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Article

Composite Hurwitz Rings as PF-Rings and PP-Rings

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Abstract: Let $R \subseteq T$ be an extension of commutative rings with identity and $\mathrm{H}(R,T)$ (respectively, $\mathrm{h}(R,T)$) the composite Hurwitz series ring (respectively, composite Hurwitz polynomial ring). In this article, we study equivalent conditions for the rings $\mathrm{H}(R,T)$ and $\mathrm{h}(R,T)$ to be PF-rings and PP-rings. We also give some examples of PP-rings and PF-rings via the rings $\mathrm{H}(R,T)$ and $\mathrm{h}(R,T)$.

Keywords: composite Hurwitz series ring; composite Hurwitz polynomial ring; McCoy condition; PF-ring; PP-ring; annihilator; idempotent element; torsion-free $\mathbb{Z}$-module

1. Introduction

1.1. Composite Hurwitz Rings

Let R be a commutative ring with identity and let $\mathrm{H}(R)$ be the set of formal expressions of the type $\sum_{n=0}^{\infty} a_n X^n$, where $a_n \in R$ for all $n \geq 0$. Define addition and $*$-product on $\mathrm{H}(R)$ as follows: for $f = \sum_{n=0}^{\infty} a_n X^n, g = \sum_{n=0}^{\infty} b_n X^n \in \mathrm{H}(R)$,

$$f + g = \sum_{n=0}^{\infty} (a_n + b_n) X^n \text{ and } f * g = \sum_{n=0}^{\infty} c_n X^n,$$

where $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$. Then, $\mathrm{H}(R)$ becomes a commutative ring with identity containing R under these two operations. The ring $\mathrm{H}(R)$ is called the *Hurwitz series ring* over R. The *Hurwitz polynomial ring* $\mathrm{h}(R)$ is the subring of $\mathrm{H}(R)$ consisting of formal expressions of the form $\sum_{i=0}^{n} a_i X^i$. The Hurwitz rings were first introduced by Keigher [1] to study differential algebra.

Let $R \subseteq T$ be an extension of commutative rings with identity. Let $\mathrm{H}(R,T) = \{f \in \mathrm{H}(T) \mid$ be the constant term of f belonging to $R\}$ and let $\mathrm{h}(R,T) = \{f \in \mathrm{h}(T) \mid$ be the constant term of f belonging to $R\}$. Then, $\mathrm{H}(R,T)$ and $\mathrm{h}(R,T)$ are commutative rings with identity satisfying $\mathrm{H}(R) \subseteq \mathrm{H}(R,T) \subseteq \mathrm{H}(T)$ and $\mathrm{h}(R) \subseteq \mathrm{h}(R,T) \subseteq \mathrm{h}(T)$. The rings $\mathrm{H}(R,T)$ and $\mathrm{h}(R,T)$ are called the *composite Hurwitz series ring* and the *composite Hurwitz polynomial ring*, respectively. Note that, if $R \subsetneq T$, then $\mathrm{H}(R,T)$ (respectively, $\mathrm{h}(R,T)$) gives algebraic properties of Hurwitz series type rings (respectively, Hurwitz polynomial type rings) strictly between two Hurwitz series rings $\mathrm{H}(R)$ and $\mathrm{H}(T)$ (respectively, Hurwitz polynomial rings $\mathrm{h}(R)$ and $\mathrm{h}(T)$). In many cases, algebraic structures of $\mathrm{H}(R,T)$ (respectively, $\mathrm{h}(R,T)$) are completely different from those of $\mathrm{H}(R)$ and $\mathrm{H}(T)$ (respectively, $\mathrm{h}(R)$ and $\mathrm{h}(T)$); thus, these kinds of rings have been studied by several mathematicians.

Let $R \subseteq T$ be an extension of commutative rings with identity, $u : R \to T$ the natural monomorphism and $v : \mathrm{H}(T) \to T$ the canonical epimorphism. Then, $\mathrm{H}(R,T)$ can be understood as a pullback of R and $\mathrm{H}(T)$ as follows:

$$\begin{array}{ccc} \mathrm{H}(R,T) = R \times_T \mathrm{H}(T) & \longrightarrow & R \\ \big\downarrow & & \big\downarrow u \\ \mathrm{H}(T) & \xrightarrow{v} & T \cong \mathrm{H}(T)/XT[\![X]\!]. \end{array}$$

Similarly, $\mathrm{h}(R,T)$ is also considered as a pullback of R and $\mathrm{h}(T)$.

The readers can refer to [1–4] for the Hurwitz rings and to [5–7] for the composite Hurwitz rings.

1.2. PP-Rings and PF-Rings

Let R be a commutative ring with identity and, for each $a \in R$, set $\mathrm{ann}_R(a) = \{r \in R \mid ra = 0\}$. Then, $\mathrm{ann}_R(a)$ is an ideal of R. Recall that R is a *PP-ring* if every principal ideal of R is projective as an R-module; and R is a *PF-ring* if every principal ideal of R is flat as an R-module. Since every projective module is flat, every PP-ring is a PF-ring. In fact, the concepts of PP-rings and PF-rings were first introduced by Hattori in a noncommutative setting [8] (page 151). In [9] (page 687), Evans mentioned that R is a PP-ring if and only if, for each $a \in R$, $\mathrm{ann}_R(a)$ is generated by an idempotent element of R. In [10] (Theorem 1), Al-Ezeh proved that R is a PF-ring if and only if for each $a \in R$, $\mathrm{ann}_R(a)$ is a pure ideal of R (or, equivalently, for each $a \in R$ and $b \in \mathrm{ann}_R(a)$, there exists an element $c \in \mathrm{ann}_R(a)$ such that $b = bc$). (Recall that an ideal I of R is *pure* if, for any $a \in I$, there exists an element $b \in I$ such that $ab = a$.) It is well known that R is a PP-ring if and only if R is a PF-ring and $\mathrm{Min}(R)$ with the induced Zariski topology is compact, where $\mathrm{Min}(R)$ is the set of minimal prime ideals of R [11] (Theorem 4.2.10). In addition, it was shown in [12] (Lemma 2.2) that a PF-ring is a reduced ring. (Recall that the ring R is a *reduced ring* if R has no nonzero nilpotent elements.)

In this paper, we study equivalence conditions for composite Hurwitz rings $\mathrm{H}(R,T)$ and $\mathrm{h}(R,T)$ to be PF-rings and PP-rings, where $R \subseteq T$ is an extension of commutative rings with identity. In Section 2, we study the McCoy condition in the composite Hurwitz rings. We show that, if T is both a reduced ring and a torsion-free $\mathbb{Z}$-module and $f = \sum_{i=0}^{\infty} a_i X^i$ and $g = \sum_{i=0}^{\infty} b_i X^i$ are elements of $\mathrm{H}(R,T)$ with $f * g = 0$, then $a_i b_j = 0$ for all $i, j \in \mathbb{N}_0$. We also prove that T is a torsion-free $\mathbb{Z}$-module if and only if for each zero-divisor f of $\mathrm{h}(R,T)$, there exists a nonzero element $t \in T$ such that $t * f = 0$ in $\mathrm{h}(T)$. In Section 3, we investigate when the composite Hurwitz rings $\mathrm{H}(R,T)$ and $\mathrm{h}(R,T)$ are PF-rings. We show that $\mathrm{H}(R,T)$ (respectively, $\mathrm{h}(R,T)$) is a PF-ring if and only if for each $f, g \in \mathrm{H}(R,T)$ (respectively, $\mathrm{h}(R,T)$) with $f * g = 0$, there exists an element $r \in R$ such that $r * f = 0$ and $r * g = g$. We also prove that, if T is a Noetherian ring, then $\mathrm{H}(R,T)$ is a PF-ring if and only if $\mathrm{h}(R,T)$ is a PF-ring. In Section 4, we study when the composite Hurwitz rings $\mathrm{H}(R,T)$ and $\mathrm{h}(R,T)$ are PP-rings. We show that $\mathrm{H}(R,T)$ is a PP-ring if and only if R is a PP-ring, T is a torsion-free $\mathbb{Z}$-module, for each $a \in T$, there exists an element $e \in \mathrm{Idem}(R)$ such that $\mathrm{ann}_T(a) = eT$, and any (increasing) sequence in $\mathrm{Idem}(R)$ admits the least upper bound in $(R, \leq)$ that belongs to $\mathrm{Idem}(R)$. We also prove that $\mathrm{h}(R,T)$ is a PP-ring if and only if R is a PP-ring, T is a torsion-free $\mathbb{Z}$-module and, for each $a \in T$, there exists an element $e \in \mathrm{Idem}(R)$ such that $\mathrm{ann}_T(a) = eT$. We show that, if R is a Noetherian ring, then $\mathrm{H}(R,T)$ is a PP-ring if and only if $\mathrm{h}(R,T)$ is a PP-ring. Finally, in Section 5, we give some examples of PP-rings and PF-rings via the composite Hurwitz rings. By these examples, we indicate that the converse of some results are not generally true and the Noetherian condition in some results is essential.

2. The McCoy Condition in Composite Hurwitz Rings

We start this section with a simple result. The proof is straightforward; thus, we omit it.

