mathematics

MDPI

Article

Generalized-Fractional Tikhonov-Type Method for the Cauchy Problem of Elliptic Equation

Hongwu Zhang [1,*,†] **and Xiaoju Zhang** [2,†]

[1] School of Mathematics and Information Science, North Minzu University, Yinchuan 750021, China
[2] Center for Faculty Development, North Minzu University, Yinchuan 750021, China; zh-xiaoju@163.com
* Correspondence: zhanghw2007@lzu.edu.cn or zh-hongwu@163.com
† These authors contributed equally to this work.

Received: 20 November 2019; Accepted: 23 December 2019; Published: 1 January 2020

Abstract: This article researches an ill-posed Cauchy problem of the elliptic-type equation. By placing the a-priori restriction on the exact solution we establish conditional stability. Then, based on the generalized Tikhonov and fractional Tikhonov methods, we construct a generalized-fractional Tikhonov-type regularized solution to recover the stability of the considered problem, and some sharp-type estimates of convergence for the regularized method are derived under the a-priori and a-posteriori selection rules for the regularized parameter. Finally, we verify that the proposed method is efficient and acceptable by making the corresponding numerical experiments.

Keywords: Cauchy problem; elliptic equation; regularization method; a-priori and a-posteriori convergence estimates; numerical simulation

1. Introduction

Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a connected and bounded region, $\partial\Omega$ be the smooth boundary of Ω, $L_x : H^2(\Omega) \bigcap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ be a linear elliptic operator, which is densely defined, self-adjoint and positive-definite with regard to the variable x. Suppose that the eigenvalues of L_x are $\lambda_n (n \geq 1)$, i.e., there is one nontrivial solution $X_n \in L^2(\Omega)$, and it satisfies the below boundary problem

$$\begin{cases} L_x X_n = \lambda_n X_n & \text{in } \Omega, \\ X_n = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Further assume that the eigenvalues of L_x satisfies

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots, \quad \lim_{n\to\infty} \lambda_n = \infty. \tag{2}$$

The present article studies the problem below

$$\begin{cases} w_{yy}(y,x) - L_x w(y,x) = 0, & x \in \Omega,\ 0 < y < T, \\ w(y,x) = 0, & x \in \partial\Omega,\ 0 \leq y \leq T, \\ w(0,x) = \varphi(x), & x \in \Omega, \\ w_y(0,x) = \psi(x), & x \in \Omega, \end{cases} \tag{3}$$

which is a Cauchy problem for elliptic-type equation. We seek $u(y,\cdot)(0 < y \leq T)$ under the Cauchy datum $\varphi, \psi \in L^2(\Omega)$ are given.

In many engineering and science areas, the Cauchy problem for elliptic-type equations has many important applications, see [1–4]. Note that, as $L_x = -\frac{\partial^2}{\partial x^2}$, (3) is a special ill-posed problem—the

Cauchy problem of the Laplace equation. If $L_x = -\frac{\partial^2}{\partial x^2} + k^2 (k > 0)$, we know that (3) becomes the Cauchy problem of modified Helmholtz equation.

In fact, we can divided (3) into two linear problem

$$\begin{cases} u_{yy}(y,x) - L_x u(y,x) = 0, & x \in \Omega,\ 0 < y < T, \\ u(y,x) = 0, & x \in \partial\Omega,\ 0 \le y \le T, \\ u(0,x) = \varphi(x), & x \in \Omega, \\ u_y(0,x) = 0, & x \in \Omega, \end{cases} \tag{4}$$

and

$$\begin{cases} v_{yy}(y,x) - L_x v(y,x) = 0, & x \in \Omega,\ 0 < y < T, \\ v(y,x) = 0, & x \in \partial\Omega,\ 0 \le y \le T, \\ v(0,x) = 0, & x \in \Omega, \\ v_y(0,x) = \psi(x), & x \in \Omega, \end{cases} \tag{5}$$

thus, the solution of (3) can be expressed as $w = u + v$.

