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On Fixed Point Results in G_b-Metric Spaces

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Abstract: The purpose of this paper is to consider various results in the context of G_b-metric spaces that have been recently published after the paper (Aghajani, A.; Abbas, M.; Roshan, J.R. Common fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces. *Filomat* 2014, *28*, 1087–1101). Our new results improve, complement, unify, enrich and generalize already well known results on G_b-metric spaces. Moreover, some coupled and tripled coincidence point results have been provided.

Keywords: fixed point; coupled tripled; coincidence point; Cauchy sequence; convergent; G_b-metric; b-metric

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1. Introduction and Preliminaries

In the last few decades, fixed point theory has been one of the most significant research fields in nonlinear functional analysis. It has wide applications in many disciplines such as studying the existence of solutions for nonlinear equations (algebraic, differential and integral), a system of linear (nonlinear) equations and convergence of many computational methods, in economics, sports, medical sciences, etc. In 1922, Banach [1] proved that each contraction map in a complete metric space has a unique fixed point. It is worth mentioninng that this Banach contraction principle has been generalized in two directions, by acting on the contraction (expansive) condition, or changing the topology of the space. Among these generalizations are partial metric spaces [2], b-metric spaces [3,4], partial b-metric spaces [5], G-metric spaces [6], G_p-metric spaces [7] and G_b-metric spaces [8]. The topological concepts of these spaces are different, thus the approach of existence of fixed points is different. Many researchers tried to generalize new contractive mappings to demonstrate the existence of fixed point results (for examples, see [9–19]).

Partial metric spaces were introduced by Matthews [2] in 1986 as follows:

Definition 1. *Let X be a nonempty set. A partial metric (or a p-metric) is a function $p : X \times X \to [0, +\infty)$ satisfying*

(p1) $x = y$ *if and only if* $p(x, x) = p(x, y) = p(y, y)$;
(p2) $p(x, x) \leq p(x, y)$, *for all* $x, y \in X$;
(p3) $p(x, y) = p(y, x)$, *for all* $x, y \in X$; *and*
(p4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$, *for all* $x, y, z \in X$.

The pair (X, p) is called a partial metric space.

It is clear that each metric space is a partial metric space. However, the converse is not true, in general. For example, if $X = [0, \infty)$ and $p(x, y) = \max\{x, y\}$. In this case, p is a $p-$metric, but it is not a metric on X. Many authors obtained variant fixed point results in partial metric spaces for different contractive conditions (see [20–23]).

In 1989, Bakhtin [3] and, in 1993, Czerwik [4] introduced a new distance on a non-empty set, which is called a b-metric. A b-metric space is an attempt to generalize the metric space.

Definition 2. *Let X be a non-empty set and $s \geq 1$ a given real number. A function $d : X^2 \to [0, \infty)$ is called a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:*

(b1) $d(x, y) = 0$ *if and only if* $x = y$;
(b2) $d(x, y) = d(y, x)$; *and*
(b3) $d(x, z) \leq s(d(x, y) + d(y, z))$.

The pair (X, d) is called a b-metric space. The real number $s \geq 1$ is called the coefficient of (X, d). We stress that any metric space is a b-metric space with coefficient $s = 1$. Generally, a b-metric space is not a metric space. Let $X = \mathbb{R}$ be the set of real numbers, then the mapping $d : X \times X \to [0, \infty)$ defined by $d(x, y) = (x - y)^2$ is a b-metric, but (X, d) is not a metric space. For some fixed point results on b-metric spaces, see [10,24,25].

In 2014, Shukla [5] introduced the notion of a partial b-metric space.

Definition 3. *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $p_b : X \times X \to [0, \infty)$ is called a partial b-metric if for all $x, y, z \in X$, the following conditions are satisfied:*

(pb1) $x = y$ *if and only if* $p_b(x, x) = p_b(x, y) = p_b(y, y)$;
(pb2) $p_b(x, x) \leq p_b(x, y)$;
(pb3) $p_b(x, y) = p_b(y, x)$; *and*
(pb4) $p_b(x, y) \leq s[p_b(x, z) + p_b(z, y)] - p_b(z, z)$.

The pair (X, p_b) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of (X, p_b).

The following example shows that a partial b-metric on X does not need to be a partial metric or a b-metric on X (see also [5]).

Example 1. *Let $X = [0, \infty)$. Define the function $p_b : X \times X \to [0, \infty)$ by $p_b(x, y) = [\max(x, y)]^2 + |x - y|^2$ for all $x, y \in X$. Then, (X, p_b) is a partial b-metric space on X with the coefficient $s = 2 > 1$. However, p_b is neither a b-metric nor a partial metric on X.*

Mustafa and Sims [6] generalized the concept of metric spaces by introducing new class of spaces, so-called G-metric spaces. Based on this new concept, Mustafa et al. [26] obtained some fixed point results for mappings satisfying different contractive conditions. Abbas and Rhoades [27] initiated the study of common fixed point theorems in $G-$metric spaces. Since then, many authors obtained fixed and common fixed point results in the setup of G-metric spaces (e.g., see [19,28–55]).

Definition 4 ([6]). *Let X be a nonempty set. A generalized metric or a G-metric is a function $G : X^3 \to [0, \infty)$ satisfying the following properties:*

(G1) *for all $x, y, z \in X$, $x = y = z$ if and only if $G(x, y, z) = 0$;*
(G2) *$0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;*
(G3) *$G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$;*
(G4) *$G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, (symmetry in all three variables); and*
(G5) *$G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).*

The pair (X, G) is called a G-metric space.

Example 2 ([6]). *Let $X = \mathbb{R}$. Then, G-metric G be defined by $G(x, y, z) = \frac{1}{3}(|x - y| + |y - z| + |x - z|)$, for all $x, y, z \in \mathbb{R}$.*

Definition 5. *A G-metric G is said to be symmetric if $G(x, x, y) = G(x, y, y)$ for all $x, y \in X$.*

In 2011, Ahmadi et al. [7] attempted to introduce a new generalization of both partial metric spaces and G-metric spaces, by defining the notion of G_p-metric spaces in the following manner.

Definition 6. *Let X be a nonempty set. A mapping $G_p : X^3 \to [0, +\infty)$ is called a G_p-metric if the following conditions are satisfied:*

(G_p1) for $x, y, z \in X$, if $G_p(x, y, z) = G_p(z, z, z) = G_p(y, y, y) = G_p(x, x, x)$ then $x = y = z$;
(G_p2) $G_p(x, x, x) \leq G_p(x, x, y) \leq G_p(x, y, z)$, for all $x, y, z \in X$;
(G_p3) $G_p(x, y, z) = G_p(x, z, y) = G_p(y, z, x) = \ldots$, (symmetry in all three variables); and
(G_p4) $G_p(x, y, z) \leq G_p(x, a, a) + G_p(a, y, z) - G_p(a, a, a)$, for all $x, y, z, a \in X$ (rectangle inequality).

The pair (X, G_p) is called a G_p-metric space.

For more other details with some improvements, see [37].

On the other hand, combining the concepts of G-metrics and b-metrics, Aghajani et al. [8] initiated the concept of G_b-metrics.

