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Existence of Solutions for Kirchhoff-Type Fractional Dirichlet Problem with p-Laplacian

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Abstract: In this paper, we investigate the existence of solutions for a class of p-Laplacian fractional order Kirchhoff-type system with Riemann–Liouville fractional derivatives and a parameter λ. By mountain pass theorem, we obtain that system has at least one non-trivial weak solution u_λ under some local conditions for each given large parameter λ. We get a concrete lower bound of the parameter λ, and then obtain two estimates of weak solutions u_λ. We also obtain that $u_\lambda \to 0$ if λ tends to ∞. Finally, we present an example as an application of our results.

Keywords: Kirchhoff-type system; fractional p-Laplacian; local superquadratic nonlinearity; mountain pass theorem; existence

MSC: 34B15; 34B10

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1. Introduction and Main Results

In recent decades, the subjects about fractional calculus have been investigated extensively because of their applications to many fields. Among all these subjects, ordinary and partial fractional differential equations have attracted considerable attentions in both mathematical aspects and their applications. It has been proved that fractional differential equations can provide a natural framework in the modeling of many complex real phenomena in many fields including mechanics, quantum field theory, electromagnetic theory, transport theory, fractal, biology, robotics, chemical processes, control theory, and so on ([1–20] and references therein). In this paper, we are concerned with the following system

$$\begin{cases} A(u(t))[{}_tD_T^\alpha \phi_p({}_0D_t^\alpha u(t)) + V(t)\phi_p(u(t))] = \lambda \nabla F(t,u(t)), \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1}$$

where

$$A(u(t)) = \left[a + b\int_0^T (|{}_0D_t^\alpha u(t)|^p + V(t)|u(t)|^p)dt\right]^{p-1},$$

$a, b, \lambda > 0$, $p > 1$ and $1/p < \alpha \le 1$ are constants, p is an integer, $u(t) = (u_1(t), \cdots, u_N(t))^\tau \in \mathbb{R}^N$ for a.e. $t \in [0,T]$, $T > 0$, and N is a given positive integer, $(\cdot)^\tau$ denote the transpose of a vector, $V(t) \in C([0,T],\mathbb{R})$ with $\min_{t\in[0,T]} V(t) > 0$, ${}_0D_t^\alpha$ and ${}_tD_T^\alpha$ are the left and right Riemann–Liouville fractional derivatives, respectively, $\phi_p(s) := |s|^{p-2}s$, $\nabla F(t,x)$ is the gradient of F with respect to $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, that is, $\nabla F(t,x) = (\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_N})^\tau$, and $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following condition:

(H0) there exists a constant $\delta > 0$ such that $F(t,x)$ is continuously differentiable in $x \in \mathbb{R}^N$ with $|x| \le \delta$ for a.e. $t \in [0,T]$, measurable in t for every $x \in \mathbb{R}^N$ with $|x| \le \delta$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1([0,T];\mathbb{R}^+)$ such that

$$|F(t,x)|, |\nabla F(t,x)| \le a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ with $|x| \le \delta$ and a.e. $t \in [0,T]$.

When $\alpha = 1$, the operator ${}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t))$ reduces to the usual second order differential operator $-d^2/dt^2$. Hence, if $\alpha = 1$, $p = 2$, $N = 1$, $\lambda = 1$ and $V(t) = 0$ for a.e. $t \in [0,T]$, system (1) becomes the equation with Dirichlet boundary condition

$$\begin{cases} -\left(a + b\int_0^T |u'(t)|^2 dt\right) u''(t) = f(t,u(t)), \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{2}$$

where $f(t,x) = \frac{\partial F(t,x)}{\partial x}$ and $F : [0,T] \times \mathbb{R} \to \mathbb{R}$. It is well known that Equation (2) is related to the stationary problem of a classical model introduced by Kirchhoff [21]. To be precise, in [21], Kirchhoff introduced the model

$$\rho \frac{\partial^2 u}{\partial t^2} = \left(P_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial y}\right)^2 dy \right) \frac{\partial^2 u}{\partial y^2}, \tag{3}$$

where $0 \le y \le L$, $t \ge 0$, u is the lateral deflection, ρ is the mass density, h is the cross-sectional area, L is the length, E is the Young's modulus and P_0 is the initial axial tension. (Notations: in model (3), (7) and (8) below, t is time variable and y is spatial variable, which are conventional notations in partial differential equations. One needs to distinguish them to t in (1), (2), (4)–(6) below, where t corresponds to the spatial variable x). The model (3) is used to describe small vibrations of an elastic stretched string. Equation (3) has been studied extensively, for instance, [22–34] and references therein. For $p > 1$, the reader can consult [35–39] and references therein.

When $\alpha < 1$, ${}_0D_t^{\alpha}$ and ${}_tD_T^{\alpha}$ are the left and right Riemann–Liouville fractional derivatives, respectively, which have been given some physical interpretations in [40]. Moreover, they are also applied to describe the anomalous diffusion, Lévy flights and traps in [41,42]. In [43], Jiao and Zhou considered the system

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \nabla F(t,u(t)), \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0. \end{cases} \tag{4}$$

They successfully applied critical point theory to investigate the existence of weak solutions for system (4). To be precise, they obtained that system (4) has at least one weak solution when F has a quadratic growth or a superquadratic growth by using the least action principle and mountain pass theorem. Subsequently, this topic related to system (4) attracted lots of attentions, for example, Ref. [44–49] and references therein. It is obvious that system (1) is much more complicated than system (4) since the appearance of nonlocal term $A(u(t))$ and p-Laplacian term $\phi_p(s)$. Recently, in [50], the following fractional Kirchhoff equation with Dirichlet boundary condition was investigated

$$\begin{cases} \left(a + b\int_0^T |{}_0D_t^{\alpha}u(t)|^2 dt\right) {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) + \lambda V(t)u(t) = f(t,u(t)), \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{5}$$

where $a, b, \lambda > 0$, $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$. By using the mountain pass theorem in [51] and the linking theorem in [52], the authors established some existence results of nontrivial solutions for system (5) if

f satisfies

(f1) there exist constants $\mu > 4$, $0 < \tau < 2$ and a nonnegative function $g \in L^{\frac{2}{2-\tau}}$ such that

$$F(t,x) - \frac{1}{\mu} f(t,x)x \le g(t)|x|^{\tau}, \quad \text{for a.e. } t \in [0,T], x \in \mathbb{R};$$

(f2) there exists $\theta > 2$ such that $\lim_{|x|\to\infty} \inf_{t\in[0,T]} \frac{F(t,x)}{|x|^{\theta}} > 0$;

(or (f2)′ there exists $\theta > 4$ such that $\lim_{|x|\to\infty} \inf_{t\in[0,T]} \frac{F(t,x)}{|x|^{\theta}} > 0$);

(f3) there exists $\sigma > 2$ such that $\lim_{|x|\to 0} \sup_{t\in[0,T]} \frac{F(t,x)}{|x|^{\sigma}} < \infty$,

and some other reasonable conditions.