It can be found that many scholars have researched this kind of problem in the aspect of theory and algorithm, such as [5–17], and so on. In [14], the author solved (4), (5) by constructing a generalized Tikhonov regularization method, meanwhile derived the a-priori convergence estimate of regularized method. In [18], Hochstenbach-Reichel studied the ill-posed problems of discrete type by using a fractional Tikhonov regularization method. In the present paper, by placing the a-priori restrictions on the exact solutions, we establish and prove the conditional stability of problems (4), (5). Meanwhile, inspired by the methods in [14,18], we develop a generalized-fractional Tikhonov-type regularization method to do with (4), (5), then adopt the a-priori and a-posteriori rules to choose the regularized parameters, and give and prove some sharp estimates of convergence for our method. In fact, the proposed method is a generalization for the nonlocal boundary value problem method (or quasi-boundary value method), the boundary (or modified) Tikhonov method, and the generalized Tikhonov method, it also can be regarded as a modification on the fractional Tikhonov method (see Remark 1). So far, there are no related references where this method is proposed to study (4), (5). In the numerical simulation aspect, we also notice that the computation effect of this method is also efficient and feasible.

We arrange the structure of the article as below. We give the conditional stabilities of problems (4), (5) in Section 2. Section 3 describes the construction procedure of regularization method, and Section 4 provides some necessary preparation knowledge. We give the results of convergence estimate in Section 5. Section 6 is arranged as the numerical simulation part. A summary of the article and further outlook are given in Section 7. The proofs of related Lemmas and Theorems are arranged in Appendix A.

2. Conditional Stability

The conditional stability means that the solution is continuously dependent on the given datum under certain additional condition [5,19,20]. This section establishes the conditional stability of problems (4), (5) by imposing the corresponding a-priori conditions for the exact solutions.

Define

$$\mathcal{D}^{\xi}_{\gamma,q} = \left\{ \xi \in L^2(\Omega); \sum_{n=1}^{\infty} \lambda_n^{2\gamma} e^{2qT\sqrt{\lambda_n}} |< \xi, X_n >|^2 < +\infty \right\},\ \gamma \ge 1,\ q > 1, \tag{6}$$

where $< \cdot, \cdot >$ is the inner product in $L^2(\Omega)$, $X_n = X_n(x)$ are the eigenfunctions of L_x, and it is the orthogonal basis of space $L^2(\Omega)$. We define the norm of $\mathcal{D}^{\xi}_{\gamma,q}$ as

$$\|\xi\|_{\mathcal{D}^{\xi}_{\gamma,q}} = \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma} e^{2qT\sqrt{\lambda_n}} |< \xi, X_n >|^2 \right)^{1/2}. \tag{7}$$

By adopting the Fourier method, we respectively can express the solutions of (4) and (5) as

$$u(y,x) = \sum_{n=1}^{\infty} \cosh(\sqrt{\lambda_n} y)\varphi_n X_n, \ \varphi_n = \langle \varphi, X_n \rangle, \tag{8}$$

$$v(y,x) = \sum_{n=1}^{\infty} \frac{\sinh(\sqrt{\lambda_n} y)}{\sqrt{\lambda_n}} \psi_n X_n, \ \psi_n = \langle \psi, X_n \rangle. \tag{9}$$

In 1996, [14] considered problems (4) and (5). For (4), the author formulated the problem as

$$A_1(y)u(y,x) = u(0,x) = \varphi(x), \tag{10}$$

here the linear operator $A_1(y) = 1/\cosh(\sqrt{L_x} y) : L^2(\Omega) \to L^2(\Omega)$ is self-adjoint and compact, and $1/\cosh(\sqrt{\lambda_n} y)$ are its eigenvalues, X_n are the corresponding eigenelements. For (5), the author formulated the problem as

$$A_2(y)v(y,x) = v_y(0,x) = \psi(x), \tag{11}$$

here $A_2(y) = \sqrt{L_x}/\sinh(\sqrt{L_x} y) : L^2(\Omega) \to L^2(\Omega)$ is self-adjoint, linear bounded, and compact, $\sqrt{\lambda_n}/\sinh(\sqrt{\lambda_n} y)$ are its eigenvalues, X_n are the corresponding eigenelements.