Definition 7 ([8]). *Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that $G_b : X^3 \to [0, +\infty)$ satisfies*

(G_b1) $G_b(x, y, z) = 0$ if $x = y = z$;
(G_b2) $0 < G_b(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(G_b3) $G_b(x, x, y) \leq G_b(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
(G_b4) $G_b(x, y, z) = G_b(x, z, y) = G_b(y, z, x) = \ldots$, (symmetry in all three variables); and
(G_b5) $G_b(x, y, z) \leq s(G_b(x, a, a) + G_b(a, y, z))$ for all $x, y, z, a \in X$ (triangle inequality).

Then, G_b is called a generalized b-metric and the pair (X, G_b) is called a generalized b-metric space or G_b-metric space.

Example 3 ([8]). *Let (X, G) be a G-metric space. Consider $G_b(x, y, z) = (G(x, y, z))^p$, where $p > 1$ is a real number. Then, G_b is a G_b-metric with $s = 2^{p-1}$.*

Each G-metric space is a G_b-metric space with $s = 1$. The following example shows that a G_b-metric on X does not need to be a G-metric on X.

Example 4. *The function defined by $G_b(x, y, z) = \frac{1}{9}(|x - y| + |y - z| + |x - z|)^2$ is a G_b-metric on $X = \mathbb{R}$ with $s = 2$, but it is not a G-metric on $\mathbb{R}$. Indeed, by taking $x = 3$, $y = 5$, $z = 7$ and $a = \frac{7}{2}$, we have $G_b(3, 5, 7) = \frac{64}{9}$, $G_b\left(3, \frac{7}{2}, \frac{7}{2}\right) = \frac{1}{9}$. While, $G_b\left(\frac{7}{2}, 5, 7\right) = \frac{49}{9}$, so $G_b(3, 5, 7) = \frac{64}{9} > \frac{50}{9} = G_b\left(3, \frac{7}{2}, \frac{7}{2}\right) + G_b\left(\frac{7}{2}, 5, 7\right)$.*

The following properties are consequences of Definition 7.

Proposition 1 ([8]). *Let (X, G_b) be a G_b-metric space, then for each $x, y, z, a \in X$, we have:*

(1) if $G_b(x,y,z) = 0$, then $x = y = z$;
(2) $G_b(x,y,z) \le s(G_b(x,x,y) + G_b(x,x,z)$;
(3) $G_b(x,y,y) \le 2sG_b(y,x,x)$; and
(4) $G_b(x,y,z) \le s(G_b(x,a,z) + G_b(a,y,z))$.

Proof.
(1) By (G_b3) and (G_b2), we get a contradiction. Indeed,

$$0 = G_b(x,y,z) \ge G_b(x,x,y) > 0,$$

when $x \ne y$ and $y \ne z$. In addition,

$$0 = G_b(x,y,z) = G_b(y,y,z) > 0,$$

when $x = y$ and $y \ne z$.
(2) Properties (G_b4) and (G_b5) imply

$$G_b(x,y,z) = G_b(y,x,z) \le s(G_b(y,x,x) + G_b(x,x,z)) = s(G_b(x,x,y) + G_b(x,x,z)).$$

(3) By (G_b4) and (G_b5), it follows that

$$G_b(x,y,y) = G_b(y,x,y) \le s(G_b(y,x,x) + G_b(x,x,y)) = 2sG_b(y,x,x).$$

(4) By (G_b4), (G_b5) and (G_b3), we get that

$$G_b(x,y,z) \le s(G_b(x,a,a) + G_b(a,y,z)) \le s(G_b(a,x,z) + G_b(a,y,z)),$$

when $x \ne z$. □

Definition 8 ([8]). *A G_b-metric G_b is said to be symmetric if $G_b(x,x,y) = G_b(x,y,y)$ for all $x, y \in X$.*

Remark 1. *From Example 3, it follows that $G_b = (G(x,y,z))^p$, $p > 1$ is symmetric if $G(x,y,z)$ is symmetric.*

Definition 9 ([8]). *Let (X, G_b) be a G_b-metric space, then for $x_0 \in X$, $r > 0$, the G_b-ball with center x_0 and radius r is*

$$B_{G_b}(x_0, r) = \{y \in X : G_b(x_0, y, y) < r\}.$$

Aghajani et al. [8] proved that every G_b-metric space is topologically equivalent to a b-metric space. This allows us to readily transport many concepts and results from b-metric spaces into G_b-metric spaces.

Definition 10 ([8]). *Let X be a G_b-metric space. A sequence $\{x_n\}$ in X is said to be*

(1) G_b-Cauchy sequence if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m, n, l \ge n_0$, $G_b(x_n, x_m, x_l) < \varepsilon$; and
(2) G_b-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m, n \ge n_0$, $G_b(x_n, x_m, x) < \varepsilon$.

Using above definitions, we prove the following two significant propositions.

Proposition 2. *Let (X, G_b) be a G_b-metric space. Then, the following are equivalent:*

(1) the sequence $\{x_n\}$ is G_b-Cauchy; and
(2) for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G_b(x_n, x_m, x_m) < \varepsilon$, for all $m, n \geq n_0$.

Proof. (1) $\Rightarrow$ (2). In (1) of Definition 10, we take $l = m$.

(2) $\Rightarrow$ (1). Let $\varepsilon > 0$ and $\varepsilon_1 = \frac{\varepsilon}{2s}$. By (2), there exists $n_0 \in \mathbb{N}$ such that $G_b(x_n, x_m, x_m) < \varepsilon_1$, for all $m, n \geq n_0$. By (G_b5),

$$G_b(x_n, x_m, x_l) \leq s(G_b(x_n, x_m, x_m) + G_b(x_m, x_m, x_l)) < 2s\varepsilon_1 \leq \varepsilon,$$

for all $m, n, l \geq n_0$. $\square$

Proposition 3. *Let (X, G_b) be a G_b-metric space. Then, the following are equivalent:*

(i) $\{x_n\}$ is G_b-convergent to x;
(ii) $G_b(x_n, x_n, x) \to 0$ as $n \to +\infty$; and
(iii) $G_b(x_n, x, x) \to 0$ as $n \to +\infty$.

Proof. (i) $\Rightarrow$ (ii). In (ii) of Definition 10, we take $m = n$.

(ii) $\Rightarrow$ (iii). Let $\varepsilon > 0$ and $\varepsilon_1 = \frac{\varepsilon}{2s}$. By (ii), there exists $n_0 \in \mathbb{N}$ such that $G_b(x_n, x_n, x) < \varepsilon_1$, for all $n \geq n_0$. Then, by (3) of Proposition 1, we have

$$G_b(x_n, x, x) \leq 2sG_b(x, x_n, x_n) < 2s\varepsilon_1 \leq \varepsilon,$$

for all $m, n \geq n_0$.