In [53], Chen-Liu investigated the Kirchhoff-type fractional Dirichlet problem with p-Laplacian

$$\begin{cases} \left(a + b\int_0^T |{}_0D_t^{\alpha}u(t)|^p dt\right)^{p-1} {}_tD_T^{\alpha}\phi_p({}_0D_t^{\alpha}u(t)) = f(t,u(t)), & t \in (0,T), \\ u(0) = u(T) = 0. \end{cases} \tag{6}$$

where $a, b, \lambda > 0$, $f \in C^1([0,T] \times \mathbb{R}, \mathbb{R})$. By the Nehari method, they established the existence result of ground state solution for system (6) if f satisfies

(f4) $f(t,x) = o(|x|^{p-1})$ as $|x| \to 0$ uniformly for all $t \in [0,T]$,

and the well-known Ambrosetti–Rabinowitz (AR for short) condition

(AR) there exist two constants $\mu > p^2$, $R > 0$ such that

$$0 < \mu F(t,x) \le x f(t,x), \quad \text{for } \forall t \in [0,T], x \in \mathbb{R} \text{ with } |x| \ge R,$$

where $F(t,x) = \int_0^x f(t,s)ds$, and some additional conditions. It is easy to see that all of these conditions (f1), (f2), (f2)′ and (AR) imply that $F(t,x)$ needs to have a growth near the infinity about x, and (f3) and (f4) imply that $F(t,x)$ needs to have a growth near 0 about x.

In this paper, we investigate the existence of solutions for system (1) when the nonlinear term F has local assumptions only near 0 about x. Our work is mainly motivated by [32,54]. In [54], Costa and Wang investigated the multiplicity of both signed and sign-changing solutions for the one-parameter family of elliptic problems

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u(y) = 0 & \text{in } \partial\Omega, \end{cases} \tag{7}$$

where $\lambda > 0$ is a parameter, Ω is a bounded smooth domain in $\mathbb{R}^N (N \ge 3)$ and $f \in C^1(\mathbb{R}, \mathbb{R})$. They assumed that the nonlinearity $f(u)$ has superlinear growth only in a neighborhood of $u = 0$ and then obtained the number of signed and sign-changing solutions which are dependent on the parameter λ. They used a cut-off technique together with energy estimates given by minimax methods. The idea in [54] has been applied to some different problems, for example, [55,56] for quasilinear elliptic problems with p-Laplacian operator, [57] for an elliptic problem with fractional Laplacian operator, Ref. [58] for Schrödinger equations, [59] for Neumann problem with nonhomogeneous differential operator and critical growth, and [60] for quasilinear Schrödinger equations. Especially, in [32], Li and Su investigated the Kirchhoff-type equations

$$\begin{cases} -\left[1 + \int_{\mathbb{R}^3}(|\nabla u|^2 + V(y)u^2)dy\right][\Delta u + V(y)u] = \lambda Q(y)f(u), & y \in \mathbb{R}^3, \\ u(y) \to 0, \text{ as } |y| \to \infty, \end{cases} \tag{8}$$

where $\lambda > 0$, V, Q are radial functions and $f \in C((-\delta_0, \delta_0), \mathbb{R})$ for some $\delta_0 > 0$. Via the idea in [54], they also established the existence result of solutions when $f(u)$ has superlinear growth in a neighborhood of $u = 0$. It is worthy to note that λ usually needs to be sufficiently large, that is, λ has a lower bound λ^*. However, the concrete values of λ^* are not given in these references. Similar to

Equation (8), comparing with Equations (5) and (6), we add a nonlocal term $\int_0^T V(t)|u(t)|^p dt$ in system (1) where $\min_{t\in[0,T]} V(t) > 0$, and multiply $V(t)\phi_p(u(t))$ by the nonlocal part $A(u(t))$. Moreover, we consider the high-dimensional case, that is, $N \geq 1$. Since $\min_{t\in[0,T]} V(t) > 0$, system (1) is different from Equations (2), (5), (6) and system (4). More importantly, we present a concrete value of the lower bound λ^* for system (1) and then obtain two estimates of the solutions family $\{u_\lambda\}$ for all $\lambda > \lambda^*$. Next, we make some assumptions for F.

(H1) there exist constants $q_1 > p^2$, $q_2 \in (p^2, q_1)$, $M_1 > 0$ and $M_2 > 0$ such that

$$M_1|x|^{q_1} \leq F(t,x) \leq M_2|x|^{q_2}$$

for all $x \in \mathbb{R}^N$ with $|x| \leq \delta$ and a.e. $t \in [0,T]$;

(H2) there exists a constant $\beta > p^2$ such that

$$0 \leq \beta F(t,x) \leq (\nabla F(t,x), x)$$

for all $x \in \mathbb{R}^N$ with $|x| \leq \delta$ and a.e. $t \in [0,T]$.

Theorem 1. *Suppose that (H0)–(H2) hold. Then system (1) has at least a nontrivial weak solution u_λ for all $\lambda > \lambda^* := \max\{\Lambda_1, \Lambda_2, \Lambda_3\}$ and*

$$\begin{aligned}
\|u_\lambda\|_V^p &\leq \frac{p^2\theta}{a^{p-1}(\theta - p^2)} C_* \lambda^{-\frac{p-1}{q_1-p}} \leq \frac{p^2\theta}{a^{p-1}(\theta - p^2)} C_* \max\{\Lambda_1, \Lambda_2, \Lambda_3\}^{-\frac{p-1}{q_1-p}}, \\
\|u_\lambda\|_\infty &\leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}} \cdot \frac{p^2\theta}{a^{p-1}(\theta - p^2)} C_* \lambda^{-\frac{p-1}{q_1-p}} \\
&\leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}} \cdot \frac{p^2\theta}{a^{p-1}(\theta - p^2)} C_* \max\{\Lambda_1, \Lambda_2, \Lambda_3\}^{-\frac{p-1}{q_1-p}}, \\
\lim_{\lambda\to\infty} &\|u_\lambda\|_V = 0 = \lim_{\lambda\to\infty} \|u_\lambda\|_\infty,
\end{aligned}$$

where $\theta = \min\{\beta, q_2\}$, $q = \frac{p}{p-1}$,

$$\|u_\lambda\|_V = \left(\int_0^T |{}_0D_t^\alpha u_\lambda(t)|^p dt + \int_0^T V(t)|u_\lambda(t)|^p dt\right)^{1/p}, \quad \|u_\lambda\|_\infty = \max_{t\in[0,T]} u_\lambda(t), \tag{9}$$

$$\Lambda_1 = \max\left\{ \frac{V_\infty a^{p-1}(\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}} G_0)^{q_2-p}}{2p^2 M_2 T^{(\alpha-\frac{1}{p})(q_2-p)}(\delta \min\{1, V_\infty\} D)^{q_2-p}}, \frac{\frac{1}{bp^2}(a + \frac{b\delta^p}{G_0^p}\max\{1, V^\infty\}(D^p + G^p))^p}{M_1 \frac{\delta^{q_1}}{G_0^{q_1}} T^{1-\frac{q_1}{p}} D^{q_1}} \right\}, \tag{10}$$

$$\Lambda_2 = \left[a + b[\max\{1, V^\infty\}]^p \frac{\delta^p}{G_0^p}(D^p + G^p)\right]^{q_1(p-1)}, \tag{11}$$

$$\Lambda_3 = \left(\frac{T^{p\alpha-1}}{\left[\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}\right]^p} \cdot \frac{p^2\theta C_*}{a^{p-1}(\theta - p^2)} \cdot \frac{2^p}{\delta^p} \right)^{\frac{q_1-p}{p-1}}, \tag{12}$$