Lemma 1. *If $R \subseteq T$ is an extension of commutative rings with identity, then the following assertions hold.*

(1) *If T is a reduced ring, then so is R.*
(2) *If T is a torsion-free $\mathbb{Z}$-module, then so is R.*

Let $R \subseteq T$ be an extension of commutative rings with identity. We next study zero-divisors in composite Hurwitz rings $\mathrm{H}(R, T)$ and $\mathrm{h}(R, T)$.

Proposition 1. *Let $R \subseteq T$ be an extension of commutative rings with identity. Suppose that T is a reduced ring which is a torsion-free $\mathbb{Z}$-module. If $f = \sum_{i=0}^{\infty} a_i X^i$ and $g = \sum_{j=0}^{\infty} b_j X^j$ are elements of $\mathrm{H}(R, T)$ such that $f * g = 0$, then $a_i b_j = 0$ for all $i, j \in \mathbb{N}_0$.*

Proof. Suppose that $f * g = 0$. Then, $a_0 b_0 = 0$. Suppose that $a_0 b_0 = \cdots = a_0 b_n = 0$ for some $n \in \mathbb{N}_0$. Then, the coefficient of X^{n+1} in $f * g$ is $\sum_{i=0}^{n+1} \binom{n+1}{i} a_i b_{n+1-i} = 0$. Multiplying both sides by a_0, $a_0^2 b_{n+1} = 0$. Since T is a reduced ring, $a_0 b_{n+1} = 0$. By the induction, $a_0 b_j = 0$ for all $j \in \mathbb{N}_0$. This also implies that $a_0 * g = 0$.

We next suppose that $a_0 * g = \cdots = a_m X^m * g = 0$ for some $m \in \mathbb{N}_0$. Let $h = \left(\sum_{i=m+1}^{\infty} a_i X^i\right) * g$. Then, $h = f * g - \left(\sum_{i=0}^{m} a_i X^i\right) * g = 0$; thus, $a_{m+1} b_0 = 0$. Suppose that $a_{m+1} b_0 = \cdots = a_{m+1} b_n = 0$ for some $n \in \mathbb{N}_0$. Note that the coefficient of X^{m+n+2} in h is $\sum_{i=m+1}^{m+n+2} \binom{m+n+2}{i} a_i b_{m+n+2-i} = 0$. By multiplying both sides by a_{m+1}, $\binom{m+n+2}{m+1} a_{m+1}^2 b_{n+1} = 0$. Since T is a torsion free $\mathbb{Z}$-module, $a_{m+1}^2 b_{n+1} = 0$. Since T is a reduced ring, $a_{m+1} b_{n+1} = 0$. By the induction, $a_{m+1} b_j = 0$ for all $j \in \mathbb{N}_0$. This shows that $a_{m+1} X^{m+1} * g = 0$.

Thus, by the induction, $a_i b_j = 0$ for all $i, j \in \mathbb{N}_0$. $\square$

We give examples which show that two conditions "T is a reduced ring" and "T is a torsion-free $\mathbb{Z}$-module" in Proposition 1 are not superfluous.

Example 1. *Let $\mathbb{Z}_6$ be the ring of integers modulo 6. Then, $\mathbb{Z}_6$ is a reduced ring which is not a torsion-free $\mathbb{Z}$-module. Note that $2X * 5X^2 = 0$ in $\mathrm{H}(\mathbb{Z}_6)$ but $2 \cdot 5 \neq 0$ in $\mathbb{Z}_6$. Hence, the condition "T is a torsion-free $\mathbb{Z}$-module" in Proposition 1 is essential.*

Example 2. *Let $\mathbf{A} = \{Y\} \cup \{Z_n \mid n \in \mathbb{N}_0\}$ be a set of indeterminates over $\mathbb{Q}$, I the ideal of $\mathbb{Q}[\mathbf{A}]$ generated by the set $\{YZ_0\} \cup \{(n+1)Z_n + YZ_{n+1} \mid n \in \mathbb{N}_0\}$ and $R = \mathbb{Q}[\mathbf{A}]/I$. For an element $h \in \mathbb{Q}[\mathbf{A}]$, let $\overline{h}$ denote the homomorphic image of h in R.*

(1) *Suppose that there exist an integer $m \geq 2$ and an element $h \in \mathbb{Q}[\mathbf{A}]$ such that $m\overline{h} = 0$. Then, $mh \in I$. Since m is a unit in $\mathbb{Q}$, $h \in I$; thus, $\overline{h} = 0$. Hence, R is a torsion-free $\mathbb{Z}$-module.*
(2) *Note that $Y^2 Z_1 = (Z_0 + YZ_1)Y - YZ_0 \in I$; thus, $(YZ_1)^2 \in I$. Suppose to the contrary that $YZ_1 \in I$. Then, $YZ_1 = YZ_0 f_0 + \sum_{i=1}^{n}(iZ_{i-1} + YZ_i) f_i$ for some $f_0, \ldots, f_n \in \mathbb{Q}[\mathbf{A}]$; thus, by an easy calculation, the constant term of f_1 is 0 and, for each $k \in \{2, \ldots, n\}$, the coefficient of Y^{k-1} in f_k is $(-1)^k \frac{1}{k!}$. Therefore, the coefficient of $Y^n Z_n$ in YZ_1 is $(-1)^n \frac{1}{n!}$, which is absurd. Hence, $YZ_1 \notin I$. Thus, R is not a reduced ring.*
(3) *Let $f = \overline{Y} + X$ and $g = \sum_{i=0}^{\infty} \overline{Z_i} X^i$ be elements of $\mathrm{H}(R)$. Then, $f * g = 0$ but $\overline{Y}\,\overline{Z_1} \neq 0$. This shows that the condition "T is a reduced ring" in Proposition 1 is essential.*

Let R be a commutative ring with identity and $\mathrm{Z}(R)$ the set of nonzero zero-divisors of R. Recall that $\mathrm{h}(R)$ satisfies the *McCoy condition* if for any $f \in \mathrm{Z}(\mathrm{h}(R))$, there exists a nonzero element $a \in R$ such that $a * f = 0$ ([2] Section 4).

Proposition 2. *If $R \subseteq T$ is an extension of commutative rings with identity, then the following assertions are equivalent.*

(1) *For each $f \in \mathrm{Z}(\mathrm{h}(R, T))$, there exists a nonzero element $t \in T$ such that $t * f = 0$ in $\mathrm{h}(T)$.*
(2) *T is a torsion-free $\mathbb{Z}$-module.*

(3) $\mathrm{h}(T)$ *satisfies the McCoy condition.*

Proof. (1) $\Rightarrow$ (2) Suppose that T is not a torsion-free $\mathbb{Z}$-module. Then, there exist an integer $m \geq 2$ and a nonzero element a of T such that $ma = 0$; thus, $aX^{m-1} * X = maX^m = 0$. Hence, $X \in \mathrm{Z}(\mathrm{h}(R,T))$. However, $t * X \neq 0$ for any nonzero element t of T. This is a contradiction to the hypothesis. Thus, T is a torsion-free $\mathbb{Z}$-module.

(2) $\Leftrightarrow$ (3) The equivalence appears in [2] (Theorem 4.1).

(3) $\Rightarrow$ (1) Let $f \in \mathrm{Z}(\mathrm{h}(R,T))$. Then, $f \in \mathrm{Z}(\mathrm{h}(T))$. Since $\mathrm{h}(T)$ satisfies the McCoy condition, there exists a nonzero element $t \in T$ such that $t * f = 0$. $\square$

3. PF-Rings

Let $R \subseteq T$ be an extension of commutative rings with identity. In this section, we give necessary and sufficient conditions for the rings $\mathrm{H}(R,T)$ and $\mathrm{h}(R,T)$ to be PF-rings. To do this, we first study some relations among $\mathrm{H}(R,T)$, $\mathrm{h}(R,T)$, R and T in the view of PF-rings.

Lemma 2. *Let $R \subseteq T$ be an extension of commutative rings with identity and let D be either $\mathrm{H}(R,T)$ or $\mathrm{h}(R,T)$. If D is a PF-ring, then R and T are both PF-rings and torsion-free $\mathbb{Z}$-modules.*

Proof. Let $a \in T$ and $b \in \mathrm{ann}_T(a)$. Then, $aX, bX \in D$ with $bX \in \mathrm{ann}_D(aX)$. Since D is a PF-ring, there exists an element $f = \sum_{i\geq 0} c_i X^i \in \mathrm{ann}_D(aX)$ such that $bX * f = bX$. Hence, $ac_0 = 0$ and $bc_0 = b$. Thus, T is a PF-ring. A similar argument also shows that R is a PF-ring.

Let $a \in T$ and suppose that there exists an integer $m \geq 2$ such that $ma = 0$. Then, $aX^{m-1} * X = maX^m = 0$; thus, $aX^{m-1} \in \mathrm{ann}_D(X)$. Since D is a PF-ring, we can find an element $g = \sum_{i\geq 0} b_i X^i \in \mathrm{ann}_D(X)$ such that $aX^{m-1} * g = aX^{m-1}$. Hence, $ab_0 = a$ and $b_0 = 0$, which indicates that $a = 0$. Thus, T is a torsion-free $\mathbb{Z}$-module. By Lemma 1(2), R is also a torsion-free $\mathbb{Z}$-module. $\square$

Let R be a commutative ring with identity. Then, it is obvious that, if R is a torsion-free $\mathbb{Z}$-module, then $\mathrm{char}(R) = 0$. Hence, by Lemma 2, we obtain

Corollary 1. *Let $R \subseteq T$ be an extension of commutative rings with identity. If $\mathrm{char}(T) > 0$, then neither $\mathrm{H}(R,T)$ nor $\mathrm{h}(R,T)$ is a PF-ring.*

We give necessary and sufficient conditions for the composite Hurwitz series ring to be a PF-ring.

