “Default and Recovery rates for project bank loans, 1983–2014”, the ultimate recovery rates average 80%, which is roughly consistent with existing works of [27] (84.14%), [28] (81.12%), [29] (80%), and [30] (87%). For the purpose of this study, we consider loan recovery rates estimated by Moody’s Investors Service.

4.3. Loan Valuation and Credit Risk

In this section, the year-ahead market value of loans and credit risk are estimated. Reference is made to the approach employed by [4] and followed accordingly. Table 2 contains entries of $(j - 1)$ -year forward rates, $f_{cr}^{1,j}$, which are spot rates from now with credit rating cr .

Table 2. One-year forward zero curve for each credit rating category (%). Adapted from [4] with permission from authors.

Category	Year 1 ($f_{cr}^{1,2}$)	Year 2 ($f_{cr}^{1,3}$)	Year 3 ($f_{cr}^{1,4}$)	Year 4 ($f_{cr}^{1,5}$)
AAA	3.60	4.17	4.73	5.12
AA	3.65	4.22	4.78	5.17
A	3.72	4.32	4.93	5.32
BBB	4.10	4.67	5.25	5.63
BB	5.55	6.02	6.78	7.27
B	6.05	7.02	8.03	8.52
CCC/C	15.05	15.02	14.03	13.52

Please refer to [4] for the mathematical expression of the forward value of a unit capital invested in the k^{th} loan with either a default (ζ_{kdt}) or non-default (ζ_{kt}) migration path, expected forward loan value ($E(\zeta_k)$), and variance of unit capital invested ($Var(\zeta_k)$).

Credit risk arises because loan values can vary depending on the credit quality changes, and so, any reasonable risk measure must reflect this variability. CreditMetrics proposed two measures to characterize credit risk: variance or standard deviation and percentile level [31,32]. The risk measures reflect the portfolio distribution, and both contribute in the effort to quantify risk. The two risk measures also reflect potential losses from the same portfolio distribution. The variance measure reflects how different the expected value of the loans will be from the actual value, and the percentile levels aid in arriving at the unexpected losses of the portfolio.

The specified percentile level is interpreted as the lowest value the loan portfolio will achieve for say 5% of the time: the fifth percentile. Therefore, the likelihood that the true loan portfolio value is less than the calculated fifth percentile level is only 5%. Given the full distribution of loan portfolio values, ζ_{kt} and ζ_{kdt} , one can derive the percentile level. It is important to note that the two credit risk measures (variance and the use of percentile level) give different values and must be interpreted in a different manner. According to the CreditMetrics Technical document [32], the process of estimating credit risk via percentile levels, for e.g. fifth percentile, is as follows: First, the fifth-percentile level number is obtained from the loan value distribution, and second, using the fifth percentile, the amount of credit risk is estimated by taking the difference of the mean portfolio and the fifth percentile level number.

The percentile level approaches often used in the literature are Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR), as they assess risk within the full context of a portfolio. The percentile level is naturally appealing to employ as it shows precisely the likelihood that the portfolio value will fall below that number.

This study in the context of credit risk infers VaR and CVaR from a distribution of the value of the portfolio and not from a distribution of the losses in the portfolio. Credit VaR is defined as the distance from the percentile to the mean of the forward distribution, at the desired confidence level. Credit CVaR is the average distance beyond VaR from the percentile to the mean of the forward distribution, at the desired confidence level. Both reflect (average) unexpected credit loss at the desired confidence level. However, there is another way to look at credit VaR if one considers the distribution of losses: it can also be interpreted as percentile loss itself, i.e., including expected losses.

This paper uses the credit VaR and credit CVaR to replace the market VaR and market CVaR used in a modified safety-first framework described in [9] to avert extreme unexpected credit losses and control the downside risk.

5. Model Development

The asset-liability optimization model based on the chance constraint of capital to the risk (weighted) asset requirement is structured and presented in this section. Objective functions consider maximizing the annual interest rate and expected portfolio value, minimizing credit risk, and providing a safety-net against extreme losses.