$$V^\infty = \max_{t\in[0,T]} V(t), \quad V_\infty = \min_{t\in[0,T]} V(t),$$

$$C_* = \left(\frac{1}{p(M_1 q_1)^{\frac{p}{q_1-p}}} - \frac{M_1}{(M_1 q_1)^{\frac{q_1}{q_1-p}}} \right) \left(\frac{[\max\{1, V^\infty\}]^{1/p}(D^p + G^p)^{1/p}}{T^{\frac{1}{q_1}-\frac{1}{p}} D} \right)^{\frac{pq_1}{q_1-p}}, \tag{13}$$

$$D = \begin{cases} \left(\frac{T^{p+1}}{\pi^{p+1}} \cdot \frac{2(p-1)!!}{p!!}\right)^{\frac{1}{p}}, & \text{if } p \text{ is odd,} \\ \left(\frac{T^{p+1}}{\pi^{p}} \cdot \frac{(p-1)!!}{p!!}\right)^{\frac{1}{p}}, & \text{if } p \text{ is even,} \end{cases}$$

$$G = \left(\frac{T^{p+1-p\alpha}}{[\Gamma(2-\alpha)]^p(p+1-p\alpha)}\right)^{1/p},$$

$$G_0 = \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}} G,$$

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \ \ (\text{for all } z > 0).$$

We organize this paper as follows. In Section 2, we recall some preliminary results including the definitions of Riemann–Liouville fractional derivatives and working spaces, some conclusions for the working spaces and mountain pass theorem. In Section 3, we complete the proof of Theorem 1. In Section 4, we apply Theorem 1 to an example and compute the value of lower bound λ^* in the example.

2. Preliminaries

In this section, we mainly recall some basic definitions and results.

Definition 1 (Left and Right Riemann–Liouville Fractional Integrals [44,61])**.** *Let f be a function defined on $[a, b]$. The left and right Riemann–Liouville fractional integrals of order $\gamma > 0$ for function f denoted by ${}_aD_t^{-\gamma} f(t)$ and ${}_tD_b^{-\gamma} f(t)$, respectively, are defined by*

$$
\begin{aligned}
{}_aD_t^{-\gamma} f(t) &= \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s)ds, \ t \in [a,b], \gamma > 0, \\
{}_tD_b^{-\gamma} f(t) &= \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} f(s)ds, \ t \in [a,b], \gamma > 0,
\end{aligned}
$$

provided the right-hand sides are pointwise defined on $[a, b]$, where $\Gamma > 0$ is the Gamma function.

Definition 2 (Left and Right Riemann–Liouville Fractional Derivatives [44,61])**.** *Let f be a function defined on $[a, b]$. The left and right Riemann–Liouville fractional derivatives of order $\gamma > 0$ for function f denoted by ${}_aD_t^{\gamma} f(t)$ and ${}_tD_b^{\gamma} f(t)$, respectively, are defined by*

$$
\begin{aligned}
{}_aD_t^{\gamma} f(t) &= \frac{d^n}{dt^n} {}_aD_t^{\gamma-n} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \left(\int_a^t (t-s)^{n-\gamma-1} f(s)ds \right), \\
{}_tD_b^{\gamma} f(t) &= (-1)^n \frac{d^n}{dt^n} {}_tD_b^{\gamma-n} f(t) = \frac{(-1)^n}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \left(\int_t^b (s-t)^{n-\gamma-1} f(s)ds \right),
\end{aligned}
$$

where $t \in [a, b], n - 1 \le \gamma < n$ and $n \in \mathbb{N}$.

Definition 3 ([43])**.** *Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty([0, T], \mathbb{R}^N)$ with the norm*

$$\|u\| = \left(\int_0^T |{}_0D_t^\alpha u(t)|^p dt + \int_0^T |u(t)|^p dt\right)^{1/p}, \ \ \forall\, u \in E_0^{\alpha,p}.$$

From the definition of $E_0^{\alpha,p}$, it is apparent that the fractional derivative space $E_0^{\alpha,p}$ is the space of functions $u : [0,T] \to \mathbb{R}^N$ which is absolutely continuous and has an α-order left Riemann–Liouville fractional derivative ${}_0D_t^\alpha u \in L^p([0,T],\mathbb{R}^N)$ and $u(0) = u(T) = 0$ and one can define the norm on $L^p([0,T],\mathbb{R}^N)$ as

$$\|u\|_{L^p} = \left(\int_0^T |u(t)|^p dt\right)^{1/p}.$$

$E_0^{\alpha,p}$ is uniformly convex by the uniform convexity of L^p ([43]).

Remark 1. *It is easy to see that $\|u\|_V$ defined by (9) is also a norm on $E_0^{\alpha,p}$ and $\|u\|_V$ and $\|u\|$ are equivalent and*

$$\min\{1, V_\infty\}\|u\|^p \le \|u\|_V^p \le \max\{1, V^\infty\}\|u\|^p. \tag{14}$$

Lemma 1 ([43])**.** *Let $0 < \alpha \le 1$ and $1 < p < \infty$. $E_0^{\alpha,p}$ is a reflexive and separable Banach space.*

Lemma 2 ([43])**.** *Let $0 < \alpha \le 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, there has*

$$\|u\|_{L^p} \le C_p \|{}_0D_t^\alpha u\|_{L^p},$$

where

$$C_p = \frac{T^\alpha}{\Gamma(\alpha+1)} > 0.$$

Moreover, if $\alpha > \frac{1}{p}$, then

$$\|u\|_\infty \le \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}} \|{}_0D_t^\alpha u\|_{L^p}, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{15}$$

Lemma 3 ([43])**.** *Let $1/p < \alpha \le 1$ and $1 < p < \infty$. The imbedding of $E_0^{\alpha,p}$ in $C([0,T],\mathbb{R}^N)$ is compact.*

Let X be a Banach space. $\varphi \in C^1(X,\mathbb{R})$ and $c \in \mathbb{R}$. A sequence $\{u_n\} \subset X$ is called $(\mathrm{PS})_c$ sequence (named after Palais and Smale) if the sequence $\{u_n\}$ satisfies

$$\varphi(u_n) \to c, \quad \varphi'(u_n) \to 0.$$

Lemma 4 (Mountain Pass Theorem [62,63])**.** *Let X be a Banach space, $\varphi \in C^1(X,\mathbb{R})$, $w \in X$ and $r > 0$ be such that $\|w\| > r$ and*

$$b := \inf_{\|u\|=r} \varphi(u) > \varphi(0) \ge \varphi(w).$$

Then there exists a $(PS)_c$ sequence with

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t)),$$

$$\Gamma := \{\gamma \in ([0,1], X]) : \gamma(0) = 0, \gamma(1) = w\}.$$

As in [53], for each $\lambda > 0$, we can define the functional $I_\lambda : E_0^{\alpha,p} \to \mathbb{R}$ as

$$I_\lambda(u) = \frac{1}{bp^2}\left(a + b\int_0^T (|{}_0D_t^\alpha u(t)|^p + V(t)|u(t)|^p)dt\right)^p - \lambda\int_0^T F(t,u(t))dt - \frac{a^p}{bp^2}.$$