Theorem 1. *If $R \subseteq T$ is an extension of commutative rings with identity, then the following statements are equivalent.*

(1) $\mathrm{H}(R,T)$ *is a PF-ring.*
(2) *T is a torsion-free $\mathbb{Z}$-module and if $a_0, b_0 \in R$ and $a_i, b_i \in T$ for all $i \geq 1$ satisfy $a_i b_j = 0$ for all $i, j \in \mathbb{N}_0$, then there exists an element $r \in R$ such that $ra_i = 0$ and $rb_i = b_i$ for all $i \in \mathbb{N}_0$.*
(3) *For each $f, g \in \mathrm{H}(R,T)$ such that $f * g = 0$, there exists an element $r \in R$ such that $r * f = 0$ and $r * g = g$.*

Proof. (1) $\Rightarrow$ (2) Let $f = \sum_{i=0}^{\infty} a_i X^i$ and $g = \sum_{i=0}^{\infty} b_i X^i$. Then, by the assumption, $f, g \in \mathrm{H}(R,T)$ and $f * g = 0$. Since $\mathrm{H}(R,T)$ is a PF-ring, there exists an element $h = \sum_{i=0}^{\infty} c_i X^i \in \mathrm{H}(R,T)$ such that $f * h = 0$ and $g * h = g$. By Lemma 2 and [12] (Lemma 2.2), T is both a reduced ring and a torsion-free $\mathbb{Z}$-module. Thus, by Proposition 1, $c_0 a_i = 0$ and $c_0 b_i = b_i$ for all $i \in \mathbb{N}_0$.

(2) $\Rightarrow$ (3) Let $f = \sum_{i=0}^{\infty} a_i X^i$ and $g = \sum_{i=0}^{\infty} b_i X^i$ be elements of $\mathrm{H}(R,T)$ such that $f * g = 0$. Note that, by the assumption, T is a PF-ring; thus, T is a reduced ring. Since T is a torsion-free $\mathbb{Z}$-module, by Proposition 1, $a_i b_j = 0$ for all $i, j \in \mathbb{N}_0$. Hence, we can find an element $r \in R$ such that $ra_i = 0$ and $rb_i = b_i$ for all $i \in \mathbb{N}_0$. Thus, $r * f = 0$ and $r * g = g$.

(3) $\Rightarrow$ (1) This implication comes from the definition of PF-rings. $\square$

Let R be a commutative ring with identity and let D be either $\mathrm{H}(R)$ or $\mathrm{h}(R)$. Recall that R is a *Noetherian ring* if every ideal of R is finitely generated. For an $f = \sum_{i\geq 0} a_iX^i \in D$, the *content ideal* of f is the ideal of R generated by the set $\{a_i \mid i \in \mathbb{N}_0\}$ and is denoted by $c_R(f)$.

Corollary 2. *Let $R \subseteq T$ be an extension of commutative rings with identity. If T is a Noetherian ring, then the following assertions are equivalent.*

(1) $\mathrm{H}(R,T)$ *is a PF-ring.*
(2) *T is a torsion-free $\mathbb{Z}$-module and, for each $a \in T$ and $b \in \mathrm{ann}_T(a)$, there exists an element $r \in R$ such that $ra = 0$ and $rb = b$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $\mathrm{H}(R,T)$ is a PF-ring. Then, by Lemma 2, T is a torsion-free $\mathbb{Z}$-module. Let $a \in T$ and $b \in \mathrm{ann}_T(a)$. Then, $aX, bX \in \mathrm{H}(R,T)$ with $aX * bX = 0$; thus, by Theorem 1, there exists an element $r \in R$ such that $r * aX = 0$ and $r * bX = bX$. Thus, $ra = 0$ and $rb = b$.

(2) $\Rightarrow$ (1) Let $f = \sum_{i=0}^{\infty} a_iX^i \in \mathrm{H}(R,T)$ and $g = \sum_{i=0}^{\infty} b_iX^i \in \mathrm{ann}_{\mathrm{H}(R,T)}(f)$. Since T is a Noetherian ring, $c_T(f) = (a_0,\ldots,a_m)$ and $c_T(g) = (b_0,\ldots,b_n)$ for some $m,n \in \mathbb{N}_0$. Note that, by the hypothesis, T is a PF-ring; thus, T is a reduced ring. Since T is a torsion-free $\mathbb{Z}$-module, Proposition 1 indicates that $a_ib_j = 0$ for all $i \in \{0,\ldots,m\}$ and $j \in \{0,\ldots,n\}$. Therefore, by the hypothesis, for each $i \in \{0,\ldots,m\}$ and $j \in \{0,\ldots,n\}$, there exists an element $r_{ij} \in R$ such that $r_{ij}a_i = 0$ and $r_{ij}b_j = b_j$. For each $j \in \{0,\ldots,n\}$, set $d_j = r_{0j}\cdots r_{mj}$. Then, for all $i \in \{0,\ldots,m\}$ and $j \in \{0,\ldots,n\}$, $d_ja_i = 0$ and $d_jb_j = b_j$.

Let $c_0 = d_0$ and, for each $k \in \{0,\ldots,n-1\}$, let $c_{k+1} = c_k + d_{k+1} - c_kd_{k+1}$. Then, $c_k \in R$ for all $k \in \{0,\ldots,n\}$. By an iterative calculation, it can be shown that for each $k \in \{0,\ldots,n\}$, $c_ka_i = 0$ and $c_kb_j = b_j$ for all $i \in \{0,\ldots,m\}$ and $j \in \{0,\ldots,k\}$. Hence, $c_na_i = 0$ and $c_nb_j = b_j$ for all $i \in \{0,\ldots,m\}$ and $j \in \{0,\ldots,n\}$. Since $c_T(f) = (a_0,\ldots,a_m)$ and $c_T(g) = (b_0,\ldots,b_n)$, $c_n * f = 0$ and $c_n * g = g$. Thus, by Theorem 1, $\mathrm{H}(R,T)$ is a PF-ring. $\square$

By Theorem 1 and Corollary 2, we can regain

Corollary 3 ([12] (Theorem 2.5 and Corollary 2.7))**.** *If R is a commutative ring with identity, then the following assertions hold.*

(1) *$\mathrm{H}(R)$ is a PF-ring if and only if for each $f,g \in \mathrm{H}(R)$ with $f * g = 0$, there exists an element $r \in R$ such that $r * f = 0$ and $r * g = g$.*
(2) *If R is a Noetherian ring, then $\mathrm{H}(R)$ is a PF-ring if and only if R is a PF-ring which is a torsion-free $\mathbb{Z}$-module.*

Corollary 4. *Let $R \subseteq T$ be an extension of commutative rings with identity. If $\mathrm{H}(R,T)$ is a PF-ring, then $\mathrm{H}(R)$ and $\mathrm{H}(T)$ are PF-rings.*

Proof. Let f and g be elements of $\mathrm{H}(T)$ such that $f * g = 0$. Then, $f * X$ and $g * X$ are elements of $\mathrm{H}(R,T)$ such that $(f * X) * (g * X) = 0$. Since $\mathrm{H}(R,T)$ is a PF-ring, by Theorem 1, there exists an element $r \in R$ such that $r * (f * X) = 0$ and $r * (g * X) = g * X$. Note that, by Theorem 1, T is a torsion-free $\mathbb{Z}$-module; thus, $r * f = 0$ and $r * g = g$. Thus, by Corollary 3(1), $\mathrm{H}(T)$ is a PF-ring.

Let f and g be elements of $\mathrm{H}(R)$ such that $f * g = 0$. Then, f and g are elements of $\mathrm{H}(R,T)$ such that $f * g = 0$. Since $\mathrm{H}(R,T)$ is a PF-ring, by Theorem 1, there exists an element $r \in R$ such that $r * f = 0$ and $r * g = g$. Thus, by Corollary 3(1), $\mathrm{H}(T)$ is a PF-ring. $\square$

We next study when the composite Hurwitz polynomial ring is a PF-ring.

Theorem 2. *If $R \subseteq T$ is an extension of commutative rings with identity, then the following statements are equivalent.*

(1) $\mathrm{h}(R,T)$ *is a PF-ring.*

(2) *T is a torsion-free $\mathbb{Z}$-module and if $a_0, b_0 \in R$ and $a_i, b_j \in T$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ are such that $a_i b_j = 0$ for all $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$, then there exists an element $r \in R$ such that $ra_i = 0$ and $rb_j = b_j$ for all $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$.*

(3) *For each $f, g \in \mathrm{h}(R, T)$ such that $f * g = 0$, there exists an element $r \in R$ such that $r * f = 0$ and $r * g = g$.*

Proof. (1) $\Rightarrow$ (2) Let $f = \sum_{i=0}^{m} a_i X^i$ and $g = \sum_{i=0}^{n} b_i X^i$. Then, by the assumption, $f, g \in \mathrm{h}(R, T)$ and $f * g = 0$. Since $\mathrm{h}(R, T)$ is a PF-ring, we can find an element $h = \sum_{i=0}^{\ell} c_i X^i \in \mathrm{h}(R, T)$ such that $f * h = 0$ and $g * h = g$. By Lemma 2 and [12] (Lemma 2.2), T is both a torsion-free $\mathbb{Z}$-module and a reduced ring. Thus, by Proposition 1, $c_0 a_i = 0$ and $c_0 b_j = b_j$ for all $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$.