Below, we give the conditional stabilities for problems (4) and (5), respectively.

Theorem 1. *Assume that $u(T,x)$ satisfies*

$$\|u(T,x)\|_{\mathcal{D}^{u}_{\gamma,q}} \le E, \tag{12}$$

then the below stability estimate of condition can be established

$$\|u(y,x)\|_{L^2(\Omega)} \le 2^{\frac{y}{qT}} \left(\lambda_1^{\gamma} e^{\sqrt{\lambda_1} T} \right)^{-\frac{y}{qT}} E^{\frac{y}{qT}} \|\varphi\|_{L^2(\Omega)}^{1-\frac{y}{qT}}. \tag{13}$$

Theorem 2. *Suppose that $v(T,x)$ satisfies*

$$\|v(T,x)\|_{\mathcal{D}^{v}_{\gamma,q}} \le E, \tag{14}$$

then we can establish the following result of conditional stability

$$\|v(y,x)\|_{L^2(\Omega)} \le 2^{\frac{y}{qT}} \lambda_1^{\left(\frac{1}{2}-\gamma\right)-\frac{qT}{2y}} \left(e^{\sqrt{\lambda_1} T} \left(1 - e^{-2\sqrt{\lambda_1} T} \right) \right)^{-\frac{y}{qT}} E^{\frac{y}{qT}} \|\psi\|_{L^2(\Omega)}^{1-\frac{y}{qT}}. \tag{15}$$

3. Regularization Method

From (8), (9), we know that, as $n \to \infty$, the limits of the sequences $\{\cosh(\sqrt{\lambda_n} y)\}$ and $\left\{ \frac{\sinh(\sqrt{\lambda_n} y)}{\sqrt{\lambda_n}} \right\}$ are infinities, respectively, hence problems (4), (5) are both severely ill-posed. To recover the continuous dependence of the solutions on the datum φ, ψ, we need construct the corresponding regularization solutions for (4), (5). For the references which describe the usual regularization theory, we can refer to [21,22], etc.

3.1. Regularization Method for Problem (4)

In [14], the author considered (4) and expressed it as the operator equation (10). Then let $\gamma > 0$, $\varphi^{\delta}(x) = u^{\delta}(0, x)$ be the observation data, δ is the observed error bound, μ is the regularized parameter, the author proposed a generalized Tikhonov method by solving the minimum value problem

$$\min_{u \in L^2(\Omega)} J_{\mu}(u), J_{\mu}(u) = \left\| \frac{1}{\cosh(\sqrt{L_x}y)} u - u^{\delta}(0, x) \right\|^2 + \mu \left\| L_x^{\frac{\gamma}{2}} \frac{\cosh(\sqrt{L_x}T)}{\cosh(\sqrt{L_x}y)} u \right\|^2, \tag{16}$$

denote $u_{\mu}^{\delta}(y, x)$ be the regularization solution, then it satisfies the normal equation

$$\left(\frac{1}{\cosh^2(\sqrt{L_x}y)} + \mu L_x^{\gamma} \frac{\cosh^2(\sqrt{L_x}T)}{\cosh^2(\sqrt{L_x}y)} \right) u_{\mu}^{\delta}(y, x) = \frac{1}{\cosh(\sqrt{L_x}y)} u^{\delta}(0, x), \tag{17}$$

from (17), we can get that the regularized solution is

$$u_{\mu}^{\delta}(y, x) = \sum_{n=1}^{\infty} \frac{\cosh(\sqrt{\lambda_n}y)\varphi_n^{\delta}}{1 + \mu \lambda_n^{\gamma} \cosh^2(\sqrt{\lambda_n}T)} X_n(x), \tag{18}$$

where $\varphi_n^{\delta} = \langle \varphi^{\delta}, X_n \rangle_{L^2(\Omega)}$, the error data φ^{δ} satisfies

