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Customer Exposure to Sellers, Probabilistic Optimization and Profit Research

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Abstract: This paper deals with a probabilistic problem in which there is a specific probability for a customer to meet a seller in a specified area. It is assumed that the area in which a seller acts follows an exponential distribution and affects the probability of meeting with a customer. Furthermore, the range in which a customer can meet a seller is another parameter which affects the probability of a successful meeting. The solution to the problem is based on a bomb fragmentation model using Lagrange equations. More specifically, using Lagrange equations, the abovementioned dimensions will be calculated in order to optimize the probability of a customer meeting a seller.

Keywords: probabilistic analysis; optimization; Lagrange equations; operations research; bomb fragmentation

1. Introduction

Probabilistic optimization methods may have several applications in many scientific areas, such as business management, finance, computer science, nuclear safety, and the environment [1]. Thus, probabilistic optimization methods have been used by several researchers for various cases. In the field of business, probabilistic optimization models can be used in cases such as service identification, process standardization [2], and sales modeling [3]. These methods can be useful in many decision-making cases in business as they are built upon uncertainty [4].

Based on a primary analysis of the same context [5], the aim of this study is to further analyze the range in which a seller acts as well as the range in which customers can meet with a seller and buy the products they sell. The aim of this analysis is the probability optimization of a successful meeting between a seller and a customer. The warfare problems introduced by Finn and Kent [6] and Przemieniecki [7] will be the basis of the analysis due to the fact that such problems are applied in many business cases [8,9].

The model to be developed is based on Lagrange multipliers and concerns the optimization of a cluster bomb's probability to destroy various enemy targets. The main assumptions of the mathematical model is that the area to be hit by the cluster bomb is a circle with radius R and the bomb's clusters are normally distributed [10].

2. Probabilistic Model Formulation

Based on the abovementioned model, we assume that a number of sellers (N) are spread in a circular area with radius R (e.g., the center of a park). X_1 and X_2 are assumed to be the positions of the sellers in the examined area. The aim is to calculate the maximum probability of a customer being in the area to be met with a seller.

The dispersion of each seller in the area can be a normally distributed two-dimensional random variable following the probability:

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-p^2}} e^{-\varphi(x_1,x_2)}$$

where p refers to the Pearson correlation coefficient and $\varphi(x_{1,}x_2)$ is

$$\varphi(x_1, x_2) = \frac{1}{2(1-p^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2p\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]$$

Assuming that $p = 0$, $\mu_1 = \mu_2 = 0$ and, $\sigma_1 = \sigma_2 = \sigma$, we obtain

$$f(x_1, x_2) = \frac{1}{2\pi\sigma^2} e^{-\frac{x_1^2+x_2^2}{2\sigma^2}}$$

We will now calculate the probability of a customer being inside the abovementioned circular area. Thus, the following transformations are made:

$$x_1 = r\ \cos\theta$$

$$x_2 = r\ \sin\theta$$

where

$$0 \le r \le R$$
$$0 \le \theta \le 2\pi$$

and

$$\det\left(\frac{\partial(x_1, x_2)}{\partial(r,\theta)}\right) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Therefore,

$$\iint\limits_D f(x_1, x_2)dx_1dx_2 = \int_0^{2\pi}\int_0^R \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} r\ drd\theta = 1 - e^{-\frac{R^2}{2\sigma^2}}$$

Thus, the probability of a potential customer being in the seller's area is

$$P_1 = 1 - e^{-\frac{R^2}{2\sigma^2}}$$

where R refers to the seller's action area radius.

We assume a radius (R) of a circular area in which only half of the sellers are located. The value of R where $P_1 = 0,5$ is denoted by (c). Then, we obtain

$$P_1 = 0,5$$

$$\Leftrightarrow 1 - e^{-\frac{c^2}{2\sigma^2}} = 0,5$$

$$\Leftrightarrow e^{-\frac{c^2}{2\sigma^2}} = \frac{1}{2}$$

$$\Leftrightarrow \frac{c^2}{2\sigma^2} = \ln 2$$

$$\Leftrightarrow 2\sigma^2 = \frac{c^2}{\ln 2}$$

Now, we obtain

$$P_1 = 1 - e^{-\frac{R^2 \ln 2}{c^2}}$$

We now consider A to be the area in which a customer is exposed to a seller. The probability that the customer is not in this area is

$$1 - \frac{A}{\pi R^2}$$

Based on the first assumption of the analysis that the seller number is equal to N, the probability that a customer does not meet with any of the sellers is

$$\left(1 - \frac{A}{\pi R^2}\right)^N$$

We can state

$$\left(1 - \frac{A}{\pi R^2}\right)^N \approx e^{-\frac{AN}{\pi R^2}}$$

Thus, the probability that a customer meets with at least one of the sellers is

$$P = \left(1 - e^{-\frac{R^2 \ln 2}{c^2}}\right)\left(1 - e^{-\frac{AN}{\pi R^2}}\right)$$