(2) $\Rightarrow$ (3) Let $f = \sum_{i=0}^{m} a_i X^i$ and $g = \sum_{i=0}^{n} b_i X^i$ be elements of $\mathrm{h}(R, T)$ such that $f * g = 0$. Note that, by the hypothesis, T is a PF-ring; thus, T is a reduced ring. Since T is a torsion-free $\mathbb{Z}$-module, Proposition 1 implies that $a_i b_j = 0$ for all $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$. Hence, we can find an element $r \in R$ such that $ra_i = 0$ and $rb_j = b_j$ for all $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$. Thus, $r * f = 0$ and $r * g = g$.

(3) $\Rightarrow$ (1) This implication follows directly from the definition of PF-rings. $\square$

Corollary 5. *If $R \subseteq T$ is an extension of commutative rings with identity, then the following assertions are equivalent.*

(1) *$\mathrm{h}(R, T)$ is a PF-ring.*

(2) *T is a torsion-free $\mathbb{Z}$-module and, for each $a \in T$ and $b \in \mathrm{ann}_T(a)$, there exists an element $r \in R$ such that $ra = 0$ and $rb = b$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $\mathrm{h}(R, T)$ is a PF-ring. Then, by Lemma 2, T is a torsion-free $\mathbb{Z}$-module. Let $a \in T$ and $b \in \mathrm{ann}_T(a)$. Then, $aX, bX \in \mathrm{h}(R, T)$ with $bX \in \mathrm{ann}_{\mathrm{h}(R,T)}(aX)$. Hence, by Theorem 2, there exists an element $r \in R$ such that $r * aX = 0$ and $r * bX = bX$. Thus, $ra = 0$ and $rb = b$.

(2) $\Rightarrow$ (1) Let $f = \sum_{i=0}^{m} a_i X^i$ and $g = \sum_{i=0}^{n} b_i X^i$ be elements of $\mathrm{h}(R, T)$ such that $f * g = 0$. Note that, by the hypothesis, T is a PF-ring; thus, T is a reduced ring. Since T is a torsion-free $\mathbb{Z}$-module, by Proposition 1, $a_i b_j = 0$ for all $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$. Therefore, by the hypothesis, for each $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$, we can find an element $r_{ij} \in R$ such that $r_{ij} a_i = 0$ and $r_{ij} b_j = b_j$. For each $j \in \{0, \ldots, n\}$, set $d_j = r_{0j} \cdots r_{mj}$. Then, for all $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$, $d_j a_i = 0$ and $d_j b_j = b_j$.

Let $c_0 = d_0$ and, for each $k \in \{0, \ldots, n-1\}$, let $c_{k+1} = c_k + d_{k+1} - c_k d_{k+1}$. Then, a routine calculation shows that for each $k \in \{0, \ldots, n\}$, $c_k a_i = 0$ and $c_k b_j = b_j$ for all $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, k\}$. Therefore, $c_n a_i = 0$ and $c_n b_j = b_j$ for all $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$. Hence, $c_n * f = 0$ and $c_n * g = g$. Note that $c_n \in R$. Thus, by Theorem 2, $\mathrm{h}(R, T)$ is a PF-ring. $\square$

Corollary 6. *If $R \subseteq T$ is an extension of commutative rings with identity, then the following assertions hold.*

(1) *If $\mathrm{H}(R, T)$ is a PF-ring, then $\mathrm{h}(R, T)$ is a PF-ring.*

(2) *If T is a Noetherian ring, then $\mathrm{H}(R, T)$ is a PF-ring if and only if $\mathrm{h}(R, T)$ is a PF-ring.*

Proof. (1) This is an immediate consequence of Theorems 1 and 2.

(2) The equivalence follows directly from Corollaries 2 and 5. $\square$

Corollary 7 (cf. [12] Theorem 2.6)**.** *If R is a commutative ring with identity, then the following conditions are equivalent.*

(1) *$\mathrm{h}(R)$ is a PF-ring.*

(2) *R is both a PF-ring and a torsion-free $\mathbb{Z}$-module.*

(3) *For each $f, g \in \mathrm{h}(R)$ with $f * g = 0$, there exists an element $r \in R$ such that $r * f = 0$ and $r * g = g$.*

Proof. The equivalences come directly from Theorem 2 and Corollary 5. □

Corollary 8. *Let $R \subseteq T$ be an extension of commutative rings with identity. If* $\mathrm{h}(R, T)$ *is a PF-ring, then* $\mathrm{h}(R)$ *and* $\mathrm{h}(T)$ *are PF-rings.*

Proof. The result follows from Lemma 2 and Corollary 7. □

4. PP-Rings

Let $R \subseteq T$ be an extension of commutative rings with identity. In this section, we give equivalent conditions for the rings $\mathrm{H}(R, T)$ and $\mathrm{h}(R, T)$ to be PP-rings. Our first result in this section is a characterization of idempotent elements in $\mathrm{H}(R, T)$ and $\mathrm{h}(R, T)$.

Let R be a commutative ring with identity and let $\mathrm{Idem}(R)$ be the set of idempotent elements of R.

Lemma 3. *If $R \subseteq T$ is an extension of commutative rings with identity, then* $\mathrm{Idem}(\mathrm{H}(R, T)) = \mathrm{Idem}(\mathrm{h}(R, T)) = \mathrm{Idem}(R)$.

Proof. Clearly, $\mathrm{Idem}(R) \subseteq \mathrm{Idem}(\mathrm{h}(R, T)) \subseteq \mathrm{Idem}(\mathrm{H}(R, T))$; thus, it remains to prove that $\mathrm{Idem}(\mathrm{H}(R, T)) \subseteq \mathrm{Idem}(R)$. Let $f \in \mathrm{Idem}(\mathrm{H}(R, T))$. Then, $f \in \mathrm{Idem}(\mathrm{H}(T))$. Note that $\mathrm{Idem}(\mathrm{H}(T)) = \mathrm{Idem}(T)$ [13] (Proposition 2.3). Thus, $f \in \mathrm{Idem}(T) \cap R = \mathrm{Idem}(R)$. □

Let $R \subseteq T$ be an extension of commutative rings with identity. We next study PP-properties in terms of relations among $\mathrm{H}(R, T)$, $\mathrm{h}(R, T)$, R, and T.

Lemma 4. *Let $R \subseteq T$ be an extension of commutative rings with identity and let D be either* $\mathrm{H}(R, T)$ *or* $\mathrm{h}(R, T)$. *If D is a PP-ring, then R and T are both PP-rings and torsion-free $\mathbb{Z}$-modules.*

Proof. Note that any PP-ring is a PF-ring; thus, by Lemma 2, R and T are torsion-free $\mathbb{Z}$-modules. Let $a \in T$. Since D is a PP-ring, by Lemma 3, there exists an element $e \in \mathrm{Idem}(R)$ such that $\mathrm{ann}_D(aX) = e * D$. Let $b \in \mathrm{ann}_T(a)$. Then, $bX \in \mathrm{ann}_D(aX)$; thus, $bX = e * f$ for some $f \in D$. Therefore, $b \in eT$, which means that $\mathrm{ann}_T(a) \subseteq eT$. The reverse containment is obvious. Hence, $\mathrm{ann}_T(a) = eT$. Thus, T is a PP-ring. A similar argument shows that R is a PP-ring. □

Let R be a commutative ring with identity. If $\mathrm{char}(R) > 0$, then R is not a torsion-free $\mathbb{Z}$-module. Hence, by Lemma 4, we obtain

Corollary 9. *Let $R \subseteq T$ be an extension of commutative rings with identity. If* $\mathrm{char}(T) > 0$, *then neither* $\mathrm{H}(R, T)$ *nor* $\mathrm{h}(R, T)$ *is a PP-ring.*

Lemma 5. *Let R be a commutative ring with identity. If R is a reduced ring, then the following conditions hold.*

(1) *The relation defined on R by $a \leq b$ if and only if $a^2 = ab$ is a partial order.*
(2) *Let $(e_n)_{n \geq 0}$ be a sequence of idempotent elements of R. If e is a least upper bound of $(e_n)_{n \geq 0}$, which is an idempotent element of R, and a is any upper bound of $(e_n)_{n \geq 0}$, which is an idempotent element of R, then $e \leq a$.*
(3) *If a sequence of idempotent elements of R has a least upper bound in R which is an idempotent element, then it is unique.*
(4) *If each increasing sequence of idempotent elements of R has the least upper bound in R, which is an idempotent element, then each sequence of idempotent elements of R has the least upper bound in R, which is an idempotent element.*
(5) *Any finite sequence of idempotent elements of R has the least upper bound in R which is an idempotent element.*

Proof. (1) This assertion was shown in [12] (Lemma 3.8).