$$\|\varphi^{\delta} - \varphi\|_{L^2(\Omega)} \leq \delta. \tag{19}$$

In 2011, Hochstenbach-Reichel studied the ill-posed problems of discrete type by using a fractional Tikhonov regularization method [18]. Note that, if applying the method of [18] to problem (4), we can write the regularized solution as

$$u_{\mu}^{\delta}(y, x) = \sum_{n=1}^{\infty} \frac{\cosh(\sqrt{\lambda_n}y)\varphi_n^{\delta}}{1 + \mu \cosh^q(\sqrt{\lambda_n}y)} X_n(x), \ q > 1. \tag{20}$$

Inspired by the ideas of [14,18], setting $\gamma \geq 1$, $q > 1$, α is the regularized parameter, we develop a generalized-fractional Tikhonov-type regularized solution $u_{\alpha}^{\delta}(y, x)$ by solving the minimum value problem

$$\min_{u \in L^2(\Omega)} J_{\alpha}(u), J_{\alpha}(u) = \left\| \frac{1}{\cosh(\sqrt{L_x}y)} u - u^{\delta}(0, x) \right\|^2 + \alpha \left\| L_x^{\frac{\gamma}{2}} \frac{\cosh^{\frac{q}{2}}(\sqrt{L_x}T)}{\cosh(\sqrt{L_x}y)} u \right\|^2, \tag{21}$$

from the necessary condition of the first order, we know that $u_{\alpha}^{\delta}(y, x)$ satisfies the normal equation

$$\left(\frac{1}{\cosh^2(\sqrt{L_x}y)} + \alpha L_x^{\gamma} \frac{\cosh^q(\sqrt{L_x}T)}{\cosh^2(\sqrt{L_x}y)} \right) u_{\alpha}^{\delta}(y, x) = \frac{1}{\cosh(\sqrt{L_x}y)} u^{\delta}(0, x), \tag{22}$$

from (22), we can express the regularized solution of (4) as

$$u_{\alpha}^{\delta}(y, x) = \sum_{n=1}^{\infty} \frac{\cosh(\sqrt{\lambda_n}y)\varphi_n^{\delta}}{1 + \alpha \lambda_n^{\gamma} \cosh^q(\sqrt{\lambda_n}T)} X_n(x), \ q > 1. \tag{23}$$

Remark 1. *In the process of definition for the regularization solution (23), we always assume that $\gamma \geq 1$ and $q > 1$, this assumption mainly is used to derive the sharp convergence estimate of regularization method. In fact, if $\gamma = 0$, $q = 1$, the method in this paper becomes the nonlocal boundary value problem method (or quasi-boundary value method) in [13]. If $\gamma = 0$, $q > 1$, compared with (20), (23) can be seen as a modification on fractional Tikhonov method in [18]. When $\gamma = 0$, $q = 2$, our method is the boundary (or modified) Tikhonov method (you* *can see [23,24], etc). As $\gamma > 0$, $q = 2$, our method is the generalized Tikhonov method in [14]. In summary, the proposed method in this paper is a meaningful expansion and extension on the these existing works.*

3.2. Regularization Method for Problem (5)

For (5), [14] expressed it as the operator equation (11) equivalently. Then let $\gamma > 0$, μ is the regularized parameter, δ is the bound of observed error, $\psi^\delta(x) = v_y^\delta(0,x)$ is the observed data, the author designed a generalized Tikhonov regularized method by solving the minimum value problem below

$$\min_{v\in L^2(\Omega)} J_\mu(v), J_\mu(v) = \left\| \frac{\sqrt{L_x}}{\sinh(\sqrt{L_x}y)} v - v_y^\delta(0,x) \right\|^2 + \mu \left\| L_x^{\frac{\gamma}{2}} \frac{\sinh(\sqrt{L_x}T)}{\sinh(\sqrt{L_x}y)} v \right\|^2, \tag{24}$$

denote $v_\mu^\delta(y,x)$ be the regularized solution, then it satisfies the normal equation