(iii) $\Rightarrow$ (i). Let $\varepsilon > 0$ and $\varepsilon_1 = \frac{\varepsilon}{2s}$. By (iii), there exists $n_0 \in \mathbb{N}$ such that $G_b(x_n, x, x) < \varepsilon_1$, for all $n \geq n_0$. By (G_b5), we get

$$G_b(x_n, x_m, x) \leq s(G_b(x_n, x, x) + G_b(x, x_m, x)) < 2s\varepsilon_1 \leq \varepsilon,$$

for all $m, n \geq n_0$. $\square$

Definition 11 ([8])**.** *A G_b-metric space X is called G_b-complete if every G_b-Cauchy sequence is G_b-convergent in X.*

Proposition 4 ([8])**.** *Let (X, G_b) be a G_b-metric space. Then, the function given as $d_{G_b}(x, y) = G_b(x, y, y) + G_b(x, x, y)$, defines a $b-$metric on X. We call it a b-metric induced by the G_b-metric G_b.*

Proof. Let us prove that conditions of Definition 2 are fulfilled for $d_{G_b}(x, y)$.

(b1) If $d_{G_b}(x, y) = 0$ then $G_b(x, x, y) = 0$ and by (1) of Proposition 1, it follows that $x = y$.
If $x = y$, then (by (G_b1)) $G_b(x, x, y) = G_b(x, y, y) = 0$ and $d_{G_b}(x, y) = 0$.

(b2) Property (G_b4) implies that

$$d_{G_b}(x, y) = G_b(x, y, y) + G_b(x, x, y) = G_b(y, x, x) + G_b(y, y, x) = d_{G_b}(y, x).$$

(b3) By (G_b5), it follows that

$$d_{G_b}(x, z) = G_b(x, z, z) + G_b(x, x, z) \leq s(G_b(x, y, y) + G_b(y, z, z)) + s(G_b(z, y, y) +$$

$$G_b(y, x, x))$$

$$s(G_b(x, y, y) + G_b(x, x, y)) + s(G_b(y, z, z) + G_b(y, y, z)) = s(d_{G_b}(x, y) + d_{G_b}(y, z)).$$

$\square$

Theorem 1 ([8], Theorem 2.1). *Let $(X, \preceq)$ be a partially ordered set. Suppose that there exists a symmetric G_b-metric on X such that (X, G) is a complete G_b-metric space. In addition, let $f, g, h, S, T, R : X \to X$ satisfy the following*

$$\psi(2s^4 G(fx, gy, hz)) \le \psi(M_s(x, y, z)) - \varphi(M_s(x, y, z)) \tag{1}$$

for all comparable elements $x, y, z \in X$, where $\varphi, \psi : [0, \infty) \to [0, \infty)$ are such that ψ is continuous, nondecreasing and φ is lower semi-continuous with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$, and

$$M_s(x, y, z) = \max\{G(Rx, Ty, Sz), G(Rx, Ty, gy), G(Ty, Sz, hz), G(Sz, Rx, fx),$$
$$\frac{G(fx, Rx, gy) + G(fx, Sz, hz) + G(gy, Ty, hz)}{3s}\}.$$

If f, g and h are dominated, then S, T and R are also dominating with $fX \subseteq TX, gX \subseteq SX$ and $hX \subseteq RX$, and for a nonincreasing sequence $\{x_n\}$ such that $y_n \preceq x_n$ for each n and $y_n \to u$, then $u \preceq x_n$ and

(a) one of fX, gX or hX is a closed subset of X; and
(b) the pairs $(f, R), (g, T)$ and (h, S) are weakly compatible. Then, f, g, h, S, T and R have a common fixed point in X. Moreover, the set of common fixed points of f, g, h, S, T and R is well ordered if and only if f, g, h, S, T and R have one and only one common fixed point.

Theorem 2 ([42]). *Let (X, G) be a complete symmetric G_b-metric space with parameter $s \ge 1$ and let the mappings $S, T, R : X^2 \to X$ satisfy*

$$\begin{aligned} G(S(x,y), T(u,v), R(a,b)) &\le a_1 \frac{G(x,u,a) + G(y,v,b)}{2} \\ &+ a_2 \frac{G(S(x,y), T(u,v), R(a,b)) G(x,u,a)}{1 + G(x,u,a) + G(y,v,b)} + a_3 \frac{G(S(x,y), T(u,v), R(a,b)) G(y,v,b)}{1 + G(x,u,a) + G(y,v,b)} \\ &+ a_4 \frac{G(x,x,S(x,y)) G(x,u,a)}{1 + G(x,u,a) + G(y,v,b)} + a_5 \frac{G(x,x,S(x,y)) G(y,v,b)}{1 + G(x,u,a) + G(y,v,b)} \\ &+ a_6 \frac{G(u,u,T(u,v)) G(x,u,a)}{1 + G(x,u,a) + G(y,v,b)} + a_7 \frac{G(u,u,T(u,v)) G(y,v,b)}{1 + G(x,u,a) + G(y,v,b)} \\ &+ a_8 \frac{G(a,a,R(a,b)) G(x,u,a)}{1 + G(x,u,a) + G(y,v,b)} + a_9 \frac{G(a,a,R(a,b)) G(y,v,b)}{1 + G(x,u,a) + G(y,v,b)}, \end{aligned} \tag{2}$$

for all $x, y, u, v, a, b \in X$ and $a_1, \dots, a_9 \ge 0$ such that $a_1 + a_2 + a_3 + 2(a_4 + a_5) + a_6 + a_7 + a_8 + a_9 < 1$. Then, S, T and R have a unique common coupled fixed point in X.

Lemma 1 ([56]). *Every sequence $\{x_n\}_{n \in N}$ of elements in a b-metric space (X, d) with a real coefficient $s \ge 1$, satisfying*

$$d(x_{n+1}, x_n) \le \gamma d(x_n, x_{n-1}),$$

for every $n \in N$, where $\gamma \in [0, 1)$, is Cauchy.

2. Main Results

Using Lemma 1, we have the following result.

Lemma 2. *Let (X, G_b) be a G_b-metric space (with a coefficient $s \ge 1$) and $\{x_n\}$ be a sequence such that*

$$G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_n, x_n, x_{n+1}) \le \lambda(G_b(x_{n-1}, x_n, x_n) + G_b(x_{n-1}, x_{n-1}, x_n)), \tag{3}$$

for all $n \in \mathbb{N}$, where $\lambda \in [0, 1)$ is a Lipschitz type constant. Then, $\{x_n\}$ is G_b-Cauchy.

Proof. The condition in Equation (3), together with Proposition 4, implies that

$$d_{G_b}(x_{n+1}, x_n) \leq \lambda d_{G_b}(x_n, x_{n-1}), \tag{4}$$

for every $n \in N$. By Lemma 1, it follows that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the b-metric space (X, d_{G_b}). Since (X, d_{G_b}) and (X, G_b) are topologically equivalent, we have that $\{x_n\}_{n\in N}$ is also Cauchy in (X, G_b). □

In the sequel, we improve and generalize some already announced results (see ([57], Lemma 3.1. and Theorem 3.21.)). Namely, we have the following result.

Lemma 3. *Let $\{x_n\}$ be a sequence in a G_b-metric space $(X, G_b, s \geq 1)$ such that*

$$G_b\left(x_{n+1}, x_{n+2}, x_{n+2}\right) \leq \lambda G_b\left(x_n, x_{n+1}, x_{n+1}\right), \tag{5}$$

where $\lambda \in [0, 1)$ and $n = 1, 2, \ldots$. Then, $\{x_n\}$ is a G_b-Cauchy sequence.