It is easy to see that the assumption (H0)–(H2) can not ensure that I_λ is well defined on $E_0^{\alpha,p}$. So we follow the idea in [54] and simply sketch the outline of proof here. We use Lemma 4 to complete the proof. Since F satisfies the growth condition only near 0 by (H0)-(H2), in order to use the conditions globally, we modify and extend F to $\bar{F}$ defined in section 3, and the corresponding functional is defined as $\bar{I}_\lambda$. Next we prove that $\bar{I}_\lambda$ has mountain pass geometry on $E_0^{\alpha,p}$. Then Lemma 4 implies that $\bar{I}_\lambda$ has a $(\mathrm{PS})_{c_\lambda}$ sequence. Then by a standard analysis, a convergent subsequence of the $(\mathrm{PS})_{c_\lambda}$ sequence is obtained to ensure that c_λ is the critical value of $\bar{I}_\lambda$. Finally, by an estimate about $\|u_\lambda\|_\infty$, we obtain that the critical point u_λ of $\bar{I}_\lambda$ with $\|u_\lambda\|_\infty \le \delta/2$ is just right the solution of system (1) for all $\lambda > \lambda^*$ for some concrete λ^*.

3. Proofs

Define $m(s) \in C^1(\mathbb{R}, [0,1])$ as an even cut-off function satisfying $sm'(s) \le 0$ and

$$m(s) = \begin{cases} 1, & \text{if} \quad |s| \leqslant \delta/2, \\ 0, & \text{if} \quad |s| \geqslant \delta. \end{cases} \tag{16}$$

Define $\bar{F} : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ as

$$\bar{F}(t,x) = m(|x|)F(t,x) + (1 - m(|x|))M_2|x|^{q_2}.$$

We define the variational functional corresponding to $\bar{F}$ as

$$\begin{aligned} \bar{I}_\lambda(u) &= \frac{1}{bp^2}\left(a + b\int_0^T (|{}_0D_t^\alpha u(t)|^p + V(t)|u(t)|^p)dt\right)^p - \lambda\int_0^T \bar{F}(t,u(t))dt - \frac{a^p}{bp^2} \\ &= \frac{1}{bp^2}\left(a + b\|u\|_V^p\right)^p - \lambda\int_0^T \bar{F}(t,u(t))dt - \frac{a^p}{bp^2} \end{aligned} \tag{17}$$

for all $u \in E_0^{\alpha,p}$. By (H0) and the definition of $\bar{F}$, it is easy to obtain that $\bar{F}$ satisfies
(H0)′ $\bar{F}(t,x)$ is continuously differentiable in $\mathbb{R}^N$ for a.e. $t \in [0,T]$, measurable in t for every $x \in \mathbb{R}^N$, and there exists $b \in L^1([0,T];\mathbb{R}^+)$ such that

$$\begin{aligned} &|\bar{F}(t,x)| \le a_0 b(t) + M_2|x|^{q_2}, \\ &|\nabla\bar{F}(t,x)| \le (1+m_0)a_0 b(t) + M_2 q_2|x|^{q_2-1} + m_0 M_2|x|^{q_2} \end{aligned}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, $a_0 = \max_{s\in[0,\delta]} a(s)$ and $m_0 = \max_{s\in[\frac{\delta}{2},\delta]} |m'(s)|$.

Hence, a standard argument shows that $\bar{I}_\lambda \in C^1(E_0^{\alpha,p}, \mathbb{R})$ and

$$\begin{aligned} \langle \bar{I}'_\lambda(u), v\rangle = &\left(a + b\|u\|_V^p\right)^{p-1}\left(\int_0^T [|{}_0D_t^\alpha u(t)|^{p-2}({}_0D_t^\alpha u(t),\ {}_0D_t^\alpha v(t))dt + V(t)|u(t)|^{p-2}(u(t),v(t))]dt\right) \\ &-\lambda\int_0^T (\nabla\bar{F}(t,u(t)), v(t))dt \end{aligned}$$

for all $u, v \in E_0^{\alpha,p}$. Hence

$$\langle \bar{I}'_\lambda(u), u\rangle = \left(a + b\|u\|_V^p\right)^{p-1}\|u\|_V^p - \lambda\int_0^T (\nabla\bar{F}(t,u(t)), u(t))dt$$

for all $u \in E_0^{\alpha,p}$.

Lemma 5. *Assume that (H1)–(H2) hold. Then*
(H1)′

$$0 \le \bar{F}(t,x) \le M_2|x|^{q_2}, \ \text{for all } x \in \mathbb{R}^N;$$

(H2)′

$$0 < \theta\bar{F}(t,x) \le (\nabla\bar{F}(t,x), x), \ \text{for all } x \in \mathbb{R}^N/\{0\},$$

where $\theta = \min\{q_2, \beta\}$.

Proof.
- If $|x| \le \frac{\delta}{2}$, then by (H1), the conclusion (H1)$'$ holds;
  If $\frac{\delta}{2} < |x| \le \delta$, by (H1), we have

$$0 \le \bar{F}(t,x) = m(|x|)F(t,x) + (1-m(|x|))M_2|x|^{q_2} \le m(|x|)M_2|x|^{q_2} + (1-m(|x|))M_2|x|^{q_2} = M_2|x|^{q_2};$$

  If $|x| \ge \delta$, then by the definition of m, we have $\bar{F}(t,x) = M_2|x|^{q_2}$.
- For all $x \in \mathbb{R}^N/\{0\}$, we have

$$\nabla\bar{F}(t,x) = m'(|x|)\tfrac{x}{|x|}F(t,x) + m(|x|)\nabla F(t,x) + (1-m(|x|))q_2M_2|x|^{q_2-2}x - m'(|x|)\tfrac{x}{|x|}M_2|x|^{q_2}.$$

Then

$$(\nabla\bar{F}(t,x),x) \quad = \quad |x|m'(|x|)(F(t,x) - M_2|x|^{q_2}) + m(|x|)(\nabla F(t,x),x) + (1-m(|x|))q_2M_2|x|^{q_2}.$$

and

$$\begin{aligned}\theta\bar{F}(t,x) - (\nabla\bar{F}(t,x),x) &= m(|x|)(\theta F(t,x) - (\nabla F(t,x),x)) + (\theta - q_2)(1-m(|x|))M_2|x|^{q_2} \\ &\quad - |x|m'(|x|)(F(t,x) - M_2|x|^{q_2}).\end{aligned}$$

Apparently, the conclusion holds for $0 \le |x| \le \delta/2$ and $|x| \ge \delta$. If $\delta/2 < |x| < \delta$, by using $\theta \le q_2$, the conclusion (H1), (H2) and the fact $sm'(s) \le 0$ for all $s \in \mathbb{R}$, we can get the conclusion (H2)$'$. □

Lemma 6. *$\bar{I}_\lambda$ satisfies the mountain pass geometry for all $\lambda > \Lambda_1$, where Λ_1 is defined in (10).*