Given definitions of input parameters and replacing chance Constraint (1) with its robust constraint counterpart (2) provide the following distributionally-robust chance-constrained problem:

$$\begin{aligned}
 & \underset{x}{\text{maximize}} && \mathbf{R}(x) \\
 & \underset{x}{\text{maximize}} && \mu(x) \\
 & \underset{x}{\text{minimize}} && \sigma^2(x) \\
 & \underset{x}{\text{minimize}} && \frac{(1 - \alpha)\text{creditCVaR}(x)}{\text{creditVaR}(x) - R} \\
 & \text{subject to} && \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{P}\{y^0(x) + y(x)^T \zeta \leq 0\} \geq \alpha \\
 & && \sum_{k=1}^{u+v} x_k = 1 \\
 & && x_1, x_2, \dots, x_{u+v} \geq 0
 \end{aligned} \tag{13}$$

Based on Theorem 1, the robust chance constraint is equivalent to a deterministic convex counterpart. Therefore, the multi-objective robust chance-constrained optimization model is:

$$\begin{aligned}
 & \underset{x}{\text{maximize}} && \mathbf{R}(x) \\
 & \underset{x}{\text{maximize}} && \mu(x) \\
 & \underset{x}{\text{minimize}} && \sigma^2(x) \\
 & \underset{x}{\text{minimize}} && \frac{(1 - \alpha)\text{creditCVaR}(x)}{\text{creditVaR}(x) - R} \\
 & \text{subject to} && \hat{\phi}^2(z) \geq \sigma^2(z) \frac{\alpha}{(1 - \alpha)} \\
 & && \sum_{k=1}^{u+v} x_k = 1 \\
 & && x_1, x_2, \dots, x_{u+v} \geq 0
 \end{aligned} \tag{14}$$

where $\alpha \in (0, 1)$, $\hat{\phi}(z) = \hat{r}^T \tilde{z}$, $\sigma(z) = \tilde{z}^T \Gamma \tilde{z}$, and $\tilde{z} = [y^0(x) \quad y(x)^T]^T$.

Given a portfolio x for u number of loans, the probability of the loss function not exceeding an acceptable threshold δ is:

$$\Xi(x, \delta) = \int_{f(x, \zeta) \leq \delta} p(\zeta) d\zeta. \tag{15}$$

For a confidence level α , the VaR of portfolio x is given by:

$$\text{VaR}_\alpha(x) = \min\{\delta \mid \Xi(x, \delta) \geq \alpha\}. \tag{16}$$

The corresponding CVaR is expressed as conditional expectation of the loss of the portfolio exceeding or equal to VaR, i.e., when all random values are continuous, the following is derived:

$$\begin{aligned}
 \text{CVaR}_\alpha(x) &= \mathbb{E}\{f(x, \zeta) \mid f(x, \zeta) \geq \text{VaR}_\alpha(x)\} \\
 &= \frac{1}{1 - \alpha} \int_{f(x, \zeta) \geq \text{VaR}_\alpha(x)} f(x, \zeta) p(\zeta) d\zeta.
 \end{aligned} \tag{17}$$

The work in [33] proposed an equivalent function for CVaR. They expressed their idea as:

$$\text{CVaR}_\alpha(x) = \min F_\alpha(x, \delta), \tag{18}$$

where:

$$F_\alpha(x, \delta) = \delta + \frac{1}{1 - \alpha} \int_{f(x, \zeta) \geq \delta} (f(x, \zeta) - \delta) p(\zeta) d\zeta.$$

One can find the optimal CVaR by solving the right-hand side of Equation (18). In order to minimize CVaR over x , we minimized the auxiliary function with respect to x and δ :

$$\min_x \text{CVaR}_\alpha(x) = \min_{x, \delta} F_\alpha(x, \delta). \tag{19}$$

The work in [33] presented an approximation to the auxiliary function $F_\alpha(x, \delta)$ via the sampling method:

$$\hat{F}_\alpha(x, \delta) = \delta + \frac{1}{(1 - \alpha)u} \sum_{i=1}^u \max(f(x, \zeta_i) - \delta, 0). \tag{20}$$