Proof. The case that $s = 1$ is standard. Otherwise, take $s > 1$. If $\lambda \in [0, \frac{1}{s})$, the proof is the same as in [57] (pp. 663, 664). Therefore, let $\lambda \in [\frac{1}{s}, 1)$. Firstly, define the self mapping T on X as $Tx_n = x_{n+1}$ for $n = 1, 2, \ldots$ and $Tx = z$ if $x \notin \{x_n\}_{n=1}^{+\infty}$ where $z \in X$ is an arbitrary point. Now, Equation (5) becomes

$$G_b\left(Tx_n, Tx_{n+1}, Tx_{n+1}\right) \leq \lambda G_b\left(x_n, x_{n+1}, x_{n+1}\right),\ n = 1, 2, \ldots. \tag{6}$$

Further, we have that $\lambda^n \to 0$ as $n \to \infty$. Then, there is $n_0 \in N$ such that $\lambda^{n_0} < \frac{1}{s}$, that is,

$$G_b\left(T^{n_0}x_n, T^{n_0}x_{n+1}, T^{n_0}x_{n+1}\right) \leq \lambda^{n_0} G_b\left(x_n, x_{n+1}, x_{n+1}\right). \tag{7}$$

Thus, in the case $\lambda \in [0, \frac{1}{s})$, Equation (7) shows that

$$\left\{T^{n_0+n}x_1\right\}_{n=1}^{+\infty} := \left\{x_{n_0+1}, x_{n_0+2}, \ldots, x_{n_0+n}, \ldots\right\}$$

is a G_b-Cauchy sequence.

Since

$$\{x_n\}_{n=1}^{+\infty} = \{x_1, x_2, \ldots, x_{n_0}\} \cup \left\{x_{n_0+1}, x_{n_0+2}, \ldots, x_{n_0+n}, \ldots\right\},$$

we get that the sequence $\{x_n\}$ is G_b-Cauchy. □

Remark 2. *The previous lemma generalizes ([57], Lemma 3.1.) and it can be compared with ([56], Lemma 2.2.). In addition, it is true if we replace Equation (5) by*

$$G_b\left(x_{n+1}, x_{n+1}, x_{n+2}\right) \leq \lambda G_b\left(x_n, x_n, x_{n+1}\right),\ n \in \mathbb{N},\ \lambda \in [0, 1). \tag{8}$$

In ([58], Theorem 3.3.) and ([23], Theorem 12.2.), the analog of Banach contraction principle, which is,

$$d(fx, fy) \leq \lambda d(x, y),\ x, y \in X, 0 < \lambda < \frac{1}{s},$$

is used to prove the following fixed point theorems in G_b-metric spaces.

In the following result, we extend ([58], Theorem 3.3.) and ([23], Theorem 12.2.) by considering $\lambda \in [0, 1)$ instead of $\lambda \in [0, \frac{1}{s})$.

Theorem 3. *Let $(X, G_b, s \geq 1)$ be a G_b-complete G_b-metric space. Suppose the mapping $T : X \to X$ satisfies*

$$G_b\left(Tx, Ty, Tz\right) \leq \lambda G_b\left(x, y, z\right), \tag{9}$$

for all $x, y, z \in X$, where $\lambda \in [0, 1)$ is a given constant. Then, T has a unique fixed point (say u) in X and, for $x \in X$, the Picard sequence $\{T^n x\}_{n=1}^{+\infty}$ converges to u.

Proof. The condition in Equation (9) immediately implies

$$G_b\left(Tx, Tx, Tz\right) \leq \lambda G_b\left(x, x, z\right)$$

and

$$G_b\left(Tx, Tz, Tz\right) \leq \lambda G_b\left(x, z, z\right).$$

A summation of the two last inequalities yields that

$$d_{G_b}\left(Tx, Tz\right) \leq \lambda d_{G_b}\left(x, z\right),$$

for all $x, z \in X$.

The result further follows according to ([23], Theorem 12.2.). □

Remark 3. *The relation in Equation (9) is in fact the Banach contraction principle in the context of G_b-metric spaces.*

(Ref. [11], Corollary 2.8.) treated a Boyd–Wong type result in complete $b-$rectangular metric spaces with coefficient $s > 1$. The following condition was used:

$$sd(fx, fy) \leq \phi(d(x, y)), \; x, y \in X,$$

where $\phi : [0, \infty) \to [0, \infty)$ is upper semi-continuous from the right, $\phi(0) = 0$ and $\phi(t) < t$ for each $t > 0$. The related result in G_b-metric spaces is

Theorem 4. *Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s > 1$ and $f : X \to X$. If*

$$sG_b(fx, fy, fz) \leq \phi(G_b(x, y, z)) \tag{10}$$

for every $x, y, z \in X$, where ϕ is described above, then there exists a unique fixed point of f.

Proof. Since $\phi(t) < t$ and $s > 1$, the condition in Equation (9) directly follows from Equation (10). □

The Meir–Keeler type theorem in complete $b-$rectangular metric spaces with coefficient $s > 1$ is studied in ([59], Theorem 2.1.). It is assumed that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \; \text{ implies } \; sd(fx, fy) < \varepsilon.$$

The related Meir–Keeler result in G_b-metric spaces is

Theorem 5. *Let (X, G_b) be a G_b-complete G_b-metric space and $f : X \to X$. If*

$$\varepsilon \leq G_b(x, y, z) < \varepsilon + \delta \; \textit{ implies } \; sG_b(fx, fy, fz) < \varepsilon, \tag{11}$$

for all $x, y \in X$ and $s > 1$, then there exists a unique fixed point of f.

Proof. By Equation (11), we have

$$sG_b(fx, fy, fz) < \varepsilon \leq G_b(x, y, z), \; x, y, z \in X,$$

and the condition in Equation (9) is fulfilled. □

(Ref. [11], Corollary 2.2.) was a Geraghty type result in complete b-metric spaces with coefficient $s > 1$. It was supposed that the function $f : X \to X$ satisfies the following condition:

$$d(fx, fy) \leq \beta(d(x, y))d(x, y),$$

for all $x, y \in X$, where $\beta : [0, \infty) \to [0, \frac{1}{s})$ is such that

$$\lim_{n\to\infty} \beta(t_n) = \frac{1}{s} \quad \text{implies} \quad \lim_{n\to\infty} t_n = 0. \tag{12}$$

The related Geraghty type result in G_b-metric spaces is

Theorem 6. *Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s > 1$ and $f : X \to X$. Given $\beta : [0, \infty) \to [0, \frac{1}{s})$ satisfying Equation (12). If*

$$G_b(fx, fy, fz) \leq \beta(G_b(x, y, z))G_b(x, y, z), \tag{13}$$

for all $x, y, z \in X$, then there exists a unique fixed point of f.