(2) By the assumption, $e_n = e_n e$ and $e_n = e_n a$ for all $n \in \mathbb{N}_0$; thus, $e_n^2 = e_n^2 ea = e_n ea$ for all $n \in \mathbb{N}_0$. Therefore, ea is an upper bound of $(e_n)_{n\geq 0}$. Note that $(ea)^2 = ea = e^2 a = ea^2$; thus, $ea \leq e$ and $ea \leq a$. Hence, $e = ea$ by the minimality of e. Thus, $e \leq a$.

(3) This is an immediate consequence of (2).

(4) This appears in [14] (Lemma 2.5).

(5) The result can be shown by a similar argument as in the proof of [14] (Lemma 2.5). □

Example 3. *Let $T = \prod_{i=1}^{\infty} \mathbb{Z}$, $1_T = (1, 1, \dots)$ and R the subring of T generated by $\bigoplus_{i=1}^{\infty} \mathbb{Z}$ and 1_T.*

(1) Let $(t_i)_{i\geq 0}$ be an increasing sequence in $\mathrm{Idem}(T)$ and, for each $i \geq 0$, let $t_i = (t_{i1}, t_{i2}, \dots)$. For each $j \geq 1$, let

$$e_j = \begin{cases} 1 & \text{if } t_{ij} \neq 0 \text{ for some } i \geq 0 \\ 0 & \text{if } t_{ij} = 0 \text{ for all } i \geq 0. \end{cases}$$

Let $e = (e_1, e_2, \dots)$. Then, $e \in \mathrm{Idem}(T)$ such that e is the least upper bound of $(t_i)_{i\geq 0}$.

(2) For each $i \in \mathbb{N}_0$ and $j \in \mathbb{N}$, let

$$e_{ij} = \begin{cases} 1 & \text{if } j \leq 2i+1 \text{ and } j \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

For each $i \in \mathbb{N}_0$, let $e_i = (e_{i1}, e_{i2}, \dots)$. Then, $(e_i)_{i\geq 0}$ is an increasing sequence in $\mathrm{Idem}(R)$. Suppose to the contrary that there exists the least upper bound $a = (a_1, a_2, \dots)$ of $(e_i)_{i\geq 0}$ in $\mathrm{Idem}(R)$. Then, $e_i^2 = e_i a$ for all $i \geq 0$; thus, $a_{2i+1} = 1$ for all $i \in \mathbb{N}_0$. Since $a \in R$, we can find an integer $n \geq 1$ such that $a_k = 1$ for all $k \geq n$. Let m be the smallest even integer such that $m \geq n$ and, for each $i \geq 1$, let

$$b_i = \begin{cases} a_i & \text{if } i \neq m \\ 0 & \text{if } i = m. \end{cases}$$

Let $b = (b_1, b_2, \dots)$. Then, $b \in \mathrm{Idem}(R)$ and $e_i \leq b \lneq a$ for all $i \in \mathbb{N}_0$. This contradicts the fact that a is the least upper bound of $(e_i)_{i\geq 0}$. Thus, $(e_i)_{i\geq 0}$ does not have a least upper bound in $\mathrm{Idem}(R)$.

Let R be a commutative ring with identity. Then, $\mathrm{Reg}(R)$ denotes the set of regular elements of R.

Lemma 6. *If $R \subseteq T$ is an extension of commutative rings with identity, then the following assertions are equivalent.*

(1) For each $a \in T$, there exists an element $e \in \mathrm{Idem}(R)$ such that $\mathrm{ann}_T(a) = eT$.

(2) For each $a \in T$, there exist $e \in \mathrm{Idem}(R)$ and $t \in \mathrm{Reg}(T)$ such that $a = et$.

Proof. (1) $\Rightarrow$ (2) This implication was shown in the remark in the proof of [14] (Proposition 2.6).

(2) $\Rightarrow$ (1) Let $a \in T$. Then, there exist $e \in \mathrm{Idem}(R)$ and $t \in \mathrm{Reg}(T)$ such that $a = et$. Let $b \in \mathrm{ann}_T(a)$. Then, $ab = 0$; thus, $ebt = 0$. Since t is regular in T, $eb = 0$; thus, $b = b(1-e) \in (1-e)T$. Note that $(1-e)a = (1-e)et = 0$; thus, $1 - e \in \mathrm{ann}_T(a)$. Thus, $\mathrm{ann}_T(a) = (1-e)T$. □

We are ready to give a necessary and sufficient conditions for the composite Hurwitz series ring to be a PP-ring.

Theorem 3. *If $R \subseteq T$ is an extension of commutative rings with identity, then the following statements are equivalent.*

(1) $\mathrm{H}(R, T)$ is a PP-ring.

(2) *R is a PP-ring, T is a torsion-free $\mathbb{Z}$-module, for each $a \in T$, there exists an element $e \in \mathrm{Idem}(R)$ such that $\mathrm{ann}_T(a) = eT$, and any sequence in* $\mathrm{Idem}(R)$ *admits the least upper bound in $(R, \leq)$ that belongs to* $\mathrm{Idem}(R)$.

(3) *R is a PP-ring, T is a torsion-free $\mathbb{Z}$-module, for each $a \in T$, there exists an element $e \in \mathrm{Idem}(R)$ such that $\mathrm{ann}_T(a) = eT$, and any increasing sequence in* $\mathrm{Idem}(R)$ *admits the least upper bound in $(R, \leq)$ that belongs to* $\mathrm{Idem}(R)$.

Proof. (1) $\Rightarrow$ (2) Suppose that $\mathrm{H}(R,T)$ is a PP-ring. Then, by Lemma 4, R is a PP-ring and T is a torsion-free $\mathbb{Z}$-module. Let $a \in T$. Then, by Lemma 3, there exists an element $e \in \mathrm{Idem}(R)$ such that $\mathrm{ann}_{\mathrm{H}(R,T)}(aX) = e * \mathrm{H}(R,T)$; thus, $e * aX = 0$. Therefore, $ea = 0$. Hence, $eT \subseteq \mathrm{ann}_T(a)$. Let $b \in \mathrm{ann}_T(a)$. Then, $bX \in \mathrm{ann}_{\mathrm{H}(R,T)}(aX)$; thus, $bX = e * f$ for some $f \in \mathrm{H}(R,T)$. Therefore, $b \in eT$. Hence, $\mathrm{ann}_T(a) \subseteq eT$. Thus, $\mathrm{ann}_T(a) = eT$.

Let $(e_n)_{n\geq 0}$ be a sequence of idempotent elements of R and let $f = \sum_{n=0}^{\infty} e_n X^n$. Then, $f \in \mathrm{H}(R,T)$. Since $\mathrm{H}(R,T)$ is a PP-ring, by Lemma 3, there exists an element $e \in \mathrm{Idem}(R)$ such that $\mathrm{ann}_{\mathrm{H}(R,T)}(f) = e * \mathrm{H}(R,T)$. Now, we show that $1-e$ is the least upper bound of $(e_n)_{n\geq 0}$. Since $e * f = 0$, $ee_n = 0$ for all $n \in \mathbb{N}_0$; thus, $e_n(1-e) = e_n = e_n^2$. Hence, $e_n \leq 1-e$ for all $n \in \mathbb{N}_0$. Let $r \in R$ be such that $e_n \leq r$ for all $n \in \mathbb{N}_0$. Then, $e_n r = e_n$; thus, $(1-r) * f = 0$. Therefore, $1-r \in \mathrm{ann}_{\mathrm{H}(R,T)}(f)$. Let $g \in \mathrm{H}(R,T)$ be such that $1-r = e * g$. Then, $1-r = eg(0)$; thus, $r(1-e) = (1-eg(0))(1-e) = 1-e = (1-e)^2$. Hence, $1-e \leq r$. Thus, $1-e$ is the least upper bound of $(e_n)_{n\geq 0}$.

(2) $\Rightarrow$ (1) Let $f = \sum_{n=0}^{\infty} a_n X^n \in \mathrm{H}(R,T)$. Then, by the hypothesis and Lemma 6, for each $n \in \mathbb{N}_0$, there exist $e_n \in \mathrm{Idem}(R)$ and $t_n \in \mathrm{Reg}(T)$ such that $a_n = e_n t_n$. In addition, by the assumption, there exist an element $e \in \mathrm{Idem}(R)$ such that e is the least upper bound of $(e_n)_{n\geq 0}$; thus, $(1-e)e_n = 0$ for all $n \geq 0$. Therefore, $(1-e)a_n = 0$ for all $n \geq 0$. Hence, $(1-e) * f = 0$, which implies that $(1-e) * \mathrm{H}(R,T) \subseteq \mathrm{ann}_{\mathrm{H}(R,T)}(f)$. For the reverse containment, let $g = \sum_{n=0}^{\infty} b_n X^n \in \mathrm{ann}_{\mathrm{H}(R,T)}(f)$. Note that, by the hypothesis, T is a PP-ring; thus, T is a reduced ring. Since T is a torsion-free $\mathbb{Z}$-module, by Proposition 1, $b_i a_j = 0$ for all $i, j \in \mathbb{N}_0$. Note that, by the assumption and Lemma 6, for each $n \geq 0$, there exist $d_n \in \mathrm{Idem}(R)$ and $x_n \in \mathrm{Reg}(T)$ such that $b_n = d_n x_n$; thus, for all $i, j \in \mathbb{N}_0$, $d_i x_i e_j t_j = b_i a_j = 0$. Since x_i and t_j are regular elements of T for all $i, j \in \mathbb{N}_0$, $d_i e_j = 0$ for all $i, j \in \mathbb{N}_0$; thus, $(1-d_i)e_j = e_j$ for all $i, j \in \mathbb{N}_0$. This shows that $e_j \leq 1-d_i$ for all $i, j \in \mathbb{N}_0$. Since e is the least upper bound of $(e_n)_{n\geq 0}$, Lemma 5(2) indicates that $e \leq 1-d_i$ for all $i \geq 0$; thus, $e(1-d_i) = e$ for all $i \geq 0$. Therefore, $ed_i = 0$ for all $i \geq 0$, which shows that $eb_i = 0$ for all $i \geq 0$. Hence, $e * g = 0$, which implies that $g = (1-e) * g \in (1-e) * \mathrm{H}(R,T)$. Consequently, $\mathrm{ann}_{\mathrm{H}(R,T)}(f) = (1-e) * \mathrm{H}(R,T)$. Thus, $\mathrm{H}(R,T)$ is a PP-ring.