Comparing Equation (20) to Equation (18), the problem $\min_x \text{CVaR}_\alpha(x)$ can be approximated by replacing $F_\alpha(x, \delta)$ with $\hat{F}_\alpha(x, \delta)$ in Equation (19):

$$\min_{x, \delta} \delta + \frac{1}{(1 - \alpha)u} \sum_{i=1}^u \max(f(x, \zeta_i) - \delta, 0). \tag{21}$$

To solve this optimization problem, one can replace $\max(f(x, \zeta_i) - \delta, 0)$ with artificial variables \mathbf{z}_i and impose constraints $\mathbf{z}_i \geq f(x, \zeta_i) - \delta$ and $\mathbf{z}_i \geq 0$:

$$\begin{aligned} \min_{x, \mathbf{z}, \delta} \quad & \delta + \frac{1}{(1 - \alpha)u} \sum_{i=1}^u \mathbf{z}_i \\ \text{subject to} \quad & \mathbf{z}_i \geq 0, \quad i = 1, \dots, u, \\ & \mathbf{z}_i \geq f(x_i, \zeta_i) - \delta, \quad i = 1, \dots, u, \\ & x \in \Lambda. \end{aligned} \tag{22}$$

For a portfolio of u loans, we assumed $\mu_i \in \mathbb{R}^u$ is the random vector of the expected returns value of ζ with a probability density function $p(\mu)$. To determine the mean loss of the portfolio, this study defines the loss function as $f(x_i, \mu_i) = -\sum_{i=1}^u x_i \mu_i = -[\mu_1 x_1 + \dots + \mu_u x_u]$. Since the loss function is convex, then the auxiliary function $F_\alpha(x, \delta)$ is also a convex function and can be solved using well-known optimization techniques.

Representing each objective function by $P_1(x), \dots, P_4(x)$, the multi-objective robust chance-constrained optimization problem (23) can be approximated via the approximation technique (22) as shown below:

$$\begin{aligned}
 & \underset{x}{\text{maximize}} && P_1(x) = \mathbf{R}(x) \\
 & \underset{x}{\text{maximize}} && P_2(x) = \mu(x) \\
 & \underset{x}{\text{minimize}} && P_3(x) = \sigma^2(x) \\
 & \underset{x}{\text{minimize}} && P_4(x) = \frac{(1 - \alpha)(\delta + \frac{1}{(1-\alpha)u} \sum_{i=1}^u \mathbf{z}_i)}{\delta - R} \\
 & \text{subject to} && \hat{\phi}^2(z) \geq \sigma^2(z) \frac{\alpha}{(1 - \alpha)}, \\
 & && \delta \geq R, \\
 & && \mathbf{z}_i \geq 0, \quad i = 1, \dots, u, \\
 & && \mathbf{z}_i + x_i^T \mu_i + \delta \geq 0, \quad i = 1, \dots, u, \\
 & && \sum_{k=1}^{u+v} x_k = 1, \\
 & && x_1, x_2, \dots, x_{u+v} \geq 0,
 \end{aligned} \tag{23}$$

where $\alpha \in (0, 1)$, $\hat{\phi}(z) = \hat{r}^T \tilde{z}$, $\sigma^2(z) = \tilde{z}^T \Gamma \tilde{z}$, and $\tilde{z} = [y^0(x) \quad y(x)^T]^T$.

Transformation and Solution to the Multi-objective Model

The multi-objective portfolio optimization model (23) addresses the trade-off between conflicting or competing objectives. This type of problem is referred to as a Polynomial Goal Programming (PGP) problem. The idea behind such an approach is to obtain smaller computable elements of the problem and then find solutions iteratively that meet individual goals.