Proof. Using $\beta(t) < \frac{1}{s}$ and Equation (13), we get the condition in Equation (9), that is,

$$G_b(fx, fy, fz) \leq \frac{1}{s}G_b(x, y, z), \ x, y, z \in X.$$

□

The following theorem is a Hardy–Rogers type result in the context of G_b-metric spaces. In a b-metric space, this type of contraction is studied in ([19], Theorem 2.7) as follows:

$$d(fx, gy) \leq \frac{1}{s^4}(a_1 d(Sx, Ty) + a_2 d(fx, Ty) + a_3 d(Sx, gy) + a_4 d(fx, Sx) + a_5 d(gy, Ty)),$$

where $a_1 + \alpha a_2 + \beta a_3 + a_4 + a_5 < 1$ and $\alpha + \beta = 2$.

The related Hardy–Rogers type fixed point result in G_b-metric spaces is

Theorem 7. *Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$ and $f : X \to X$. Let $x, y, z \in X$ be such that*

$$\begin{aligned} G_b(fx, fy, fz) \leq{}& a_1 G_b(x, y, z) + a_2 G_b(x, fx, fx) + a_3 G_b(y, fy, fy) + a_4 G_b(z, fz, fz) \\ &+ a_5 G_b(x, fy, fy) + a_6 G_b(y, fz, fz) + a_7 G_b(z, fx, fx), \end{aligned} \tag{14}$$

where $a_1 + a_2 + a_3 + a_4 + 2sa_5 + a_6 + a_7 < 1$, $s(a_3 + a_4 + a_5 + a_6) < 1$. Then, there exists a unique fixed point of f.

Proof. For an arbitrary $x_0 \in X$, define a sequence $\{x_n\}$ in X by $x_n = fx_{n-1} = f^n x_0$ for all $n \geq 1$. Let $x = x_{n-1}$ and $y = z = x_n$. By Equation (14), for $n \in N$,

$$\begin{aligned} G_b(x_n, x_{n+1}, x_{n+1}) &\leq a_1 G_b(x_{n-1}, x_n, x_n) + a_2 G_b(x_{n-1}, x_n, x_n) + a_3 G_b(x_n, x_{n+1}, x_{n+1}) \\ &+ a_4 G_b(x_n, x_{n+1}, x_{n+1}) + a_5 G_b(x_{n-1}, x_{n+1}, x_{n+1}) + a_6 G_b(x_n, x_{n+1}, x_{n+1}) + a_7 G_b(x_n, x_n, x_n) \\ &\leq (a_1 + a_2)G_b(x_{n-1}, x_n, x_n) + (a_3 + a_4 + a_6)G_b(x_n, x_{n+1}, x_{n+1}) \\ &+ sa_5(G_b(x_{n-1}, x_n, x_n) + G_b(x_n, x_{n+1}, x_{n+1})). \end{aligned}$$

Now, we have

$$G_b(x_n, x_{n+1}, x_{n+1}) \leq \frac{a_1 + a_2 + sa_5}{1 - a_3 - a_4 - sa_5 - a_6} G_b(x_{n-1}, x_n, x_n),\ n \in N. \tag{15}$$

The condition in Equation (5) is satisfied if $a_1 + a_2 + a_3 + a_4 + 2sa_5 + a_6 < 1$.
Thus, $\{x_n\}$ is G_b-convergent to some $x \in X$. Note that

$$G_b(x, fx, fx) \leq s(G_b(x, x_n, x_n) + G_b(x_n, fx, fx)), n \in N.$$

By Equation (14), for $n \in N$,

$$G_b(x_n, fx, fx) \leq a_1 G_b(x_{n-1}, x, x) + a_2 G_b(x_{n-1}, x_n, x_n) + a_3 G_b(x, fx, fx) + a_4 G_b(x, fx, fx)$$

$$+ sa_5(G_b(x_{n-1}, x_n, x_n) + G_b(x_n, fx, fx)) + a_6 G_b(x, fx, fx) + a_7 G_b(x, x_n, x_n).$$

Taking $n \to \infty$ in the last relations, we get

$$G_b(x_n, fx, fx) \leq \frac{a_3 + a_4 + a_6}{1 - sa_5} G_b(x, fx, fx)$$

and

$$(1 - s\frac{a_3 + a_4 + a_6}{1 - sa_5})G_b(x, fx, fx) \leq 0.$$

By condition $s(a_3 + a_4 + a_5 + a_6) < 1$, it follows that $fx = x$.
Let $z \in X$ such that $fz = z$. By Equation (14),

$$G_b(fx, fz, fx) \leq a_1 G_b(x, z, x) + a_2 G_b(x, fx, fx) + a_3 G_b(z, fz, fz) + a_4 G_b(x, fx, fx) + a_5 G_b(x, fz, fz)$$

$$+ sa_5(G_b(z, x, x) + G_b(x, x, z)) + a_6 G_b(z, fx, fx) + a_7 G_b(x, fx, fx).$$

Now,

$$(1 - a_1 - 2sa_5 - a_6)G_b(x, x, z) \leq 0$$

and by assumption of theorem, it follows that $G_b(x, x, z) = 0$, i.e., $x = z$. □

In ([60], Theorem 2.4), a Kannan type contraction in rectangular b-metric space with $s > 1$, is studied. Here,

$$d(fx, fy) \leq \alpha(d(x, fx) + d(y, fy)),$$

for all $x, y \in X$ and $\alpha \in [0, \frac{1}{s+1}]$. The same condition as $\alpha \in [0, \frac{1}{2})$ is used in ([61], Theorem 2), where a fixed point result in complete b-metric spaces is presented.

The Chatterjea type contraction defined as

$$d(Tx, Ty) \leq \lambda(d(x, Ty) + d(y, Tx)),\ x, y \in X,\ s\lambda \in [0, \frac{1}{2}],$$

is treated in ([61], Theorem 3) and a fixed point result in complete b-metric spaces is presented.

A fixed point theorem of Reich type contraction:

$$d(fx, fy) \leq ad(x, fx) + bd(y, dy) + cd(x, y),\ x, y \in X,\ a + s(b + c) < 1,$$

in complete b-metric spaces (with $s > 1$) was studied in ([62], Theorem 3.2).

The related Kannnan, Chatterjea and Reich type fixed point result in G_b-metric spaces are just corollaries of Theorem 7.

Corollary 1. *Let $(X, G_b, s \geq 1)$ be a G_b-complete G_b-metric space and $f : X \to X$. Suppose there exists $0 \leq \lambda < \frac{1}{\max\{2s,3\}}$ such that*

$$G_b(fx, fy, fz) \leq \lambda(G_b(x, fx, fx) + G_b(y, fy, fy) + G_b(z, fz, fz)), \tag{16}$$

for all $x, y, z \in X$. Then, there exists a unique fixed point of f.

Proof. The assertion follows if we take $a_1 = a_5 = a_6 = a_7 = 0$ and $a_2 = a_3 = a_4 = \lambda$ in Theorem 7. □

Corollary 2. *Let $(X, G_b, s \geq 1)$ be a G_b-complete G_b-metric space and $f : X \to X$. Suppose there exists $0 \leq \lambda < \frac{1}{2s+1}$ such that*

$$G_b(fx, fy, fz) \leq \lambda(G_b(x, fy, fy) + G_b(y, fz, fz) + G_b(z, fx, fx)), \tag{17}$$

for all $x, y, z \in X$. Then, there exists a unique fixed point of f.