Proof. Note that $q_2 > p^2 > p$. By Lemma 5 and (15) , we have

$$\begin{aligned}\bar{I}_\lambda(u) &= \frac{1}{bp^2}(a + b\|u\|_V^p)^p - \lambda\int_0^T \bar{F}(t,u(t))dt - \frac{a^p}{bp^2} \\ &\ge \frac{a^p}{bp^2} + \frac{a^{p-1}}{p^2}\|u\|_V^p - \lambda M_2\int_0^T |u(t)|^{q_2}dt - \frac{a^p}{bp^2} \\ &\ge \frac{a^{p-1}}{p^2}\|u\|_V^p - \lambda M_2\|u\|_\infty^{q_2-p}\int_0^T |u(t)|^p dt \\ &\ge \frac{a^{p-1}}{p^2}\|u\|_V^p - \lambda M_2\left(\frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}}\right)^{q_2-p}\|u\|_V^{q_2-p}\|u\|_{L^p}^p \\ &\ge \frac{a^{p-1}}{p^2}\|u\|_V^p - \lambda\frac{M_2}{V_\infty}\left(\frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}}\right)^{q_2-p}\|u\|_V^{q_2}.\end{aligned}$$

We choose $\nu_\lambda = \left(\dfrac{a^{p-1}V_\infty}{2p^2\lambda M_2\left(\frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q-q+1)^{\frac{1}{q}}}\right)^{q_2-p}}\right)^{\frac{1}{q_2-p}}$ for any given $\lambda > 0$. Then we have

$$\bar{I}_\lambda(u) > d_\lambda := \frac{a^{p-1}}{p^2}\nu_\lambda^p - \lambda\frac{M_2}{V_\infty}\left(\frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}}\right)^{q_2-p}\nu_\lambda^{q_2} > 0, \quad \text{for all } \|u\|_V = \nu_\lambda. \tag{18}$$

Choose

$$e = \left(\frac{T}{\pi}\sin\frac{\pi t}{T}, 0, \cdots, 0\right) \in E_0^{\alpha,p}. \tag{19}$$

Then

$$\|e\|_{L^p} = D := \begin{cases} \left(\frac{T^{p+1}}{\pi^{p+1}}\frac{2(p-1)!!}{p!!}\right)^{\frac{1}{p}}, & \text{if } p \text{ is odd,} \\ \left(\frac{T^{p+1}}{\pi^{p}}\frac{(p-1)!!}{p!!}\right)^{\frac{1}{p}}, & \text{if } p \text{ is even} \end{cases} \tag{20}$$

and

$$\|{}_0D_t^{\alpha}e\|_{L^p} \le G := \frac{T^{p+1-p\alpha}}{\Gamma^p(2-\alpha)(p+1-p\alpha)}. \tag{21}$$

By (15),

$$\|e\|_{\infty} \le \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}}\|{}_0D_t^{\alpha}e\|_{L^p} \le G_0 := \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}}G. \tag{22}$$

Note that

$$\Lambda_1 = \max\left\{\frac{V_\infty a^{p-1}(\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}G_0)^{q_2-p}}{2p^2M_2T^{(\alpha-\frac{1}{p})(q_2-p)}(\delta\min\{1, V_\infty\}D)^{q_2-p}}, \frac{\frac{1}{bp^2}(a + \frac{b\delta^p}{G_0^p}\max\{1, V^\infty\}(D^p + G^p))^p}{M_1\frac{\delta^{q_1}}{G_0^{q_1}}T^{1-\frac{q_1}{p}}D^{q_1}}\right\}.$$

Then

$$\|\frac{\delta}{G_0}e\|_V \ge \frac{\delta\min\{1, V_\infty\}}{G_0}\|e\|_{L^p} \ge \nu_\lambda$$

for all $\lambda > \Lambda_1$. By (22), we have $\|\frac{\delta}{G_0}e\|_\infty \le \delta$. By the definition of $\bar{F}$ and (H1), we have $\bar{F}(t,x) = F(t,x) \ge M_1|x|^{q_1}$ for all $|x| \le \delta/2$, and

$$\bar{F}(t,x) = m(|x|)F(t,x) + (1-m(x))M_2|x|^{q_2} \ge m(|x|)M_1|x|^{q_1} + (1-m(x))M_1|x|^{q_1} = M_1|x|^{q_1}$$

for all $\frac{\delta}{2} < |x| \le \delta$. Hence, by Hölder inequality, we have

$$\begin{aligned} \bar{I}_\lambda(\frac{\delta}{G_0}e) &= \frac{1}{bp^2}(a + b\|\frac{\delta}{G_0}e\|_V^p)^p - \lambda\int_0^T \bar{F}(t, \frac{\delta}{G_0}e(t))dt - \frac{a^p}{bp^2} \\ &\le \frac{1}{bp^2}(a + b\|\frac{\delta}{G_0}e\|_V^p)^p - \lambda M_1\int_0^T |\frac{\delta}{G_0}e(t)|^{q_1}dt - \frac{a^p}{bp^2} \\ &\le \frac{1}{bp^2}(a + \frac{b\delta^p}{G_0^p}\max\{1, V^\infty\}\|e\|^p)^p - \lambda M_1\frac{\delta^{q_1}}{G_0^{q_1}}T^{1-\frac{q_1}{p}}\|e\|_{L^p}^{q_1} \\ &\le \frac{1}{bp^2}(a + \frac{b\delta^p}{G_0^p}\max\{1, V^\infty\}(D^p + G^p))^p - \lambda M_1\frac{\delta^{q_1}}{G_0^{q_1}}T^{1-\frac{q_1}{p}}D^{q_1} \\ &< 0 \end{aligned}$$

for all $\lambda > \Lambda_1$.

Let $w = \frac{\delta}{G_0} e$ and $\varphi = \bar{I}_\lambda$. Then for any given $\lambda > \Lambda_1$, Lemmas 4 and 6 imply that $\bar{I}_\lambda$ has a $(\mathrm{PS})_{c_\lambda}$ sequence $\{u_n\} := \{u_{n,\lambda}\}$, that is, there exists a sequence $\{u_n\}$ satisfying

$$\bar{I}_\lambda(u_n) \to c_\lambda, \quad \bar{I}'_\lambda(u_n) \to 0, \text{ as } n \to \infty, \tag{23}$$

where

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \bar{I}_\lambda(\gamma(t)), \tag{24}$$

$$\Gamma := \{\gamma \in ([0,1], X]) : \gamma(0) = 0, \gamma(1) = w\}.$$

□

Lemma 7. *The $(PS)_{c_\lambda}$ sequence $\{u_n\}$ has a convergent subsequence.*