Generally, there will not be a single solution of Problem (23) that can maximize both Objective 1 and Objective 2. Alternately, the solution of the multi-objective optimization problem (23) has to be obtained in a two-step process. First, individually solve each objective $P_1(x)$, $P_2(x)$, $P_3(x)$, and $P_4(x)$ subject to the constraints. Denote optimal values (desired goals) after solving the individual objectives subject to constraints by P_1^* , P_2^* , P_3^* , and P_4^* . Second, find an optimal solution that preserves individual objectives by minimizing the deviation of each individual objective from the ideal solution. Let d_1, d_2, d_3 , and d_4 be non-negative variables that account for deviation from the desired goals, P_1^* , P_2^* , P_3^* , and P_4^* . The multi-objective optimization problem (23) transforms into a single objective problem in a specific form of the general Minkowski distance defined as $O = \left\{ \sum_{f=1}^4 \left| \frac{d_f^p}{P_f^*} \right| \right\}^{1/p}$, where P_f represents the corresponding desired goal and is used as basis for normalization of the f^{th} variable.

The single objective optimization problem derived via PGP for the multi-objective problem (23) is defined as:

$$\begin{aligned}
 &\underset{x,d}{\text{minimize}} && O(x) = \left| \frac{d_1}{P_1^*} \right| + \left| \frac{d_2}{P_2^*} \right| + \left| \frac{d_3}{P_3^*} \right| + \left| \frac{d_4}{P_4^*} \right| \\
 &\text{subject to} && \mathbf{R}(x) + d_1 = P_1^* \\
 &&& \mu(x) + d_2 = P_2^* \\
 &&& \sigma^2(x) - d_3 = P_3^* \\
 &&& \frac{(1 - \alpha)(\delta + \frac{1}{(1-\alpha)u} \sum_{i=1}^u \mathbf{z}_i)}{\delta - R} d_4 = P_4^* \\
 &&& (\hat{r}^T \tilde{z})^2 \geq \tilde{z}^T \Gamma \tilde{z} \frac{\alpha}{(1 - \alpha)}, \tag{24} \\
 &&& \delta \geq R, \\
 &&& \mathbf{z}_i \geq 0, \quad i = 1, \dots, u, \\
 &&& \mathbf{z}_i + x_i^T \mu_i + \delta \geq 0, \quad i = 1, \dots, u, \\
 &&& \sum_{k=1}^{u+v} x_k = 1 \\
 &&& d_f \geq 0, \quad f = 1, \dots, 4 \\
 &&& x_1, x_2, \dots, x_{u+v} \geq 0
 \end{aligned}$$

where $\alpha \in (0, 1)$ and $\tilde{z} = [y^0(x) \quad y(x)^T]^T$.

Given that the objective function of Problem (24) has a fractional component, an alternative equivalent program can be deduced. Under the assumption that the feasible region is non-empty and bounded, the transformation:

$$w = \frac{1}{\delta - R}, \quad w \geq 0,$$

$$\tilde{x} = xw$$

translates (24) to the equivalent single objective optimization problem via PGP as:

$$\begin{aligned}
 & \underset{\tilde{x}, d, w, z, \delta}{\text{minimize}} && \left| \frac{d_1}{P_1^*} \right| + \left| \frac{d_2}{P_2^*} \right| + \left| \frac{d_3}{P_3^*} \right| + \left| \frac{d_4}{P_4^*} \right| \\
 & \text{subject to} && \mathbf{R}^T \tilde{x} + d_1 = P_1^*, \\
 & && \mu^T \tilde{x} + d_2 = P_2^*, \\
 & && \tilde{x}^T Q \tilde{x} - d_3 = P_3^*, \\
 & && (w - \alpha w) \left(\delta + \frac{1}{(1 - \alpha) \mathbf{1}^T} \sum_{i=1}^{\mathbf{T}} \mathbf{z}_i \right) - d_4 = P_4^*, \\
 & && (\hat{r}^T \tilde{z})^2 \geq \tilde{z}^T \tilde{z} \frac{\alpha}{(1 - \alpha)}, \\
 & && \sum_{k=1}^{u+v} \tilde{x}_k = w, \\
 & && d_f \geq 0, \quad f = 1, \dots, 4, \\
 & && \delta w - R w = 1, \\
 & && w \geq 0, \\
 & && \mathbf{z}_i w \geq 0, \quad i = 1, \dots, u, \\
 & && \mathbf{z}_i w + \tilde{x}_i^T \mu_i + \delta w \geq 0, \quad i = 1, \dots, u, \\
 & && \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{u+v} \geq 0,
 \end{aligned} \tag{25}$$