Proof. If we take $a_5 = a_6 = a_7 = \lambda$ and $a_1 = a_2 = a_3 = a_4 = 0$ in Theorem 7, we obtain the condition in Equation (17) for $0 \leq \lambda < \frac{1}{2s+2}$. Moreover, the estimate in Equation (15) holds if $0 \leq \lambda < \frac{1}{2s+1}$. □

Corollary 3. *Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$ and $f : X \to X$. Let $x, y, z \in X$ be such that*

$$G_b(fx, fy, fz) \leq a_1 G_b(x, y, z) + a_2 G_b(x, fx, fx) + a_3 G_b(y, fy, fy) + a_4 G_b(z, fz, fz), \tag{18}$$

where $a_1 + a_2 + a_3 + a_4 < 1$ and $s(a_2 + a_3) < 1$. Then, there exists a unique fixed point of f.

Proof. Take $a_1 = a_2 = a_3 = a_4 = \lambda$ and $a_5 = a_6 = a_7 = 0$ in Theorem 7. □

The Ćirić type contraction (quasi-contraction) in b-metric spaces given as

$$d(Tx, Ty) \leq q \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \ x, y \in X,$$

where $q \leq \frac{1}{s^2+s}$, was treated in ([10], Corollary 2.4). The related Ćirić type fixed point result in G_b-metric spaces is

Theorem 8. *Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$ and $f : X \to X$. If there exists $0 \leq \lambda < \frac{1}{2s}$ such that*

$$\begin{aligned} G_b(fx, fy, fz) \leq \lambda \max\{&G_b(x, y, z), G_b(x, fx, fx), G_b(y, fy, fy), \\ &G_b(z, fz, fz), G_b(x, fy, fy), G_b(y, fz, fz), G_b(z, fx, fx)\}, \end{aligned} \tag{19}$$

for all $x, y, z \in X$. Then, there exists a unique fixed point of f.

Proof. For $x_0 \in X$, take $x_n = fx_{n-1} = f^n x_0$. Putting $x = x_{n-1}$ and $y = z = x_n$ in Equation (19), we have

$$\begin{aligned} G_b(x_n, x_{n+1}, x_{n+1}) &\leq \lambda \max\{G_b(x_{n-1}, x_n, x_n), G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), \\ &\quad G_b(x_n, x_{n+1}, x_{n+1}), G_b(x_{n-1}, x_{n+1}, x_{n+1}), G_b(x_n, x_n, x_n)\} \\ &\leq \lambda \max\{G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), s(G_b(x_{n-1}, x_n, x_n) + G_b(x_n, x_{n+1}, x_{n+1}))\} \\ &= s(G_b(x_{n-1}, x_n, x_n) + G_b(x_n, x_{n+1}, x_{n+1})), \end{aligned}$$

that is,

$$G_b(x_n, x_{n+1}, x_{n+1}) \leq \frac{\lambda s}{1-\lambda s} G_b(x_{n-1}, x_n, x_n),\ n \in N.$$

Thus, the condition in Equation (5) holds if $0 \leq \lambda < \frac{1}{2s}$ and $\{x_n\}$ is G_b-Cauchy sequence (in the G_b-complete G_b-metric space (X, G_b)).

Let $x \in X$ be such that $\{x_n\}$ is G_b-convergent to x. By Equation (19), it follows that

$$\begin{aligned} G_b(x_n, fx, fx) \leq \lambda \max\{&G_b(x_{n-1}, x, x), G_b(x_{n-1}, x_n, x_n), G_b(x, fx, fx), G_b(x, fx, fx), \\ &s(G_b(x_{n-1}, x_n, x_n) + G_b(x_n, fx, fx)), G_b(x, fx, fx), G_b(x, x_n, x_n)\} \\ = \lambda \max\{&G_b(x, fx, fx), sG_b(x_n, fx, fx)\}. \end{aligned}$$

Letting $n \to \infty$ implies that $G_b(x, fx, fx) \leq \lambda s G_b(x, fx, fx)$. Since $s\lambda < 1$, the above inequality holds unless $G_b(x, fx, fx) = 0$, so $x = fx$.

For the uniqueness of the fixed point of f, let $z \in X$ be such that $fz = z$. Using Equation (19), we have

$$\begin{aligned} G_b(fx, fx, fz) \leq \lambda \max\{&G_b(x, x, z), G_b(x, fx, fx), G_b(x, fx, fx), G_b(z, fz, fz), \\ &G_b(x, fx, fx), 2sG_b(x, x, z), G_b(z, fx, fx)\}. \end{aligned}$$

Then,

$$(1 - 2\lambda s) G_b(x, x, z) \leq 0.$$

Since $\lambda < \frac{1}{2s}$, again the above holds unless $G_b(x, x, z) = 0$, thus we have that $x = z$. □

In addition, we observe a Bianchini [63] type contraction:

$$d(fx, fy) \leq \lambda \max\{d(x, fx), d(y, fy)\},\ 0 \leq \lambda < 1.$$

The related Bianchini type fixed point result in G_b-metric spaces is a consequence of Theorem 8.

Corollary 4. *Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$, $f : X \to X$. If there exists $0 \leq \lambda < \frac{1}{s}$ such that*

$$G_b(fx, fy, fz) \leq \lambda \max\{G_b(x, fx, fx), G_b(y, fy, fy), G_b(z, fz, fz)\}, \tag{20}$$

for all $x, y, z \in X$. Then, there exists a unique fixed point of f.

Proof. For $x_0 \in X$, take $x_n = fx_{n-1} = f^n x_0$. Note that the condition in Equation (20) implies the condition in Equation (19). By Theorem 8, there is a unique fixed point of f in the case that $0 \leq \lambda < \frac{1}{2s}$. In the sequel, we will show that this last interval for λ could be expanded to $[0, \frac{1}{s})$.

Let $x = x_{n-1}$ and $y = z = x_n$. By Equation (20), we have

$$G_b(x_n, x_{n+1}, x_{n+1}) \leq \lambda \max\{G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), G_b(x_n, x_{n+1}, x_{n+1})\}, \tag{21}$$

for all $n \in N$. If for some n, $G_b(x_{n-1}, x_n, x_n) \leq G_b(x_n, x_{n+1}, x_{n+1})$, then by Equation (21), it follows that

$$(1 - \lambda) G_b(x_n, x_{n+1}, x_{n+1}) \leq 0,$$

i.e., $G_b(x_n, x_{n+1}, x_{n+1}) = 0$. Thus, $G_b(x_{n-1}, x_n, x_n) > G_b(x_n, x_{n+1}, x_{n+1})$, $n \in N$. The relation in Equation (21) corresponds to the condition (5).

Therefore, $\{x_n\}$ is G_b-convergent to some $x \in X$. By the condition in Equation (20), for $n \in N$, it follows that

$$G_b(x, fx, fx) \leq s(G_b(x, x_n, x_n) + G_b(x_n, fx, fx))$$

$$\leq sG_b(x, x_n, x_n) + s\lambda \max\{G_b(x_{n-1}, x_n, x_n), G_b(x, fx, fx), G_b(x, fx, fx)\}.$$

Since $G_b(x, x_n, x_n) \to 0$ and $G_b(x_{n-1}, x_n, x_n) \to 0$ as $n \to \infty$, we obtain that

$$(1 - s\lambda)G_b(x, fx, fx) \leq 0.$$

This holds unless $G_b(x, fx, fx) = 0$, thus $fx = x$.