Proof. By virtue of Lemma 5, (23) and $\theta = \min\{q_2, \beta\} > p^2$, there exists a positive constant $M > 0$ such that

$$\begin{aligned}
M + \|u_n\|_V &\geq \bar{I}_\lambda(u_n) - \frac{1}{\theta}\langle \bar{I}'_\lambda(u_n), u_n\rangle \\
&= (a + b\|u_n\|_V^p)^{p-1}\left[\frac{1}{bp^2}(a + b\|u_n\|_V^p) - \frac{1}{\theta}\|u_n\|_V^p\right] \\
&\quad -\lambda \int_0^T \left[\bar{F}(t, u_n) - \frac{1}{\theta}(\nabla \bar{F}(t, u_n), u_n)\right] dt - \frac{a^p}{bp^2} \\
&\geq (a + b\|u_n\|_V^p)^{p-1}\left[\frac{1}{bp^2}(a + b\|u_n\|_V^p) - \frac{1}{\theta}\|u_n\|_V^p\right] - \frac{a^p}{bp^2} \\
&\geq a^{p-1}\left[\frac{a}{bp^2} + \left(\frac{1}{p^2} - \frac{1}{\theta}\right)\|u_n\|_V^p\right] - \frac{a^p}{bp^2} \\
&= a^{p-1}\left(\frac{1}{p^2} - \frac{1}{\theta}\right)\|u_n\|_V^p
\end{aligned} \tag{25}$$

for n large enough, which shows that $\{u_n\}$ is bounded in $E_0^{\alpha,p}$ by $p > 1$. By Lemma 1, we can assume that, up to a subsequence, for some $u_\lambda \in E_0^{\alpha,p}$,

$$\begin{aligned}
u_n &\rightharpoonup u_\lambda \text{ in } E_0^{\alpha,p}, \\
u_n &\to u_\lambda \text{ in } C([0,T], \mathbb{R}^N).
\end{aligned} \tag{26}$$

The following argument is similar to [64] with some modifications. Since

$$\begin{aligned}
&\left\langle \bar{I}'_\lambda(u_n), u_n - u_\lambda \right\rangle \\
= &\left(a + b\|u\|_V^p\right)^{p-1}\left(\int_0^T (|{}_0D_t^\alpha u_n|^{p-2}{}_0D_t^\alpha u_n, {}_0D_t^\alpha(u_n - u_\lambda))dt\right. \\
&\left. + \int_0^T V(t)(|u_n|^{p-2}u_n, u_n - u_\lambda)dt\right) - \lambda \int_0^T (\nabla \bar{F}(t, u_n), u_n - u_\lambda)dt,
\end{aligned} \tag{27}$$

we have

$$
\begin{aligned}
& \left\langle \overline{I}'_\lambda(u_n) - \overline{I}'_\lambda(u_\lambda), u_n - u_\lambda \right\rangle \\
= \ & (a + b\|u_n\|_V^p)^{p-1}\Big(\int_0^T (|_0D_t^\alpha u_n|^{p-2}{}_0D_t^\alpha u_n, {}_0D_t^\alpha (u_n - u_\lambda))dt \\
& + \int_0^T V(t)(|u_n|^{p-2}u_n, u_n - u_\lambda)dt\Big) - \lambda \int_0^T (\nabla \bar{F}(t, u_n), u_n - u_\lambda)dt \\
& - \Big[(a + b\|u_\lambda\|_V^p)^{p-1}\Big(\int_0^T (|_0D_t^\alpha u_\lambda|^{p-2}{}_0D_t^\alpha u_\lambda, {}_0D_t^\alpha (u_n - u_\lambda))dt \\
& + \int_0^T V(t)(|u_\lambda|^{p-2}u_\lambda, u_n - u_\lambda)dt\Big) - \lambda \int_0^T (\nabla \bar{F}(t, u_\lambda), u_n - u_\lambda)dt\Big] \\
= \ & (a + b\|u_n\|_V^p)^{p-1}\Big(\|u_n\|_V^p - \int_0^T (|_0D_t^\alpha u_n|^{p-2}{}_0D_t^\alpha u_n, {}_0D_t^\alpha u_\lambda)dt \\
& - \int_0^T V(t)(|u_n|^{p-2}u_n, u_\lambda)dt\Big) - (a + b\|u_\lambda\|_V^p)^{p-1}\Big(- \|u_\lambda\|_V^p \\
& + \int_0^T (|_0D_t^\alpha u_\lambda|^{p-2}{}_0D_t^\alpha u_\lambda, {}_0D_t^\alpha u_n)dt + \int_0^T V(t)(|u_\lambda|^{p-2}u_\lambda, u_n)dt\Big) \\
& - \lambda \int_0^T (\nabla \bar{F}(t, u_n) - \nabla \bar{F}(t, u_\lambda), u_n - u_\lambda)dt \\
\geq \ & (a + b\|u_n\|_V^p)^{p-1}\|u_n\|_V^p + (a + b\|u_\lambda\|_V^p)^{p-1}\|u_\lambda\|_V^p \\
& - (a + b\|u_n\|_V^p)^{p-1}\Big[\|_0D_t^\alpha u_n\|_{L^p}^{p-1}\|_0D_t^\alpha u_\lambda\|_{L^p} \\
& + \Big(\int_0^T V(t)|u_n|^p dt\Big)^{(p-1)/p}\Big(\int_0^T V(t)|u_\lambda|^p dt\Big)^{1/p}\Big] \\
& - (a + b\|u_\lambda\|_V^p)^{p-1}\Big[\|_0D_t^\alpha u_\lambda\|_{L^p}^{p-1}\|_0D_t^\alpha u_n\|_{L^p} \\
& + \Big(\int_0^T V(t)|u_\lambda|^p dt\Big)^{(p-1)/p}\Big(\int_0^T V(t)|u_n|^p dt\Big)^{1/p}\Big] \\
& - \lambda \int_0^T (\nabla \bar{F}(t, u_n) - \nabla \bar{F}(t, u_\lambda), u_n - u_\lambda)dt \\
\geq \ & (a + b\|u_n\|_V^p)^{p-1}\|u_n\|_V^p + (a + b\|u_\lambda\|_V^p)^{p-1}\|u_\lambda\|_V^p \\
& - (a + b\|u_n\|_V^p)^{p-1}\Big(\|_0D_t^\alpha u_n\|_{L^p}^p + \int_0^T V(t)|u_n|^p dt\Big)^{(p-1)/p}\Big(\|_0D_t^\alpha u_\lambda\|_{L^p}^p + \int_0^T V(t)|u_\lambda|^p dt\Big)^{1/p} \\
& - (a + b\|u_\lambda\|_V^p)^{p-1}\Big(\|_0D_t^\alpha u_\lambda\|_{L^p}^p + \int_0^T V(t)|u_\lambda|^p dt\Big)^{(p-1)/p}\Big(\|_0D_t^\alpha u_n\|_{L^p}^p + \int_0^T V(t)|u_n|^p dt\Big)^{1/p} \\
& - \lambda \int_0^T (\nabla \bar{F}(t, u_n) - \nabla \bar{F}(t, u_\lambda), u_n - u_\lambda)dt \\
= \ & (a + b\|u_n\|_V^p)^{p-1}\|u_n\|_V^p + (a + b\|u_\lambda\|_V^p)^{p-1}\|u_\lambda\|_V^p \\
& - (a + b\|u_n\|_V^p)^{p-1}\|u_n\|_V^{p-1}\|u_\lambda\|_V - (a + b\|u_\lambda\|_V^p)^{p-1}\|u_n\|_V\|u_\lambda\|_V^{p-1} \\
& - \lambda \int_0^T (\nabla \bar{F}(t, u_n) - \nabla \bar{F}(t, u_\lambda), u_n - u_\lambda)dt \\
= \ & (a + b\|u_n\|_V^p)^{p-1}\|u_n\|_V^{p-1}(\|u_n\|_V - \|u_\lambda\|_V) \\
& + (a + b\|u_\lambda\|_V^p)^{p-1}\|u_\lambda\|_V^{p-1}(\|u_\lambda\|_V - \|u_n\|_V) - \lambda \int_0^T (\nabla \bar{F}(t, u_n) - \nabla \bar{F}(t, u_\lambda), u_n - u_\lambda)dt \\
= \ & \Big((a + b\|u_n\|_V^p)^{p-1}\|u_n\|_V^{p-1} - (a + b\|u_\lambda\|_V^p)^{p-1}\|u_\lambda\|_V^{p-1}\Big)(\|u_n\|_V - \|u_\lambda\|_V) \\
& - \lambda \int_0^T (\nabla \bar{F}(t, u_n) - \nabla \bar{F}(t, u_\lambda), u_n - u_\lambda)dt.
\end{aligned} \tag{28}
$$