where Q is covariance matrix of unit capital invested in loans, $\alpha \in (0, 1)$, and $\tilde{z} = [y^0(x) \quad y(x)^T]^T$. One can obtain an optimal solution to the problem by rescaling \tilde{x}^* so that $x^* = \frac{\tilde{x}^*}{w^*}$.

6. Numerical Examples

In this section, the proposed model is subjected to numerical experiments [34] by considering a hypothetical bank operating in the U.S. Loan data are private information and difficult to obtain. This study defines and uses references and trusted sources such as World Bank, Moody’s Investors Service, and CRISIL to back the data used for this section.

6.1. Data

Consider the asset structure of a hypothetical bank in the U.S. Let the bank’s total loan and treasury bill amount, M , be \$600,000 ; the bank’s total liability TL equal \$1,192,000, disclosed reserves DR of \$166,000, revaluation reserves Rr of \$12,000, and general loan loss provisions LP of \$15,000. The financial assets characterizing the financial environment on the bank are five types of loans, a treasury bill, fixed assets, and non-interest earning assets. Table 3 presents the information about the assets and capital funds to be allocated.

Interest rate (\mathbf{R}) for individual loans was set to lending interest rates for the U.S. quoted by the World Bank in 2014 as a reference. World Bank’s one-year treasury bill rate of 0.1% for the U.S. in 2014 was used. The standardized approach of the Basel Accord was a reference point for risk weights. Moody’s recovery rate of 80% was used for recovery rates for loan types. Let us consider correlations among loan borrowers as shown in Table 4. In financial crises, correlations of random assets tend to converge positively, maybe even to one. To represent a market in distress, we consider the correlation matrix in Table 4 to see the effects on capital adequacy.

Table 3. Asset structure of a U.S. Bank (IR, Interest Rate; RR, Recovery Rate; RW, Risk Weights; CV, Credit VaR; CCV, Credit CVaR).

Assets	Collateral	IR (%)	RR (%)	RW (%)	Mean	Variance	CV	CCV	Allocation
(ζ_1) 3-Year AAA Commercial and Industrial Loan	Inventory or account receivables	3.56	80.00	20	1.0243	4.1679×10^{-8}	0.2402	0.2829	0.2550
(ζ_2) 5-Year AA Agriculture and Farm Loan	Equipment, crops, livestock, etc.	4.17	80.00	50	1.0081	1.7196×10^{-5}	0.2558	0.2889	0.1663
(ζ_3) 2-Year BBB Personal Loan	Savings account, tangible property, etc.	3.11	80.00	75	1.0213	4.2935×10^{-4}	0.2323	0.2140	0.2665
(ζ_4) 3-Year B Small Business Loan	Land, savings account, etc	4.03	80.00	75	0.81911	4.0625×10^{-3}	0.0258	0.0689	0.0468
(ζ_5) 4-Year A Auto Loan	Savings account or car itself	4.21	80.00	75	1.0248	4.7805×10^{-5}	0.2512	0.2892	0.2386
1-Year Treasury Bill	Not Applicable	0.1	100	0	1.001	0	0	0	0.0268
Fixed Assets	Not Applicable	0	100	0	1	0	0	0	-
Non-Interest Earning Assets	Not Applicable	0	100	0	1	0	0	0	-

Table 4. Asset correlations among loans of a market in crisis.