(2) $\Rightarrow$ (3) This implication is clear.

(3) $\Rightarrow$ (2) This implication was shown in Lemma 5(4). $\square$

Lemma 7. *Let R be a commutative ring with identity. If R is a reduced Noetherian ring and $(e_m)_{m\geq 0}$ is an increasing sequence of* $\mathrm{Idem}(R)$*, then there exists an integer $n \geq 0$ such that e_n is an upper bound of $(e_m)_{m\geq 0}$.*

Proof. Let $(e_m)_{m\geq 0}$ be an increasing sequence of $\mathrm{Idem}(R)$ and I the ideal of R generated by the set $\{e_m \mid m \geq 0\}$. Since R is a Noetherian ring, $I = (e_0, \ldots, e_n)$ for some $n \in \mathbb{N}_0$. Let k be an integer greater than n. Then, $e_k = r_{k0}e_0 + \cdots + r_{kn}e_n$ for some $r_{k0}, \ldots, r_{kn} \in R$. Note that $e_i^2 = e_i e_n$ for all $i \in \{0, \ldots, n\}$; thus, $e_k e_n = (r_{k0}e_0 + \cdots + r_{kn}e_n)e_n = e_k = e_k^2$. Hence, $e_k \leq e_n$. Thus, e_n is an upper bound of $(e_m)_{m\geq 0}$. $\square$

Corollary 10. *Let $R \subseteq T$ be an extension of commutative rings with identity. If R is a Noetherian ring, then $\mathrm{H}(R,T)$ is a PP-ring if and only if R is a PP-ring, T is a torsion-free $\mathbb{Z}$-module and, for each $a \in T$, there exists an element $e \in$* $\mathrm{Idem}(R)$ *such that $\mathrm{ann}_T(a) = eT$.*

Proof. The equivalence comes directly from Theorem 3 and Lemma 7. $\square$

Corollary 11 (cf. [12] Theorem 3.10)**.** *If R is a commutative ring with identity, then the following assertions hold.*

(1) $\mathrm{H}(R)$ *is a PP-ring if and only if R is both a PP-ring and a torsion-free $\mathbb{Z}$-module and any (increasing) sequence in* $\mathrm{Idem}(R)$ *admits the least upper bound in $(R, \leq)$ that belongs to* $\mathrm{Idem}(R)$*.*

(2) *If R is a Noetherian ring, then* $\mathrm{H}(R)$ *is a PP-ring if and only if R is both a PP-ring and a torsion-free $\mathbb{Z}$-module.*

Proof. (1) The equivalence is an immediate consequence of Theorem 3.

(2) The equivalence is an immediate consequence of (1) and Lemma 7. □

Corollary 12. *If $R \subseteq T$ is an extension of commutative rings with identity, then the following assertions hold.*

(1) $\mathrm{H}(R, T)$ *is a PP-ring if and only if* $\mathrm{H}(R)$ *is a PP-ring, T is a torsion-free $\mathbb{Z}$-module and, for each $a \in T$, there exists an element $e \in$* $\mathrm{Idem}(R)$ *such that* $\mathrm{ann}_T(a) = eT$*.*

(2) *If* $\mathrm{H}(R, T)$ *is a PP-ring, then* $\mathrm{H}(T)$ *is a PP-ring.*

Proof. (1) This result follows directly from suitable combinations of Lemma 1(2), Theorem 3 and Corollary 11(1).

(2) Suppose that $\mathrm{H}(R, T)$ is a PP-ring. Then, by Lemma 4, T is both a PP-ring and a torsion-free $\mathbb{Z}$-module. Let $(e_m)_{m\geq 0}$ be a sequence in $\mathrm{Idem}(T)$ and let $f = \sum_{m=0}^{\infty} e_m X^{m+1}$. Then, $f \in \mathrm{H}(R, T)$. Since $\mathrm{H}(R, T)$ is a PP-ring, Lemma 3 guarantees the existence of an element $e \in \mathrm{Idem}(R)$ such that $\mathrm{ann}_{\mathrm{H}(R,T)}(f) = e * \mathrm{H}(R, T)$. We now show that $1 - e$ is the least upper bound of $(e_m)_{m\geq 0}$. Since $e * f = 0$, $ee_m = 0$ for all $m \in \mathbb{N}_0$; thus, $e_m(1 - e) = e_m = e_m^2$ for all $m \in \mathbb{N}_0$. Hence, $e_m \leq 1 - e$ for all $m \in \mathbb{N}_0$. Let $y \in T$ be such that $e_m \leq y$ for all $m \in \mathbb{N}_0$. Then, $e_m y = e_m$; thus, $(1 - y)X * f = 0$. Therefore, $(1 - y)X \in \mathrm{ann}_{\mathrm{H}(R,T)}(f)$. Let $g \in \mathrm{H}(R, T)$ be such that $(1 - y)X = e * g$. Then, $1 - y = ec$ for some $c \in T$. Note that $y(1 - e) = (1 - ec)(1 - e) = 1 - e = (1 - e)^2$; thus, $1 - e \leq y$. Hence, $1 - e$ is the least upper bound of $(e_m)_{m\geq 0}$ that belongs to $\mathrm{Idem}(T)$. Thus, by Corollary 11(1), $\mathrm{H}(T)$ is a PP-ring. □

We next study the equivalent condition for the composite Hurwitz polynomial ring to be a PP-ring.

Theorem 4. *If $R \subseteq T$ is an extension of commutative rings with identity, then the following statements are equivalent.*

(1) $\mathrm{h}(R, T)$ *is a PP-ring.*

(2) *R is a PP-ring, T is a torsion-free $\mathbb{Z}$-module and, for each $a \in T$, there exists an element $e \in$* $\mathrm{Idem}(R)$ *such that* $\mathrm{ann}_T(a) = eT$*.*

Proof. (1) ⇒ (2) Suppose that $\mathrm{h}(R, T)$ is a PP-ring. Then, by Lemma 4, R is a PP-ring and T is a torsion-free $\mathbb{Z}$-module. Let $a \in T$. Then, by Lemma 3, there exists an element $e \in \mathrm{Idem}(R)$ such that $\mathrm{ann}_{\mathrm{h}(R,T)}(aX) = e * \mathrm{h}(R, T)$; thus, $ea = 0$. Hence, $eT \subseteq \mathrm{ann}_T(a)$. Let $b \in \mathrm{ann}_T(a)$. Then, $bX \in \mathrm{ann}_{\mathrm{h}(R,T)}(aX)$; thus, $bX = e * f$ for some $f \in \mathrm{h}(R, T)$. Hence, $b \in eT$, which shows that $\mathrm{ann}_T(a) \subseteq eT$. Thus, $\mathrm{ann}_T(a) = eT$.