Let $z \in X$ be another fixed point of f. Using Equation (20), we get

$$G_b(fx, fx, fz) \leq \lambda \max\{G_b(x, fx, fx), G_b(x, fx, fx), G_b(z, fz, fz)\} = 0$$

and thus $x = z$. □

Further, some improvements of Theorem 1 are given, where instead of the condition in Equation (1), it is supposed that

$$2s^a G_b\left(fx, gy, hz\right) \leq M_s\left(x, y, z\right),\ s > 1,\ a > \log_s \frac{2s^2 + s}{2}, \tag{22}$$

for all $x, y, z \in X$. Moreover, the assumption that G is a symmetric G_b-metric is omitted.

Many parts of the proof of Theorem 1 are the same as in [8], thus only its modification is presented. In the first part of the proof, it is supposed that $G_{3n} \leq G_{3n+1}$. By Equation (22), we have that

$$2s^a G_{3n+1} = 2s^a G(y_{3n+1}, y_{3n+2}, y_{3n+3}) \leq M_s(x_{3n+3}, x_{3n+1}, x_{3n+2}) = G_{3n+1}$$

and so $G_{3n+1} = 0$, because of $a, s > 1$. Using the same arguments, we get that $G_n = 0, n \in N$ or $\{G_n\}$ is decreasing. Hence, $\lim\limits_{n\to\infty} G_n = 0$.

Now, we present a new proof that $\{y_n\}$ is a G_b-Cauchy sequence. Since

$$\begin{aligned}
2s^a G(y_{3n+1}, y_{3n+1}, y_{3n+2}) &\leq 2s^a G(y_{3n+1}, y_{3n+2}, y_{3n+3}) \\
&= 2s^a G_{3n+1} \leq G_{3n} = G(y_{3n}, y_{3n+1}, y_{3n+2}) \\
&\leq s(G(y_{3n}, y_{3n+1}, y_{3n+1}) + G(y_{3n+1}, y_{3n+1}, y_{3n+2})) \\
&\leq s(2sG(y_{3n}, y_{3n}, y_{3n+1}) + G(y_{3n+1}, y_{3n+1}, y_{3n+2})),\ n \in N,
\end{aligned}$$

we get

$$G(y_{3n+1}, y_{3n+1}, y_{3n+2}) \leq \frac{2s^2}{2s^a - s} G(y_{3n}, y_{3n}, y_{3n+1}),\ n \in N. \tag{23}$$

If we take $a > \log_s \frac{2s^2+s}{2}$, the condition in Equation (8) is fulfilled and by Lemma 3, and it follows that $\{y_n\}$ is a G_b-Cauchy sequence.

By the arguments presented in modified proof of Theorem 1, we can improve Condition (2.1) in ([52], Theorem 2.1) as follows.

Theorem 9. *Let (X, G) be a complete G_b-metric space and let $A, B, C : X \to X$ satisfy the following condition:*

$$s^a G(Ax, By, Cz) \leq M(x, y, z), \quad x, y, z \in X,$$

where $s > 1$, $a > log_s(2s^2 + s)$ and

$$M(x, y, z) = \max\{G(x, y, z), G(x, Ax, By), G(y, By, Cz), G(z, Cz, Ax)\}.$$

3. Coupled and Tripled Coincidence Point Results

Definition 12 ([64]). *Let X be a non-empty set, $F : X^2 \to X$ and $g : X \to X$. An element $(x, y) \in X^2$ is called a coupled coincidence point of F and g if*

$$F(x, y) = gx \quad \text{and} \quad F(y, x) = gy,$$

while $(gx, gy) \in X^2$ is called a coupled point of coincidence of mappings F and g. Moreover, (x, y) is called a coupled common fixed point of F and g if

$$F(x, y) = gx = x \quad \text{and} \quad F(y, x) = gy = y.$$

Remark 4. *Otherwise, (x, y) is a coupled coincidence point of F and g if and only if (x, y) is a coincidence point of the mappings $T_F : X^2 \to X^2$ and $T_g : X^2 \to X^2$, which are defined by*

$$T_F(x, y) = (F(x, y), F(y, x)), \ T_g(x, y) = (gx, gy).$$

The following new result is useful in the context of G_b-metric spaces and its proof is immediate.

Lemma 4. *Let (X, G_b) be a symmetric G_b-metric space. Define $G_{b+} : (X^2)^3 \to [0, +\infty)$ and $G_{b\max} : (X^2)^3 \to [0, +\infty)$ by*

$$G_{b+}((x, y), (u, v), (a, b)) = G_b(x, u, a) + G_b(y, v, b)$$

and

$$G_{b\max}((x, y), (u, v), (a, b)) = \max\{G_b(x, u, a), G_b(y, v, b)\}.$$

Then, (X^2, G_{b+}) and $(X^2, G_{b\max})$ are symmetric G_b-metric spaces.

The following example shows that the previous lemma is not true if (X, G_b) is not a symmetric G_b-metric space.

Example 5. *Let $X = \{a, b\}$ with $a \neq b$. Define*

$$\begin{aligned} G(a, a, a) &= G(b, b, b) = 0 \\ G(a, a, b) &= 1, G(a, b, b) = 2, \end{aligned}$$

and extend G to X^3 by using the symmetry in the variables. (X, G) is an asymmetric G-metric space (indeed, $G(a, a, b) \neq G(a, b, b)$).

Now, using ([8], Example 1.2) with $p = 3$ and $s = 2^{3-1} = 4$, $G_4 : X^3 \to [0, \infty)$ is given as

$$\begin{aligned} G_4(a, a, a) &= G_4(b, b, b) = 0 \\ G_4(a, a, b) &= 1^3 = 1, G_4(a, b, b) = 2^3 = 8. \end{aligned}$$

Hence, (X, G_4) is a G_4-metric space (it is asymmetric).

It is easy to see that (X^2, G_{4+}), i.e., $(X^2, G_{4\max})$ is not a G_b-metric space.

Now, we are ready to state and prove our first result.

Theorem 10. *Let (X, G_b) be a complete G_b-metric space, $F : X^2 \to X$ and $g : X \to X$. Assume that there exists $k \in (0, 1)$ such that for all $x, y, u, v, a, b \in X$,*

$$G_b(T(x, y), T(u, v), T(a, b)) \leq \frac{k}{2}(G_b(gx, gu, ga) + G_b(gy, gv, gb)). \tag{24}$$

Then, F and g have a unique coupled coincidence point.