In order to maximize this probability, we consider

$$x = \frac{R^2 \ln 2}{c} \text{ and, } y = \frac{AN}{\pi R^2} \tag{1}$$

Therefore, the probability function is

$$P(x,\ y) = \left(1 - e^{-x}\right)\left(1 - e^{-y}\right)$$

We note that

$$xy = \frac{R^2}{c^2} \ln 2 \frac{AN}{\pi R^2} = \frac{\ln 2AN}{c^2} = \text{constant}$$

We now assume that $xy = k$. Thus, we must calculate the extreme values of the function:

$$P(x,\ y) = \left(1 - e^{-x}\right)\left(1 - e^{-y}\right)$$

under the restriction $g(x, y) = xy - k = 0$.

We will use the Lagrange multipliers method. We define the Lagrange function as follows:

$$L(x, y, \lambda) = P(x, y) + \lambda g(x, y)$$

$$\Leftrightarrow L(x, y, \lambda) = \left(1 - e^{-x}\right)\left(1 - e^{-y}\right) + \lambda xy - \lambda k$$

In order to find the critical points, we solve the system of the following equations:

$$\begin{cases} \frac{\partial L}{\partial x} = 0 \Leftrightarrow e^{-x}\left(1 - e^{-y}\right) = -\lambda y \\ \frac{\partial L}{\partial y} = 0 \Leftrightarrow e^{-y}\left(1 - e^{-x}\right) = -\lambda x \\ \frac{\partial L}{\partial \lambda} = 0 \Leftrightarrow xy = k \end{cases}$$

which deduces to:

$$x = y$$

Afterwards, we calculate the following determinant:

$$D = \begin{vmatrix} 0 & \frac{\partial^2 L}{\partial\lambda\partial x} & \frac{\partial^2 L}{\partial\lambda\partial y} \\ \frac{\partial^2 L}{\partial x\partial\lambda} & \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x\partial y} \\ \frac{\partial^2 L}{\partial y\partial\lambda} & \frac{\partial^2 L}{\partial y\partial x} & \frac{\partial^2 L}{\partial y^2} \end{vmatrix} = \begin{vmatrix} 0 & y & x \\ y & -e^{-x}(1-e^{-y}) & e^{-x}e^{-y}+\lambda \\ x & e^{-x}e^{-y}+\lambda & -e^{-y}(1-e^{-x}) \end{vmatrix}$$

For $x = y$, we obtain

$$D = \begin{vmatrix} 0 & x & x \\ x & -e^{-x}(1-e^{-x}) & e^{-2x}+\lambda \\ x & e^{-2y}+\lambda & -e^{-y}(1-e^{-y}) \end{vmatrix} = 2x^2(e^{-y}+\lambda) > 0$$

The above equations mean that the maximum value of the function $P(x, y)$ at the point (x, y) is when $x = y$. Based on Equation (1), we now obtain

$$\frac{R^2 \ln 2}{c^2} = \frac{AN}{\pi R^2} \Leftrightarrow R = \sqrt[4]{\frac{c^2 AN}{\ln 2\pi}}$$

Thus, the maximum probability of a customer meeting a seller is

$$P_{max} = \left(1 - e^{-\sqrt{\frac{\ln 2AN}{c^2\pi}}}\right)^2$$

We now assume that the customer number is equal to k times the number of the sellers. Thus, the mean number of customers to meet sellers is

$$\hat{p} = vP_{max} = kNP_{max}$$

Furthermore, we assume that the mean number of customer and seller meetings follows the Poisson distribution, where

$$\lambda = N\hat{p} = N^2 kP_{max}$$

Thus, the probability for at least one of the customers meeting a seller is

$$1 - P(0) = 1 - e^{-\lambda} = 1 - e^{-kN^2(1-e^{-\sqrt{\frac{\ln 2AN}{c^2\pi}}})^2}$$

3. Operational Research Application

We assume that 10% of customer and seller meetings lead to a successful sale with a gain of 0.2 for the seller. Moreover, we assume that the seller has accommodation costs. Therefore, we conclude that the seller's profit is positive when the number of meetings is over 5N.