Risky Assets	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5
ζ_1	1	0.85	0.8	0.8	0.8
ζ_2	0.85	1	0.9	0.85	0.8
ζ_3	0.8	0.9	1	0.9	0.8
ζ_4	0.8	0.85	0.9	1	0.95
ζ_5	0.8	0.8	0.8	0.95	1

6.1.1. Objective Function

The objective function is filled with various entries of Table 3 and some other computed results. Thus,

$$\begin{aligned}
 & \underset{x}{\text{maximize}} && \mathbf{R}^T x \\
 & \underset{x}{\text{maximize}} && \mu^T x \\
 & \underset{x}{\text{minimize}} && x^T Q x \\
 & \underset{x}{\text{minimize}} && \frac{(1 - \alpha)\text{creditCVaR}(x)}{\text{creditVaR}(x) - R}, \quad \alpha = 0.95 \text{ and } R = 0.009.
 \end{aligned} \tag{26}$$

6.1.2. CRAR Constraint

This study employed the CRAR developed by Basel III as the designated regulation requirement. From the definition of CRAR, i.e.,

$$\frac{M\Omega^T x - TL + DR + Rr + LP}{M(\omega_1 \zeta_1 x_1 + \dots + \omega_u \zeta_u x_u + \omega_{u+1} \zeta_{u+1} x_{u+1} + \dots + \omega_{u+v} \zeta_{u+v} x_{u+v})} \geq \lambda$$

where λ is the total capital requirement. The chance-constrained model based on the capital to risk assets ratio can be expressed as:

$$\mathbb{P} \left\{ y^0(x) + y(x)^T \zeta \leq 0 \right\} \geq \alpha \tag{27}$$

where:

$$\begin{aligned}
 y^0(x) &= TL - DR - Rr - LP - M \sum_{k=u+1}^{u+v} (1 + \mathbf{R}_k) x_k \\
 y(x) &= M \sum_{k=1}^u (\lambda \omega_k - 1) x_k
 \end{aligned}$$

The year-ahead market value of net assets of the bank should be $600,000(\zeta^T x + \zeta_1 x_6) + 707,000 - 1,192,000 + 166,000 + 12,000 + 15,000$ where $600,000(\zeta_1 x_6) + 707,000$ represents the year-ahead market value of riskless assets. In this study, the total capital requirement ratio λ of 11% was used. The constraint based on CRAR is:

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{P} \left(\frac{600,000(\zeta^T x + \zeta_1 x_6) + 707,000 - 1,192,000 + 166,000 + 12,000 + 15,000}{1,500,000(0.2\zeta_1 x_1 + 0.5\zeta_2 x_2 + \dots + 0.75\zeta_5 x_5)} \geq 0.11 \right) \geq 0.95 \tag{28}$$

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{P} \left\{ y^0(x) + y(x)^T \zeta \leq 0 \right\} \geq 0.95 \tag{29}$$

where:

$$\begin{aligned}
 y^0(x) &= 1,192,000 - 166,000 - 12,000 - 15,000 - 600,000(1 + 0.0010)x_6 - 707,000 \\
 y(x)^T &= 600,000((0.11 \cdot 0.2 - 1)x_1 + (0.11 \cdot 0.5 - 1)x_2 + \dots + (0.11 \cdot 0.75 - 1)x_5)
 \end{aligned}$$

According to Theorem 1, Equation (28) is equivalent to:

$$(\hat{p}^T \hat{z})^2 \geq \hat{z}^T \Gamma \hat{z} \begin{pmatrix} 0.95 \\ 0.05 \end{pmatrix} \tag{30}$$

where:

$$\tilde{z} = \begin{bmatrix} 292000 - 604800x_6 \\ -586800x_1 \\ -567000x_2 \\ -550500x_3 \\ -550500x_4 \\ -550500x_5 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4.1679e-8 & 7.1146e-7 & 3.3807e-6 & 1.0403e-5 & 1.1269e-6 \\ 0 & 7.1146e-7 & 1.7196e-5 & 7.6383e-6 & 2.2199e-5 & 2.2632e-5 \\ 0 & 3.3807e-6 & 7.6383e-6 & 4.2935e-4 & 0.0012 & 1.1426e-4 \\ 0 & 1.0403e-5 & 2.2199e-5 & 0.0012 & 4.0625e-3 & 4.1755e-4 \\ 0 & 1.1269e-6 & 2.2632e-5 & 1.1426e-4 & 4.1753e-4 & 4.7805e-5 \end{bmatrix}$$

and:

$$\hat{r}^T = [1 \quad 1.0243 \quad 1.0081 \quad 1.0213 \quad 0.8911 \quad 1.0248]$$