Proof. Putting $u = a$ and $v = b$ into Equation (24), we have

$$G_b\left(T\left(x,y\right),T\left(a,b\right),T\left(a,b\right)\right) \leq \frac{k}{2}\left(G_b\left(gx,ga,ga\right)+G_b\left(gy,gb,gb\right).\right) \tag{25}$$

Putting again $u = x$ and $v = y$ into Equation (24), we have

$$G_b\left(T\left(x,y\right),T\left(x,y\right),T\left(a,b\right)\right) \leq \frac{k}{2}\left(G_b\left(gx,gx,ga\right)+G_b\left(gy,gy,gb\right)\right). \tag{26}$$

Adding Equation (25) to Equation (26) and using Definition 1.17, we get

$$d_{G_b}\left(T\left(x,y\right),T\left(a,b\right)\right) \leq \frac{k}{2}\left(d_{G_b}\left(gx,ga\right)+d_{G_b}\left(gy,gb\right)\right). \tag{27}$$

Clearly, Equation (27) implies that

$$d_{G_b}\left(T\left(x,y\right),T\left(a,b\right)\right)+d_{G_b}\left(T\left(y,x\right),T\left(b,a\right)\right) \leq k\left(d_{G_b}\left(gx,ga\right)+d_{G_b}\left(gy,gb\right)\right),$$

or equivalently

$$d_{G_b+}\left(T_F\left(U\right),T_F\left(A\right)\right) \leq k d_{G_b+}\left(T_g\left(U\right),T_g\left(A\right)\right) \tag{28}$$

for all $U = (x,y)$ and $A = (a,b)$.

Since Equation (28) corresponds to the Banach contraction condition in the context of b-metric spaces (see, for example, [23]), we obtain that the mappings T_F and T_g have a unique point of coincidence $(t,w) \in X^2$, that is, $T_F\left(t,w\right) = T_g\left(t,w\right)$. We deduce that $\left(F\left(t,w\right),F\left(w,t\right)\right) = \left(g\left(t\right),g\left(w\right)\right)$, i.e., $F\left(t,w\right) = g\left(t\right)$ and $F\left(w,t\right) = g\left(w\right)$.

At the end, we give a remark related to Theorem 2.

It is easy to see that contractive condition in Equation (2) can be relaxed, and Theorem 2 should be generalized as follows:

Theorem 11. *Let (X, G_b) be a complete G_b-metric space with parameter $s \geq 1$ and let $S, T, R : X^2 \to X$ satisfy*

$$\begin{aligned} G(S(x,y),T(u,v),R(a,b)) &\leq b_1\frac{G(x,u,a)+G(y,v,b)}{2}+b_2G(x,x,S(x,y)) \\ &+ b_3G(u,u,T(u,v))+b_4G(a,a,R(a,b)), \end{aligned}$$

for all $x,y,u,v,a,b \in X$ and $b_1+b_2+b_3+b_4 < \frac{1}{4s^2}$. Then, S, T and R have a unique common coupled fixed point in X^2.

By the usual method, for $n \in \mathbb{N}$, we get

$$G_{b+}((x_{n+1},y_{n+1}),(x_{n+2},y_{n+2}),(x_{n+3},y_{n+3})) \leq hG_{b+}((x_n,y_n),(x_{n+1},y_{n+1}),(x_{n+2},y_{n+2})),$$

where $h = b_1+b_2+b_3+b_4 < \frac{1}{4s^2}$. The next step is to show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Thus,

$$G_{b+}((x_{n+1},y_{n+1}),(x_{n+2},y_{n+2}),(x_{n+2},y_{n+2}) \leq \frac{2hs^2}{1-2hs^2}G_{b+}((x_n,y_n),(x_{n+1},y_{n+1}),(x_{n+1},y_{n+1})).$$

Since $h < \frac{1}{4s^2}$, it follows that $\frac{2hs^2}{1-2hs^2} < 1$ and using Lemma 3, we have that $\{x_n\}$ is a Cauchy sequence. Now, we show that $(x,y) = (S(x,y), S(y,x))$. Suppose the contrary. For $n \in \mathbb{N}$, we have

$$\begin{aligned}
& G_{b+}((x,y),(S(x,y),S(y,x)),(S(x,y),S(y,x))) \\
\leq\ & s(G_{b+}((x,y),(x_{n+1},y_{n+1}),(x_{n+1},y_{n+1})) + G_{b+}((x_{n+1},y_{n+1}),(S(x,y),S(y,x)),(S(x,y),S(y,x)))) \\
\leq\ & s(G_{b+}((x,y),(x_n,y_n),(x_n,y_n)) + b_1 G_{b+}((x_n,y_n),(x,y),(x,y)) \\
+\ & b_2 G_{b+}((x_n,y_n),(x_n,y_n),(x_{n+1},y_{n+1})) + b_3 G_{b+}((x,y),(x,y),(S(x,y),S(y,x))) \\
+\ & b_3 G_{b+}((x,y),(x,y),(S(x,y),S(y,x))).
\end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned}
& 0 < G_{b+}((x,y),(S(x,y),S(y,x)),(S(x,y),S(y,x))) \leq (b_3+b_4) G_{b+}((x,y),(x,y),(S(x,y),S(y,x))) \\
\leq\ & 2s(b_3+b_4) G_{b+}((x,y),(S(x,y),S(y,x)),(S(x,y),S(y,x))) \\
<\ & G_{b+}((x,y),(S(x,y),S(y,x)),(S(x,y),S(y,x))).
\end{aligned}$$

Since $b_3 + b_4 < \frac{1}{2s}$, we get a contradiction.

Definition 13 ([65,66])**.** *Let X be a non-empty set, $F : X^3 \to X$ and $g : X \to X$ be two mappings. An element $(x,y,z) \in X^3$ is called a tripled coincidence point of F and g if*

$$F(x,y,z) = gx, \quad F(y,x,y) = gy \quad \text{and} \quad F(z,y,x) = gz,$$

while $(gx, gy, gy) \in X^3$ is called a tripled point of coincidence of mappings F and g. Moreover, (x,y,z) is called a tripled fixed point of F and g if

$$F(x,y,z) = gx = x, \quad F(y,x,y) = gy = y \quad \text{and} \quad F(z,y,x) = gz = z.$$

Remark 5. *It is clear that (x,y,z) is a tripled coincidence point of F and g if and only if (x,y,z) is a coincidence point for the mappings $T_F : X^3 \to X^3$ and $T_g : X^3 \to X^3$, which are defined by*

$$T_F(x,y,z) = (F(x,y,z), F(y,x,y), F(z,y,x)) \text{ and } T_g(x,y,z) = (gx, gy, gz).$$

Similar to Theorem 10, we state the following triple fixed point theorem relating to the Banach contraction mapping theorem in G_b-metric spaces. We omit its proof.

Theorem 12. *Let (X, G_b) be a complete G_b-metric space, $T : X^3 \to X$ and $g : X \to X$. Assume that there exists $k \in (0,1)$ such that for all $x,y,z,u,v,w,a,b,c \in X$, we have*

$$G_b(T(x,y,z), T(u,v,w), T(a,b,c)) \leq \frac{k}{3}(G_b(gx,gy,gz) + G_b(gu,gv,gw) + G_b(ga,gb,gc)),$$

Then, T and g have a unique tripled coincidence point.

4. Conclusions and Perspectives

We considered various fixed point results in the context of G_b-metric spaces. Taking inspiration from [67,68], it would be interesting to investigate convex contraction mapping theorems in the class of G_b-metric spaces.

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