Note that

$$
\lambda \int_0^T (\nabla \bar{F}(t, u_n) - \nabla \bar{F}(t, u_\lambda), u_n - u_\lambda)dt \leq \lambda \int_0^T |\nabla \bar{F}(t, u_n) - \nabla \bar{F}(t, u_\lambda)||u_n - u_\lambda|dt \to 0, \tag{29}
$$

by $u_n \to u_\lambda$ in $C([0, T], \mathbb{R}^N)$ and $|\nabla \bar{F}(t, u_n) - \nabla \bar{F}(t, u_\lambda)|$ is bounded in $[0, T]$ because of (H0)′ and the boundedness of $\{u_n\}$ in $E_0^{\alpha,p}$, and (23) and (26) imply that

$$
\left\langle \overline{I}'_\lambda(u_n) - \overline{I}'_\lambda(u_\lambda), u_n - u_\lambda \right\rangle \to 0, \text{ as } n \to \infty. \tag{30}
$$

So by (28)–(30), we have

$$
\|u_n\|_V \to \|u_\lambda\|_V, \text{ as } n \to \infty.
$$

By the uniform convexity of $E_0^{\alpha,p}$ and $u_n \rightharpoonup u_\lambda$, it follows from the Kadec–Klee property (see [65]) and (14), $u_n \to u_\lambda$ in $E_0^{\alpha,p}$. $\square$

By the continuity of $\bar{I}_\lambda$, we obtain that $\bar{I}_\lambda(u) = c_\lambda$, where c_λ is defined by (24). Then (18) implies that $c_\lambda \geq d_\lambda > 0$. Hence u_λ is a nontrivial critical point of $\bar{I}_\lambda$ in $E_0^{\alpha,p}$ for any given $\lambda > \Lambda_1$.

Next, we show that u_λ precisely is the nontrivial weak solution of system (1) for any given $\lambda > \lambda^*$. In order to get this, we need to make an estimate for the critical level c_λ. We introduce the functional $\widetilde{J}_\lambda : E_0^{\alpha,p} \to \mathbb{R}$ as follows

$$\widetilde{J}_\lambda(u) \quad = \quad \frac{1}{bp^2}(a + b\|u\|_V^p)^p - \lambda M_1 \int_0^T |u(t)|^{q_1} dt - \frac{a^p}{bp^2}.$$

Lemma 8. *For all $\lambda \geq \max\{\Lambda_1, \Lambda_2\}$,*

$$c_\lambda \leq C_* \lambda^{-\frac{p-1}{q_1 - p}},$$

where C_* is defined by (13) which is obviously independent of λ.

Proof. Define $f_i : [0, \infty) \to \mathbb{R}$, $i = 1, 2$, by

$$\begin{aligned} f_1(s) &= \frac{1}{bp^2}(a + bs^p\|e_1\|_V^p)^p - \lambda^{\frac{1}{q_1}} \|e_1\|_V^p \frac{s^p}{p} - \frac{a^p}{bp^2}, \\ f_2(s) &= -\lambda M_1 s^{q_1} \int_0^T |e_1|^{q_1} dt + \lambda^{\frac{1}{q_1}} \|e_1\|_V^p \frac{s^p}{p}, \end{aligned}$$

where $e_1 = \frac{\delta}{G_0} e$ and e is defined in (19). Then $f_1(s) + f_2(s) = \widetilde{J}_\lambda(se_1)$. Let

$$f_2'(s) = -\lambda M_1 q_1 \|e_1\|_{L^{q_1}}^{q_1} s^{q_1 - 1} + \lambda^{\frac{1}{q_1}} \|e_1\|_V^p s^{p-1} = 0.$$

Thus for each given $\lambda > 0$, we have $s = \left(\dfrac{\lambda^{\frac{1}{q_1}} \|e_1\|_V^p}{\lambda M_1 q_1 \|e_1\|_{L^{q_1}}^{q_1}} \right)^{\frac{1}{q_1 - p}}$. Then

$$\max_{s \geq 0} f_2(s) = \left(\frac{1}{p(M_1 q_1)^{\frac{p}{q_1 - p}}} - \frac{M_1}{(M_1 q_1)^{\frac{q_1}{q_1 - p}}} \right) \left(\frac{\|e_1\|_V}{\|e_1\|_{L^{q_1}}} \right)^{\frac{pq_1}{q_1 - p}} \lambda^{-\frac{p-1}{q_1 - p}}.$$

Obviously, $f_1(0) = 0$ and

$$f_1'(s) = (a + bs^p\|e_1\|_V^p)^{p-1} \|e_1\|_V^p s^{p-1} - \lambda^{\frac{1}{q_1}} \|e_1\|_V^p s^{p-1}.$$

So if

$$\begin{aligned} \lambda \quad &> \quad \Lambda_2 := \left[a + b[\max\{1, V^\infty\}]^p \frac{\delta^p}{G_0^p} (D^p + G^p) \right]^{q_1(p-1)} \\ &= \left(a + b[\max\{1, V^\infty\}]^p \frac{\delta^p}{G_0^p} \|e\|^p \right)^{q_1(p-1)} \\ &\geq \left(a + bs^p \|e_1\|_V^p \right)^{q_1(p-1)}, \end{aligned}$$

$f_1(s)$ is decreasing on $s \in [0,1]$ and then $f_1(s) < 0$ for all $s \in [0,1]$. By (22), we have

$$\|se_1\|_\infty \leq \|\frac{\delta}{G_0} e\|_\infty \leq \delta \tag{31}$$

for all $s \in [0,1]$. Then for all $\lambda > \Lambda_2$, by (H1)$'$, (20), (21) and Hölder inequality, we have

$$\begin{aligned}
& c_\lambda \leq \max_{s\in[0,1]} \bar{I}_\lambda(se_1) \leq \max_{s\in[0,1]} \widetilde{J}_\lambda(se_1) \leq \max_{s\in[0,1]} f_1(s) + \max_{s\geq 0} f_2(s) \\
& \leq \max_{s\geq 0} f_2(s) = \left(\frac{1}{p(M_1 q_1)^{\frac{p}{q_1-p}}} - \frac{M_1}{(M_1 q_1)^{\frac{q_1}{q_1-p}}} \right) \left(\frac{\|e_1\|_V}{\|e_1\|_{L^{q_1}}} \right)^{\frac{pq_1}{q_1-p}} \lambda^{-\frac{p-1}{q_1-p}} \\
& \leq \left(\frac{1}{p(M_1 q_1)^{\frac{p}{q_1-p}}} - \frac{M_1}{(M_1 q_1)^{\frac{q_1}{q_1-p}}} \right) \left(\frac{[\max\{1, V^\infty\}]^{1/p} (D^p + G^p)^{1/p}}{T^{\frac{1}{q_1}-\frac{1}{p}} \|e\|_{L^p}} \right)^{\frac{pq_1}{q_1-p}} \lambda^{-\frac{p-1}{q_1-p}} \\
& = C_* \lambda^{-\frac{p-1}{q_1-p}}.
\end{aligned}$$