6.1.3. Constraint Based on Other Factors

The proportions of capital allocations must add up to one. Thus, the following holds.

$$\sum_{k=1}^6 x_k = 1 \tag{31}$$

For diversification purposes, management sets up constraints with an upper bound limit of 0.4 with respect to loan allocations:

$$0 \leq x_1, \dots, x_5 \leq 0.4 \tag{32}$$

Banks often allocate proportions to risky investment after considering riskless assets. For the purpose of this study, bank management allocates at most 5% of M to treasury bills.

$$0 \leq x_6 \leq 0.05 \tag{33}$$

The problem is transformed from a multi-objective optimization model to a single-objective model via Polynomial Goal Programming (PGP) as shown in (24). Thus,

$$\begin{aligned}
 & \underset{x,d}{\text{minimize}} && \left| \frac{d_1}{P_1^*} \right| + \left| \frac{d_2}{P_2^*} \right| + \left| \frac{d_3}{P_3^*} \right| + \left| \frac{d_4}{P_4^*} \right| \\
 & \text{subject to} && \mathbf{R}(x) + d_1 = P_1^* \\
 & && \mu(x) + d_2 = P_2^* \\
 & && \sigma^2(x) - d_3 = P_3^* \\
 & && \frac{(1 - \alpha)(\delta + \frac{1}{(1-\alpha)u} \sum_{i=1}^u \mathbf{z}_i)}{\delta - R} d_4 = P_4^* \\
 & && (\hat{r}^T \hat{z})^2 \geq \hat{z}^T \hat{z} \frac{\alpha}{(1 - \alpha)}, \\
 & && \delta \geq R, \\
 & && \mathbf{z}_i \geq 0, \quad i = 1, \dots, 5, \\
 & && \mathbf{z}_i + x_i^T \mu_i + \delta \geq 0, \quad i = 1, \dots, 5, \\
 & && \sum_{k=1}^6 x_k = 1 \\
 & && d_f \geq 0, \quad f = 1, \dots, 4 \\
 & && 0 \leq x_1, \dots, x_5 \leq 0.4 \\
 & && 0 \leq x_6 \leq 0.05
 \end{aligned} \tag{34}$$

where $\alpha = 0.95$, $R = 0.009$, and $\hat{z} = [y^0(x) \ y(x)^T]^T$. The above problem can be solved as shown in (25).

6.2. Results and Remarks

Mathematical computations were executed on a MacBook Pro (Intel(R) Core(TM) i7 @ 2.9 GHz, 16 GB RAM) with MATLAB (2017b).

Column 10 of Table 3 shows the optimal investment of the assets under consideration. The two-year BBB Personal Loan allocation value of 0.2665 is the highest amongst optimal investment proportions. This might be ascribed to its proper trade-off between the objective functions of our model. The high-risk weights are complimented by its high interest rates. The optimal solution minimizes the deviation of each objective from its ideal solution. The distributionally-robust CRAR chance constraint optimization model meets the Basel III Tier 1 capital requirements ratio of equal or more than 6% and also meets the total capital requirements of 11% considered for this study under the guidance of the Basel III capital requirements.

A sensitivity analysis was performed to determine the robustness of the proposed model by computing the CRAR value using loan values under the worst-credit migration path from Table 5. This was done with the dual aim of testing our model under the worst-case scenario, i.e., default and also to test model robustness. The value of CRAR even under the worst-case scenario confirmed that the bank was guaranteed to meet our Basel III total capital requirement of 11%.

To further investigate the results of our distributionally-robust CRAR optimization model, this study explored the CRAR chance constraint with the utilization of the optimal investment proportions and worst credit migration path values of loans. The expected values of the year-ahead market value of loans under the worst-case scenario were assigned to ζ , the optimal allocation proportions to x , and the treasury bill to ξ_1 .