$$\left(\frac{L_x}{\sinh^2(\sqrt{L_x}y)} + \mu L_x^\gamma \frac{\sinh^2(\sqrt{L_x}T)}{\sinh^2(\sqrt{L_x}y)} \right) v_\alpha^\delta(y,x) = \frac{\sqrt{L_x}}{\sinh(\sqrt{L_x}y)} v_y^\delta(0,x), \tag{25}$$

from (25), the expression of the regularized solution of (5) is

$$v_\mu^\delta(y,x) = \sum_{n=1}^{\infty} \frac{\sinh(\sqrt{\lambda_n}y)\psi_n^\delta}{\sqrt{\lambda_n}(1+\mu\lambda_n^{\gamma-1}\sinh^2(\sqrt{\lambda_n}T))} X_n(x), \tag{26}$$

where $\psi_n^\delta = \langle \psi^\delta, X_n \rangle_{L^2(\Omega)}$, and

$$\|\psi^\delta - \psi\|_{L^2(\Omega)} \le \delta. \tag{27}$$

Meanwhile, if applying the fractional Tikhonov method of Hochstenbach-Reichel [18] to problem (5), the expression of regularization solution is

$$v_\mu^\delta(y,x) = \sum_{n=1}^{\infty} \frac{\sinh(\sqrt{\lambda_n}y)\psi_n^\delta X_n(x)}{\sqrt{\lambda_n}\left(1+\mu\left(\sinh(\sqrt{\lambda_n}y)/\sqrt{\lambda_n}\right)^q\right)}, \ q > 1. \tag{28}$$

Similar with Section 3.1, let $\gamma \ge 1$, $q > 1$, β be the regularized parameter, by solving the following minimization problem we constructed a generalized-fractional Tikhonov-type regularization solution of (5)

$$\min_{v\in L^2(\Omega)} J_\beta(v), J_\beta(v) = \left\| \frac{\sqrt{L_x}}{\sinh(\sqrt{L_x}y)} v - v_y^\delta(0,x) \right\|^2 + \beta \left\| L_x^{\frac{\gamma}{2}} \frac{\sinh^{\frac{q}{2}}(\sqrt{L_x}T)}{\sinh(\sqrt{L_x}y)} v \right\|^2, \tag{29}$$

from the necessary condition of the first order, we can derive that the regularized solution $v_\beta^\delta(y,x)$ satisfies the normal equation

$$\left(\frac{L_x}{\sinh^2(\sqrt{L_x}y)} + \beta L_x^\gamma \frac{\sinh^q(\sqrt{L_x}T)}{\sinh^2(\sqrt{L_x}y)} \right) v_\beta^\delta(y,x) = \frac{\sqrt{L_x}}{\sinh(\sqrt{L_x}y)} v_y^\delta(0,x), \tag{30}$$

from (30), the expression of regularized solution of (5) is

$$v_\beta^\delta(y,x) = \sum_{n=1}^{\infty} \frac{\sinh(\sqrt{\lambda_n}y)\psi_n^\delta}{\sqrt{\lambda_n}\left(1+\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)\right)} X_n(x), \ q > 1. \tag{31}$$

4. Preparation Knowledge

We need three functions and get help from some useful inequalities, which will be used in the procedure of the proof for the convergence of regularized solutions.

Let $\alpha,\beta>0$, $0<y\le T$, the following two functions $G_1(\lambda_n)$ and $G_2(\lambda_n)$ are defined

$$G_1(\lambda_n)=\frac{e^{-(qT-y)\sqrt{\lambda_n}}}{\frac{\alpha}{2^q}\lambda_n^{\gamma}+e^{-qT\sqrt{\lambda_n}}},\ \gamma\ge 1,\ q>1, \tag{32}$$

$$G_2(\lambda_n)=\frac{e^{-(qT-y)\sqrt{\lambda_n}}}{\sqrt{\lambda_1}\left(\beta\lambda_n^{\gamma-1}\left(\frac{1-e^{-2\sqrt{\lambda_1}T}}{2}\right)^q+e^{-qT\sqrt{\lambda_n}}\right)},\ \gamma\ge 1,\ q>1. \tag{33}$$