(2) ⇒ (1) Let $f = \sum_{i=0}^{m} a_i X^i \in \mathrm{h}(R, T)$. Then, by the hypothesis and Lemma 6, for each $i \in \{0, \ldots, m\}$, there exist $d_i \in \mathrm{Idem}(R)$ and $t_i \in \mathrm{Reg}(T)$ such that $a_i = d_i t_i$. Note that, by Lemma 5(5), $(d_i)_{0\leq i\leq m}$ has the least upper bound in R which is an idempotent element. Let $d \in \mathrm{Idem}(R)$ be such that d is the least upper bound of $(d_i)_{0\leq i\leq m}$. Then, $(1 - d)d_i = 0$ for all $i \in \{0, \ldots, m\}$. Hence, $(1 - d) * f = 0$, which means that $(1 - d) * \mathrm{h}(R, T) \subseteq \mathrm{ann}_{\mathrm{h}(R,T)}(f)$. For the reverse containment, let $g = \sum_{i=0}^{n} b_i X^i \in \mathrm{ann}_{\mathrm{h}(R,T)}(f)$. Note that, by the hypothesis, T is a PP-ring; thus, T is a reduced ring. Since T is a torsion-free $\mathbb{Z}$-module, Proposition 1 indicates that $a_i b_j = 0$ for all $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$. Note that, by the hypothesis and Lemma 6, for each $j \in \{0, \ldots, n\}$, there exist $e_j \in \mathrm{Idem}(R)$ and $x_j \in \mathrm{Reg}(T)$ such that $b_j = e_j x_j$. Since t_i and x_j are regular elements of T and

$a_i b_j = 0$ for all $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$, $d_i e_j = 0$ for all $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$; thus, $d_i(1 - e_j) = d_i$, or equivalently, $d_i \le 1 - e_j$ for all $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$. Since d is the least upper bound of $(d_i)_{0 \le i \le m}$, Lemma 5(2) guarantees that $d \le 1 - e_j$ for all $j \in \{0, \ldots, n\}$; thus, $de_j = 0$ for all $j \in \{0, \ldots, n\}$. Therefore, $d * g = 0$. Hence, $g = (1-d) * g \in (1-d) * \mathrm{h}(R, T)$. Consequently, $\mathrm{ann}_{\mathrm{h}(R,T)}(f) = (1-d) * \mathrm{h}(R, T)$. Thus, $\mathrm{h}(R, T)$ is a PP-ring. □

Corollary 13. *If $R \subseteq T$ is an extension of commutative rings with identity, then the following assertions hold.*

(1) *If $\mathrm{H}(R, T)$ is a PP-ring, then $\mathrm{h}(R, T)$ is also a PP-ring.*
(2) *If R is a Noetherian ring, then $\mathrm{H}(R, T)$ is a PP-ring if and only if $\mathrm{h}(R, T)$ is a PP-ring.*

Proof. (1) The equivalence follows directly from Theorems 3 and 4.
(2) This equivalence comes from Corollary 10 and Theorem 4. □

Corollary 14 ([12] Theorem 3.7)**.** *Let R be a commutative ring with identity. Then, $\mathrm{h}(R)$ is a PP-ring if and only if R is both a PP-ring and a torsion-free $\mathbb{Z}$-module.*

Proof. The equivalence comes directly from Theorem 4. □

Corollary 15. *If $R \subseteq T$ is an extension of commutative rings with identity, then the following assertions hold.*

(1) *$\mathrm{h}(R, T)$ is a PP-ring if and only if $\mathrm{h}(R)$ is a PP-ring, T is a torsion-free $\mathbb{Z}$-module and, for each $a \in T$, there exists an element $e \in \mathrm{Idem}(R)$ such that $\mathrm{ann}_T(a) = eT$.*
(2) *If $\mathrm{h}(R, T)$ is a PP-ring, then $\mathrm{h}(T)$ is a PP-ring.*

Proof. (1) This equivalence can be obtained by suitable combinations of Lemma 1(2), Theorem 4 and Corollary 14.
(2) This result follows directly from Lemma 4 and Corollary 14. □

5. Examples

In this section, we give some examples of PF-rings and PP-rings via composite Hurwitz rings.

Example 4. *Let $R = \mathbb{Z} \oplus \mathbb{Z}$, where Y an indeterminate over R and $T = R[\![Y]\!]$. Then, $R \subsetneq T$ is an extension of commutative rings with identity.*

(1) *Note that T is a torsion-free $\mathbb{Z}$-module.*
(2) *Let $a_0, b_0 \in R$ and $a_i, b_i \in T$ for all $i \in \mathbb{N}$ such that $a_i b_j = (0, 0)$ for all $i, j \in \mathbb{N}_0$. If $a_n = (0, 0)$ for all $n \in \mathbb{N}_0$, then we take $r = (1, 1)$. Then, $r \in R$ such that $ra_n = (0, 0)$ and $rb_n = b_n$ for all $n \in \mathbb{N}_0$. If $b_n = (0, 0)$ for all $n \in \mathbb{N}_0$, then we take $r = (0, 0)$. Then, $r \in R$ such that $ra_n = (0, 0)$ and $rb_n = b_n$ for all $n \in \mathbb{N}_0$. Suppose that $(a_n)_{n \ge 0}$ and $(b_n)_{n \ge 0}$ have nonzero terms. Note that R is a reduced ring; thus, T is a reduced ring. Let i and j be fixed nonnegative integers. Then, every coefficient of a_i annihilates b_j [15] (Theorem 10); thus, by symmetry, we may assume that for all $n \in \mathbb{N}_0$, every coefficient of a_n is of the form $(\alpha, 0)$ and every coefficient of b_n is of the form $(0, \beta)$. Let $r = (0, 1)$. Then, $r \in R$ such that $ra_n = (0, 0)$ and $rb_n = b_n$ for all $n \in \mathbb{N}_0$. Thus, by (1) and Theorem 1, $\mathrm{H}(R, T)$ is a PF-ring.*
(3) *By (2) and Corollary 6(1), $\mathrm{h}(R, T)$ is a PF-ring.*
(4) *Let $(a, b) \in R$. If $a = b = 0$, then $\mathrm{ann}_R(a, b)$ is generated by $(1, 1)$. If $a = 0$ and $b \neq 0$, then $\mathrm{ann}_R(a, b)$ is generated by $(1, 0)$. If $a \neq 0$ and $b = 0$, then $\mathrm{ann}_R(a, b)$ is generated by $(0, 1)$. If $a \neq 0$ and $b \neq 0$, then $\mathrm{ann}_R(a, b)$ is generated by $(0, 0)$. Thus, R is a PP-ring.*
(5) *Let $a \in T$ and, for each $n = 1, 2$, let*

$$e_n = \begin{cases} 0 & \text{if the } n\text{th coordinate of some coefficient of } a \text{ is nonzero} \\ 1 & \text{otherwise.} \end{cases}$$

Let $e = (e_1, e_2)$. Then, $e \in \mathrm{Idem}(R)$ such that $ea = (0,0)$; thus, $eT \subseteq \mathrm{ann}_T(a)$. Let $b = \sum_{j=0}^{\infty} t_j Y^j$ be any element of $\mathrm{ann}_T(a)$. Then, $ab = (0,0)$. Since T is a reduced ring, every coefficient of ab is $(0,0)$ [15] (Theorem 10). Hence, $t_j \in eR$ for all $j \in \mathbb{N}_0$, which shows that $b \in eT$. Thus, $\mathrm{ann}_T(a) = eT$.

(6) *Note that $\mathrm{Idem}(R) = \{(0,0),(1,0),(0,1),(1,1)\}$. Clearly, $(0,0) \le (1,0) \le (1,1)$ and $(0,0) \le (0,1) \le (1,1)$; thus, any increasing sequence in $\mathrm{Idem}(R)$ admits the least upper bound in $(R, \le)$ that belongs to $\mathrm{Idem}(R)$. Thus, by (1), (4), (5), and Theorem 3, $\mathrm{H}(R,T)$ is a PP-ring.*

(7) *By (6) and Corollary 13(1), $\mathrm{h}(R,T)$ is a PP-ring.*

The next example shows that the Noetherian condition is essential in Corollaries 2, 3(2), 10 and 11(2). This also indicates that any of the converse of Corollaries 6(1) and 13(1) and [12] (Lemmas 2.2 and 3.2).

Example 5. *Let $D = \prod_{n=1}^{\infty} \mathbb{Z}$, $1_D = (1,1,\dots)$ and R the subring of D generated by $\bigoplus_{n=1}^{\infty} \mathbb{Z}$ and 1_D. Let Y be an indeterminate over R and $T = R[Y]$. Then, $R \subsetneq T$ is an extension of commutative rings with identity.*

(1) *Note that $\mathbb{Z} \oplus (0) \oplus (0) \oplus \cdots \subsetneq \mathbb{Z} \oplus \mathbb{Z} \oplus (0) \oplus \cdots \subsetneq \cdots$ is a strict ascending chain of ideals of R; thus, R is not a Noetherian ring. Hence, T is not a Noetherian ring. In addition, it is easy to see that R and T are torsion-free $\mathbb{Z}$-modules.*

(2) *For each $i, j \in \mathbb{N}$, let*

$$a_{ij} = \begin{cases} 1 & \text{if } j = 2i+1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$b_{ij} = \begin{cases} 1 & \text{if } j = 2i+2 \\ 0 & \text{otherwise.} \end{cases}$$

Let $a_0 = (1,0,0,\dots)$, $b_0 = (0,1,0,0,\dots)$ and, for each $i \in \mathbb{N}$, let $a_i = (a_{i1}, a_{i2}, \dots)$ and $b_i = (b_{i1}, b_{i2}, \dots)$. Then, $a_i, b_i \in R$ for all $i \in \mathbb{N}_0$ and $a_i b_j = (0,0,\dots)$ for all $i, j \in \mathbb{N}_0$. Suppose that $\mathrm{H}(R)$ is a PF-ring. Then, by Theorem 1, there exists an element $r = (r_1, r_2, \dots) \in R$ such that $ra_i = (0,0,\dots)$ and $rb_i = b_i$ for all $i \in \mathbb{N}_0$. Hence, we obtain

$$r_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

This is impossible. Thus, $\mathrm{H}(R)$ is not a PF-ring.