□

Proof of Theorem 1. Note that u_λ is a critical point of $\bar{I}_\lambda$ with critical value c_λ. Since $\langle \bar{I}'(u_\lambda), u_\lambda \rangle = 0$, similar to the argument in (25) and by Lemma 8, we have

$$\begin{aligned}
\|u_\lambda\|_V^p &\leq \frac{p^2\theta}{a^{p-1}(\theta - p^2)} \bar{I}_\lambda(u_\lambda) \\
&= \frac{p^2\theta}{a^{p-1}(\theta - p^2)} c_\lambda \\
&\leq \frac{p^2\theta}{a^{p-1}(\theta - p^2)} C_* \lambda^{-\frac{p-1}{q_1-p}}.
\end{aligned} \tag{32}$$

Since

$$\lambda > \Lambda_3 = \left(\frac{T^{p\alpha-1}}{\left[\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}\right]^p} \cdot \frac{p^2\theta C_*}{a^{p-1}(\theta - p^2)} \cdot \frac{2^p}{\delta^p} \right)^{\frac{q_1-p}{p-1}},$$

by (32), we have

$$\|u_\lambda\|_\infty \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}} \|u_\lambda\|_V \leq \delta/2. \tag{33}$$

So for all $\lambda > \Lambda_3$, $|u_\lambda(t)| \leq \|u_\lambda\|_\infty \leq \delta/2$ for a.e. $t \in [0,T]$ and then $\bar{F}(t, u(t)) = F(t, u(t))$ for a.e. $t \in [0,T]$. Furthermore, $\bar{I}_\lambda(u_\lambda) = I_\lambda(u_\lambda) = c_\lambda > 0$ and $\langle \bar{I}'(u_\lambda), v \rangle = \langle I'(u_\lambda), v \rangle = 0$ for all $v \in E_0^{\alpha,p}$. Thus u_λ is precisely the nontrivial weak solution of system (1) when $\lambda > \lambda^* := \max\{\Lambda_1, \Lambda_2, \Lambda_3\}$. Note that $p > 1$ and $q_1 > p$. By (32) and (33), it is obvious that

$$\lim_{\lambda\to\infty} \|u_\lambda\|_V = 0 = \lim_{\lambda\to\infty} \|u_\lambda\|_\infty.$$

□

4. Example

Assume that $N = 2, a = b = T = 1$, $p = 3$ and $\delta = 1$. Then $q = \frac{3}{2}$. Let $q_1 = 12, q_2 = 10, F(t,x) = (t+1)|x|^{11}$ for a.e. $t \in [0,1]$ and all $x \in \mathbb{R}^N$ with $|x| \leq 1$. $V(t) = 7t^2 + 1$ for all $t \in [0,1]$. Then $V^\infty = 8$ and $V_\infty = 1$. Choose $\alpha = \frac{1}{2}$. Consider the system

$$\begin{cases} A(u(t))[{}_tD_1^{1/2}\phi_3({}_0D_t^{1/2}u(t)) + (7t^2+1)\phi_3(u(t))] = 11\lambda(t+1)|u|^9 u, \quad \text{a.e. } t \in [0,1], \\ u(0) = u(1) = 0, \end{cases} \tag{34}$$

where

$$A(u(t)) = \left[1 + \int_0^1 (|{}_0D_t^{1/2}u(t)|^3 + (7t^2+1)|u(t)|^3)dt\right]^2.$$

By Theorem 1, we can obtain that system (34) has at least a nontrivial solution u_λ in $E_0^{\frac{1}{2},3}$ for each $\lambda > 183.46^{24}$ and $\lim_{\lambda\to\infty}\|u_\lambda\|_V = 0 = \lim_{\lambda\to\infty}\|u_\lambda\|_\infty$.

In fact, we can verify that $F(t,u)$ satisfies (H0)-(H2) as follows.

(i) Note that

$$|F(t,x)| = (t+1)|x|^{11}, \ \ |\nabla F(t,x)| = 11(t+1)|x|^{10}$$

for all $|x| \leq \delta$. Set $a(|x|) = |x|^{10} + 1$ for all $x \in \mathbb{R}^N$ with $|x| \leq 1$ and $b(t) = 11(t+1)$ for a.e. $t \in [0,T]$. Then assumption (H0) is satisfied.

(ii) Note that $q_1 = 12 > q_2 = 10 > p^2 = 9$, and

$$|x|^{12} \leq F(t,x) = (t+1)|x|^{11} \leq 2|x|^{10},$$

for all $|x| \leq \delta$ and a.e. $t \in [0,1]$. Set $M_1 = 1$ and $M_2 = 2$. Then assumption (H1) is also satisfied.

(iii) Let $\beta = 10 > p^2 = 9$. Then

$$0 \leq 10(t+1)|x|^{11} = \beta F(t,x) \leq 11(t+1)|x|^{11} = (\nabla F(t,x), x)$$

holds for all $x \in \mathbb{R}^2$ and a.e. $t \in [0,1]$, and so assumption (H3) is satisfied. Next, we compute the value of λ^* by the formulas in Theorem 1. Note that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(2-\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$. We obtain

$$D = \left(\frac{T^{p+1}}{\pi^{p+1}}\frac{2(p-1)!!}{p!!}\right)^{\frac{1}{p}} = \left(\frac{4}{3}\right)^{\frac{1}{3}}\pi^{-\frac{4}{3}},$$

$$G = \left(\frac{T^{p+1-p\alpha}}{\Gamma^p(2-\alpha)(p+1-p\alpha)}\right)^{1/p} = \left(\frac{16}{5}\right)^{\frac{1}{3}}\pi^{-\frac{1}{2}},$$

$$G_0 = \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}}G = 5^{-\frac{1}{3}} \cdot 16^{\frac{2}{3}}\pi^{-1}.$$

Then by $\theta = \min\{\beta, q_2\} = 10$, (10)–(12), we have

$$\Lambda_1 = \max\left\{\sqrt[3]{\frac{768}{78125}}\pi^{35/6}, \frac{16^{11/2}\cdot 3\pi^4}{625}(1+\frac{5}{24\pi}+\frac{\pi^{3/2}}{2})^3\right\},$$

$$\Lambda_2 = \left(1 + \frac{40}{3}\pi^{-1} + 32\pi^{3/2}\right)^{24},$$

$$\Lambda_3 = \left(\frac{720 \cdot 16C_*}{\pi^{3/2}}\right)^{\frac{9}{2}},$$

and by (13),

$$C_* = 16\left(\frac{1}{3 \cdot \sqrt[3]{12}} - \frac{1}{\sqrt[3]{12^4}}\right)\left(1 + \frac{12}{5}\pi^{5/2}\right)^{4/3}.$$

Compared Λ_1, Λ_2 and Λ_3, it is easy to see $\lambda^* = \Lambda_2 \approx 183.46^{24}$.

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