Simultaneously, the function $H(\eta)$ also will be used, and it is given in [13].

$$H(\eta)=\begin{cases}\eta^{\eta}(1-\eta)^{1-\eta},\eta\in(0,1).\\ 1,\eta=0,1,\end{cases} \tag{34}$$

Lemma 1. *[13] Assume that $0\le r\le s<\infty$, $s\ne 0$, $\nu>0$, thus*

$$\frac{\nu e^{-r}}{\nu+e^{-s}}\le H\left(\frac{r}{s}\right)\nu^{\frac{r}{s}}. \tag{35}$$

Theorem 3. *Assume that $\alpha>0$, $G_1(\lambda_n)$ is given by (32), hence we can derive that*

$$G_1(\lambda_n)\le 2C_1\alpha^{-\frac{y}{qT}},\ C_1=\max\{1,(\lambda_1^{\gamma})^{-1}\}. \tag{36}$$

Theorem 4. *Suppose that $\beta>0$, $G_2(\lambda_n)$ is the function given in (33), we have the result as below*

$$G_2(\lambda_n)\le 2D_1\beta^{-\frac{y}{qT}},\ D_1=C_1\lambda_1^{\frac{y}{qT}-\frac{1}{2}}\left(1-e^{-2\sqrt{\lambda_1}T}\right)^{-\frac{y}{T}}. \tag{37}$$

5. Convergence Estimate

This section gives the a-priori and a-posteriori convergence results of the proposed method.

5.1. The Estimate of Convergence for the Method in (4)

5.1.1. The Estimate of a-Priori Convergence

Theorem 5. *We denote u, u_α^δ as the exact and regularized solutions given in (8) and (23), respectively, φ^δ denotes the observed data and satisfy (19). Assume that there holds the a-priori condition*

$$\|u(T,\cdot)\|^2_{\mathcal{D}^u_{\gamma,q}}=\sum_{n=1}^{\infty}\lambda_n^{2\gamma}e^{2qT\sqrt{\lambda_n}}|\langle u(T,\cdot),X_n\rangle|^2\le E^2, \tag{38}$$

α is the regularized parameter, and we select it by

$$\alpha=\delta/E, \tag{39}$$

thus we can establish the below result of convergence

$$\|u_\alpha^\delta(y,\cdot)-u(y,\cdot)\|\le 4C_1E^{\frac{y}{qT}}\delta^{1-\frac{y}{qT}}. \tag{40}$$

5.1.2. The Estimate of a-Posteriori Convergence

In Theorem 5, we adopt the a-priori rule to select the regularized parameter α, i.e., $\alpha=\delta/E$. But in some actual problems, since the analysis form of the exact solution cannot be sought easily, the computation of E usually has certain difficulty. In view of this, it is necessary to consider the a-posteriori selection rule of α. Now, we give an a-posteriori rule and use it to choose α, which is similar to the discrepancy principle described in [25]. In this rule, the regularized parameter α only depends on the bound of observed error δ and the observed data φ^δ, we need not know the a-priori bound E.

We choose α by the equation below

$$\|u_\alpha^\delta(0,x) - \varphi^\delta(x)\| = \tau\delta,\ \tau > 1. \tag{41}$$

The following two Lemmas are necessary and meaningful in deriving the a-posteriori result of convergence.