Table 5. Year-ahead or forward market value of loans under default.

Loan Type	Worst-Credit Migration Path	Value
3-Year AAA Commercial and Industrial Loan	AAA → CCC → CCC → D	0.6712
5-Year AA Agriculture and Farm Loan	AA → CCC → CCC → CCC → CCC → D	0.6193
2-Year BBB Personal Loan	BBB → CCC → D	0.7264
3-Year B Small Business Loan	B → CCC → CCC → D	0.6800
4-Year A Auto Loan	AAA → CCC → CCC → CCC → D	0.6501

The findings of this paper explicitly disclose that in a safety-first framework, CRAR values will be greater than the Basel III minimum capital requirement at a confidence level of 95% if the model introduced is employed even if the model introduced is employed under the worst-case scenario. Therefore, the CRAR optimization model does guarantee that banks will cope with the capital requirements of Basel III with a greater likelihood of 95% irrespective of changes in the forward market value of assets. This approach also provides a safety-net against extreme losses and controls credit and capital risk.

6.2.1. Additional Remarks

To further explore our findings, this study subjected our robust approach for comparison to a stochastic model. This research used the same dataset used by [4].

We replaced the equivalent of our robust chance-constraint $((\hat{p}^T \hat{z})^2 \geq \hat{z}^T \hat{z} \frac{\alpha}{(1-\alpha)})$ in (34) with an equivalent term of the stochastic version $(F_{RG}^{-1}(\alpha) \sqrt{\hat{z}^T \hat{z}} + \hat{p}^T \hat{z} \leq 0)$ derived from Theorem 4.1 of [4]. $F_{RG}^{-1}(\alpha)$ was inverse for the standard right truncated Gaussian distribution cumulative probability function. All other parameters had the same definition in this paper as the following Table 6.

Table 6. Performance index of distributionally- and non-distributionally-robust models. CRAR, Capital to Risk Assets Ratio (CRAR).

	Distributionally Robust	Non-distributionally Robust
CRAR	12.2%	11.08%
Worst-case CRAR	8.9%	8.2%
Credit risk (variance)	0.0204	0.0203
Interest rate return	0.0565	0.0546
Portfolio value return	0.7661	0.8559

Under both CRAR chance constrained models, a Tier 1 capital ratio of equal or more than 6%, and the total capital ratio, i.e., CRAR equal to or greater than 11%, considered for this study were met. However, the robust chance constraint model meets the capital adequacy to a higher degree. Using loan values under the worst-credit migration path, the CRAR value of 8.9% was reported for the robust CRAR chance-constrained optimization model. The stochastic CRAR chance-constrained model reported a CRAR value of 8.2%. The managerial implications of the findings of this study are that the distributionally-robust model when applied will meet the capital requirements at a higher rate, maximize the interest rate return at the expense of a smaller portfolio value return while providing a better framework for treating parameter uncertainty.

7. Conclusions

This paper studied the asset-liability model, which is an extension of the single-objective CRAR chance-constrained model [4], by considering the multi-objective constraint robustness approach in a modified safety-first framework. Specifically, this study constructed a distributionally-robust optimization model in a safety-first framework under the capital to risk assets ratio chance constraint with uncertainty set based on the fact that only expectation and second marginal moment information were known. This approach, which is key to practical implications, on the one hand, provides banks

with the guarantee of meeting capital regulation with a great probability of 95%, not only controlling the credit risk but also the capital risk. On the other hand, the approach provides a safety-net against extreme losses.

The proposed distributionally-robust capital to risk asset ratio chance-constrained optimization model guarantees banks will meet the capital requirements of Basel III with a likelihood of 95% irrespective of changes in the future market value of assets. A sensitivity analysis was performed to determine the robustness of the proposed model by computing the CRAR value using loan values under the worst-credit migration path. Even under the worst-case scenario, i.e., when loans default, our proposed capital to risk asset ratio chance-constrained optimization model meets the minimum total requirements of Basel III. The findings of this research are crucial for practitioners as they showcase a coherent manner to aid banks in meeting capital requirements.

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