(3) *Let $a \in R$ and $b \in \mathrm{ann}_R(a)$. For each $n \in \mathbb{N}$, let*

$$r_n = \begin{cases} 1 & \text{if the } n\text{th coordinate of } b \text{ is nonzero} \\ 0 & \text{otherwise.} \end{cases}$$

Let $r = (r_1, r_2, \dots)$. Then, $r \in R$ such that $ra = (0,0,\dots)$ and $rb = b$. Hence, R is a PF-ring. Thus, by (1) and Corollary 7, $\mathrm{h}(R)$ is a PF-ring.

(4) *By (1), (2), and (3), the condition that R is a Noetherian ring is essential in Corollary 3(2).*

(5) *Suppose that $f = \sum_{n=0}^{\infty} a_n X^n \in \mathrm{H}(R)$ is nilpotent. Then, a_0 is nilpotent and for all $n \in \mathbb{N}$, some power of a_n is with torsion [2] (Theorem 2.6) (or [16] Proposition 1.3(1)). Since R is a torsion-free $\mathbb{Z}$-module and a reduced ring, $a_n = 0$ for all $n \in \mathbb{N}_0$. Therefore, $f = 0$. Hence, $\mathrm{H}(R)$ is a reduced ring. Thus, by (2), a reduced ring need not be a PF-ring. This shows that the converse of [12] (Lemma 3.2) is not generally true.*

(6) *By (2) and Corollary 4, $\mathrm{H}(R,T)$ is not a PF-ring.*

(7) *Let $a \in T$ and $b \in \mathrm{ann}_T(a)$. Since R is a reduced ring, every coefficient of a annihilates b (cf. [15] (Theorem 10)). For each $n \in \mathbb{N}$, let*

$$r_n = \begin{cases} 1 & \text{if the nth coordinate of some coefficient of } b \text{ is nonzero} \\ 0 & \text{otherwise.} \end{cases}$$

Let $r = (r_1, r_2, \dots)$. Then, $r \in R$ such that $ra = (0, 0, \dots)$ and $rb = b$. Thus, by (1) and Corollary 5, $\mathrm{h}(R, T)$ is a PF-ring.

(8) *By (1), (6), and (7), the condition that T is a Noetherian ring is essential in Corollary 2.*

(9) *By (6) and (7), the converse of Corollary 6(1) is not true in general.*

(10) *By (2) and [12] (Lemma 3.2), $\mathrm{H}(R)$ is not a PP-ring.*

(11) *Let $a \in R$ and, for each $n \in \mathbb{N}$, let*

$$e_n = \begin{cases} 0 & \text{if the nth coordinate of } a \text{ is nonzero} \\ 1 & \text{otherwise.} \end{cases}$$

Let $e = (e_1, e_2, \dots)$. Then, $e \in \mathrm{Idem}(R)$ such that $\mathrm{ann}_R(a) = eR$. Hence, R is a PP-ring. Thus, by (1) and Corollary 14, $\mathrm{h}(R)$ is a PP-ring.

(12) *By (1), (10) and (11), the condition that R is a Noetherian ring is not superfluous in Corollary 11(2).*

(13) *By (5) and (10), a reduced ring need not be a PP-ring. This shows that the converse of [12] (Lemma 2.2) does not hold in general.*

(14) *By (10) and Corollary 12(1), $\mathrm{H}(R, T)$ is not a PP-ring.*

(15) *Let $a \in T$ and, for each $n \in \mathbb{N}$, let*

$$e_n = \begin{cases} 0 & \text{if the nth coordinate of some coefficient of } a \text{ is nonzero} \\ 1 & \text{otherwise.} \end{cases}$$

Let $e = (e_1, e_2, \dots)$. Then, $e \in \mathrm{Idem}(R)$ such that $\mathrm{ann}_T(a) = eT$. Thus, by (1), (11), and Corollary 15(1), $\mathrm{h}(R, T)$ is a PP-ring.

(16) *By (1), (11), (14), and (15), the condition that R is a Noetherian ring is essential in Corollary 10.*

(17) *By (14) and (15), the converse of Corollary 13(1) is not true in general.*

The final example shows that any of the converse of Lemmas 2 and 4 and Corollaries 4, 8, 12(2), and 15(2) does not hold in general. This also indicates that the annihilator condition in Corollaries 12(1) and 15(1) is not superfluous.

Example 6. *Let $T = \mathbb{Z} \oplus \mathbb{Z}$ and $1_T = (1, 1)$. Let R be the subring of T generated by 1_T. Then, $R \subsetneq T$ is an extension of commutative rings with identity. Let D be either $\mathrm{H}(R, T)$ or $\mathrm{h}(R, T)$.*

(1) *Clearly, R and T are both Noetherian rings and torsion-free $\mathbb{Z}$-modules.*

(2) *Note that R is isomorphic to $\mathbb{Z}$; thus, R is a PP-ring. In addition, by Example 4(4), T is a PP-ring. Hence, R and T are PF-rings [12] (Lemma 3.2).*

(3) *Let $a = (1, 0) \in T$ and $b = (0, 1) \in \mathrm{ann}_T(a)$. Suppose that D is a PF-ring. Then, by (1) and Corollaries 2 and 5, there exists an element $r = (r_1, r_2) \in R$ such that $ra = (0, 0)$ and $rb = b$. Hence, $r = (0, 1)$. This is a contradiction. Thus, D is not a PF-ring.*

(4) *By (1), (2), and (3), the converse of Lemma 2 does not hold in general.*

(5) *By (1), (2), and Corollary 3(2), $\mathrm{H}(R)$ and $\mathrm{H}(T)$ are PF-rings. In addition, by (1), (2), and Corollary 7, $\mathrm{h}(R)$ and $\mathrm{h}(T)$ are PF-rings. Hence, by (3), any of the converse of Corollaries 4 and 8 does not generally hold.*

(6) *While the fact that D is not a PP-ring comes from (3) and [12] (Lemma 3.2), we insert the proof for the sake of completeness. Suppose that D is a PP-ring and let $a = (1, 0) \in T$. Then, by Theorems 3 and 4,*

there exists an element $e \in \mathrm{Idem}(R)$ such that $\mathrm{ann}_T(a) = eT$. By an easy calculation, $e = (0, 1)$. This is absurd. Thus, D is not a PP-ring.

(7) *By (1), (2), and (6), the converse of Lemma 4 does not hold in general.*

(8) *By (1), (2), and Corollary 11(2), $\mathrm{H}(R)$ and $\mathrm{H}(T)$ are PP-rings. Hence, by (1) and (6), the annihilator condition in Corollary 12(1) is essential. In addition, by (6), the converse of Corollary 12(2) does not hold in general.*

(9) *By (1), (2), and Corollary 14, $\mathrm{h}(R)$ and $\mathrm{h}(T)$ are PP-rings. Hence, by (1) and (6), the annihilator condition in Corollary 15(1) is not superfluous. In addition, by (6), the converse of Corollary 15(2) does not generally hold.*

6. Conclusions

The Hurwitz series ring (respectively, Hurwitz polynomial ring) is a kind of power series ring (respectively, polynomial ring) which has different algebraic structures from the usual power series ring (respectively, polynomial ring); thus, after Keigher's research [1], many mathematicians have studied Hurwitz rings. In [6], the authors introduced the notion of the composite Hurwitz rings and, in [7], they investigated further research. In contradistinction to the Hurwitz rings, all algebraic structures of composite Hurwitz rings are determined by two rings; thus, the composite Hurwitz rings also have completely different ring theoretic properties from the usual Hurwitz rings. In this paper, we study equivalent conditions for the composite Hurwitz rings $\mathrm{H}(R, T)$ and $\mathrm{h}(R, T)$ to be PP-rings and PF-rings. From our study, we find the interplay between PP- and PF-properties of composite Hurwitz rings $\mathrm{H}(R, T)$ and $\mathrm{h}(R, T)$ and properties of zero-divisors and idempotent elements in R and T. Moreover, we give some examples that show many conditions are not superfluous. It makes clear the relationship between PF- and PP-properties of composite Hurwitz rings and properties of zero-divisors and idempotent elements in R and T.

Let $\mathcal{R} = (R_n)_{n\geq 0}$ be an ascending chain of commutative rings with identity, $R = \bigcup_{n\geq 0} R_n$ and $\mathrm{H}(\mathcal{R}) = \{\sum_{n=0}^{\infty} a_n X^n \in \mathrm{H}(R) \,|\, a_n \in R_n \text{ for all } n \geq 0\}$. Then, $\mathrm{H}(\mathcal{R})$ is a subring of $\mathrm{H}(R)$ containing $\mathrm{H}(R_0)$ and is called the *generalized composite Hurwitz series ring* with respect to $\mathcal{R}$. Let $\mathrm{h}(\mathcal{R})$ be the subset of all polynomials in $\mathrm{H}(\mathcal{R})$. Then, $\mathrm{h}(\mathcal{R})$ is a subring of $\mathrm{h}(R)$ and is called the *generalized composite Hurwitz polynomial ring* with respect to $\mathcal{R}$. If $R_0 = R$ and $R_n = T$ for all $n \geq 1$, then $\mathrm{H}(\mathcal{R}) = \mathrm{H}(R, T)$ and $\mathrm{h}(\mathcal{R}) = \mathrm{h}(R, T)$. In the next work, we are going to study when the generalized composite Hurwitz rings $\mathrm{H}(\mathcal{R})$ and $\mathrm{h}(\mathcal{R})$ are PF-rings and PP-rings.

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