Lemma 2. *Denote the function $\rho(\alpha) = \|u_\alpha^\delta(0,x) - \varphi^\delta(x)\|$, then there hold the following assertions: (i) In the interval $(0,+\infty)$, $\rho(\alpha)$ is consecutive; (ii) $\lim_{\alpha\to 0}\rho(\alpha) = 0$; (iii) $\lim_{\alpha\to+\infty}\rho(\alpha) = \|\varphi^\delta\|$; (iv) For $\alpha \in (0,+\infty)$, $\rho(\alpha)$ is increasing and strictly incremental.*

Lemma 3. *Let $\tau > 1$, the regularized parameter $\alpha = \alpha(\delta,\varphi^\delta)$ determined by (41) satisfies that $\alpha \geq \frac{(\tau-1)e^{\sqrt{\lambda_1}T}}{2}\frac{\delta}{E}$.*

Theorem 6. *Let u, u_α^δ be the exact and regularized solutions given in (8) and (23), respectively, φ^δ denotes the observed data and satisfy (19). Assume that there holds the a-priori condition (38), the regularized parameter is selected by (41), then the below a-posteriori estimate of convergence can be established*

$$\|u_\alpha^\delta(y,\cdot) - u(y,\cdot)\| \leq C_2 E^{\frac{y}{qT}}\delta^{1-\frac{y}{qT}}, \tag{42}$$

where $C_2 = \max\left\{2C_1\left((\tau-1)e^{\sqrt{\lambda_1}T}/2\right)^{-\frac{y}{qT}}, 2^{\frac{y}{qT}}\left(\lambda_1^\gamma e^{\sqrt{\lambda_1}T}\right)^{-\frac{y}{qT}}(\tau+1)^{1-\frac{y}{qT}}\right\}$.

Remark 2. *We all know that, as we use one regularized method to solve the ill-posed operator equation $Kf = g$, let δ be the observed error bound, g^δ denotes the error data which satisfies $\|g^\delta - g\| < \delta$, f_α^δ is the regularization solution, the basic idea of selecting the regularized parameter by a-posteriori rule is to seek α by $\|Kf_\alpha^\delta - g^\delta\| = \tau\delta$, $(\tau > 1)$. Based on this description, we can verify that the a-posteriori selection rule (41) actually can written as $\|A_1(y)u_\alpha^\delta(y,\cdot) - \varphi^\delta(x)\| = \tau\delta$, so it is consistent with the idea of [25].*

5.2. The Estimate of Convergence for the Method in (5)

5.2.1. The Estimate of a-Priori Convergence

Theorem 7. *Let v given by (9), v_β^δ defined by (31) be the exact and regularized solutions, respectively, ψ^δ is the observation data and (27) is satisfied. Assume that the a-priori condition*

$$\|v(T,\cdot)\|^2_{\mathcal{D}^v_{\gamma,q}} = \sum_{n=1}^{\infty}\lambda_n^{2\gamma}e^{2qT\sqrt{\lambda_n}}|\langle v(T,\cdot),X_n\rangle|^2 \leq E^2, \tag{43}$$

is valid, and we choose the regularized parameter β as

$$\beta = \delta/E, \tag{44}$$

thus, the below estimate of convergence can be derived

$$\|v_\beta^\delta(y,\cdot) - v(y,\cdot)\| \leq 2\left(1 + 1/(\sqrt{\lambda_1}e^{\sqrt{\lambda_1}y})\right)D_1 E^{\frac{y}{qT}}\delta^{1-\frac{y}{qT}}. \tag{45}$$

5.2.2. The Estimate of a-Posteriori Convergence

Let us seek β by the equation below

$$\|(v_\beta^\delta)_y(0,x) - \psi^\delta(x)\| = \tau\delta, \ \tau > 1. \tag{46}$$

Lemma 4. *Denote $\varrho(\beta) = \|(v_\beta^\delta)_y(0,x) - \psi^\delta(x)\|$, there hold the following assertions: (i) $\forall\beta \in (0,+\infty)$, $\varrho(\beta)$ is consecutive; (ii) $\lim_{\beta\to 0}\varrho(\beta) = 0$; (iii) $\lim_{\beta\to+\infty}\varrho(\beta) = \|\psi^\delta\|$; (iv) $\varrho(\beta)$ increases strictly in the interval $(0,+\infty)$.*

Lemma 5. *Let $\tau > 1$ is a constant, the regularized parameter $\beta = \beta(\delta,\psi^\delta)$ determined by (46) satisfies that $\beta \ge \sqrt{\lambda_1}\sinh(\sqrt{\lambda_1}T)(\tau-1)\frac{\delta}{E}$.*