We can calculate the probability of x meetings taking place by using the formula of Poisson distribution:

$$\lambda = N^2 k\left(1 - e^{-\sqrt{\frac{\ln 2AN}{c^2\pi}}}\right)^2$$

In Figure 1, we see that the probability of the number of meetings is x, assuming that the surface area around the seller is A = 100, the radius of the circular disc is c = 100 and the number of sellers is N = 10.

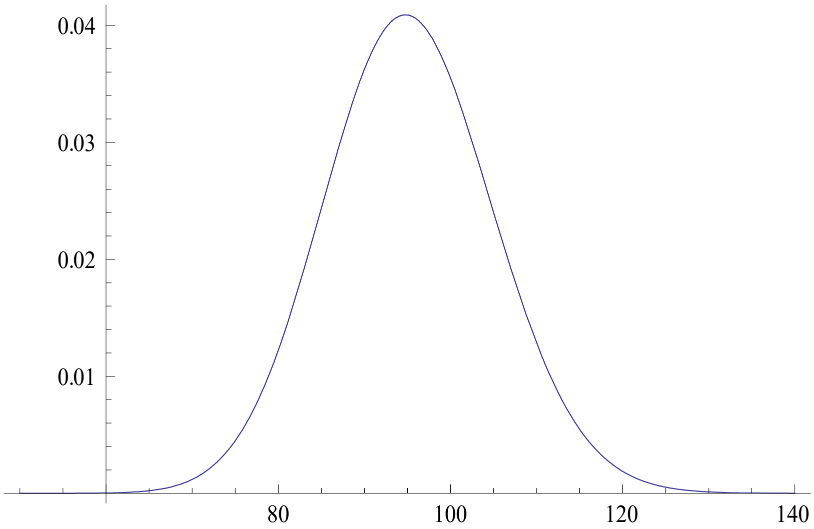

Figure 1. Probability of the number of meetings.

In Tables 1–5, we calculate the probability of customers and sellers meetings to be >5N, so that sellers have profit. We assume that $A = 50m^2$ and $c = 100m$.

Table 1. Number of customers = 10N (k = 10).

N	5N	P(X>5N)
16	80	$1.14847 \cdot 10^{-8}$
17	85	$5.2996 \cdot 10^{-7}$
18	90	0.0000174362
19	95	0.00038038
20	100	0.00516161
21	105	0.0413728
22	110	0.189822
23	115	0.501292
24	120	0.818503
25	125	1

According to the above table, when N is 23, it is more possible for a seller to make a profit (the probability exceeds 50%).

Table 2. Number of customers = 20N (k = 20).

N	5N	P(X>5N)
10	50	$1.03905 \cdot 10^{-8}$
11	55	$7.18658 \cdot 10^{-7}$
12	60	0.0000357685
13	65	0.00108754
14	70	0.0175679
15	75	0.135727
16	80	0.481138
17	85	0.852159
18	90	1

Following the same analysis, we create tables for k = 30, 40 and 50.

Table 3. Number of customers = 30N (k = 30).

N	5N	P(X>5N)
9	45	$9.64421 \cdot 10^{-6}$
10	50	0.000457264
11	55	0.0112619
12	60	0.119434
13	65	0.495747
14	70	0.887378
15	75	1

Table 4. Number of customers = 40N (k = 40).

N	5N	P(X>5N)
9	45	0.00383084
10	50	0.0652488
11	55	0.391667
12	60	0.849736
13	65	1

Table 5. Number of customers = 50N (k = 50).

N	5N	P(X>5N)
9	45	0.093595
10	50	0.502147
11	55	0.921185
12	60	1

4. Conclusions and Future Research

The aim of the above analysis was to propose a mathematical model which calculates the optimal range in which sellers should act in an area as well as the optimal range in which customers can meet with the sellers and buy their products.

The optimization of the probability of customers meeting sellers can be applied in several operational research problems. Due to the fact that, in this paper, a mathematical model is proposed, future research may concern the model's empirical application (e.g., the success of moving advertisements) which would lead to good fit confirmation as well.

Moreover, future research may concern further generalizations of the model assuming different distribution functions than Poisson distribution. More specifically, despite the fact that Poisson or two-dimensional normal distribution are the most relevant distributions in a case such as the one examined [11,12], they are not the only ones. For example, a probabilistic model, such as that used by Bass [13], could be used.

Finally, despite the fact that probabilistic models are built upon uncertainty, which is taken into account, there could be differences between the theoretical models and the observed values due to a variety of potential constraints mainly associated with the external environment [14]. The more variability a probabilistic optimization includes, the better results would be obtained [14]. Thus, the highest possible number of variables that can affect the model should be taken into consideration.

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