Theorem 8. *Let v, v_β^δ be given in (9), (31), respectively, ψ^δ is the observed data and (27) is satisfied. We assume (43) to be true, and choose the regularized parameter β by a-posteriori rule (46), then the a-posteriori estimate of convergence below can be established*

$$\|v_\beta^\delta(y,\cdot) - v(y,\cdot)\| \le D_2 E^{\frac{y}{qT}}\delta^{1-\frac{y}{qT}}, \tag{47}$$

where $D_2 = \max\left\{2D_1\left(\sqrt{\lambda_1}\sinh(\sqrt{\lambda_1}T)(\tau-1)\right)^{-\frac{y}{qT}}, 2^{\frac{y}{qT}}\lambda_1^{\left(\frac{1}{2}-\gamma\right)-\frac{qT}{2y}}\left(e^{\sqrt{\lambda_1}T}\left(1-e^{-2\sqrt{\lambda_1}T}\right)\right)^{-\frac{y}{qT}}(\tau+1)^{1-\frac{y}{qT}}\right\}$.

Remark 3. *We can verify that the a-posteriori selection rule (46) is equivalent to the form $\|A_2(y)v_\beta^\delta(y,\cdot) - \psi^\delta(x)\| = \tau\delta$, which is also consistent with the idea of [25].*

Remark 4. *We derive the sharp results of convergence (40), (42), (45), and (47) by imposing the a-priori assumptions (38), (43) and applying the conclusions of Theorems 3, 4. We know that (38) and (43) are the stronger conditions, however we can verify that they can be accepted and natural, we can set some functions that satisfy this two conditions. For example, in order to verify the feasibility of (38), we take $T = 1$, $L_x = -\frac{\partial^2}{\partial x^2} : H^2(0,\pi)\bigcap H_0^1(0,\pi) \subset L^2(0,\pi) \to L^2(0,\pi)$, here $\lambda_n = n^2$, $X_n(x) = \sqrt{2/\pi}\sin(nx)$. Taking $u(y,x) = \sin(x)\cosh(y)$, then*

$$\begin{aligned}\|u(T,\cdot)\|^2_{\mathcal{D}^u_{\gamma,q}} &= \sum_{n=1}^\infty n^{4\gamma}e^{2nqT}|\langle u(T,\cdot),X_n\rangle|^2 = \sum_{n=1}^\infty n^{4\gamma}e^{2nqT}\left|\left\langle\sqrt{\frac{\pi}{2}}\cosh(T)\sqrt{\frac{2}{\pi}}\sin(x),\sqrt{\frac{2}{\pi}}\sin(nx)\right\rangle\right|^2 \\ &= e^{2qT}\left|\left\langle\sqrt{\frac{\pi}{2}}\cosh(T)\sqrt{\frac{2}{\pi}}\sin(x),\sqrt{\frac{2}{\pi}}\sin(x)\right\rangle\right|^2 = (2/\pi)e^{2qT}\cosh^2(T)\left(\int_0^\pi \sin^2 x dx\right)^2 = (\pi/2)e^{2qT}\cosh^2(T).\end{aligned}$$

For all $\gamma \ge 1$, $q > 1$, there always exists a positive number μ, subject to $\mu > q$, hence it can be obtained

$$\|u(T,\cdot)\|^2_{\mathcal{D}^u_{\gamma,q}} = \sum_{n=1}^\infty n^{4\gamma}e^{2nqT}|\langle u(T,\cdot),X_n\rangle|^2 = (\pi/2)e^{2qT}\cosh^2(T) \le (\pi/2)e^{2\mu T}\cosh^2(T).$$

Choosing $E = E(\mu) = \sqrt{\pi/2}e^{\mu T}\cosh(T)$, we can find that $u(y,x) = \sin(x)\cosh(y)$ satisfies (38). Note that, as setting $u(y,x) = \sin(lx)\cosh(ly)$, these functions also satisfy (38). For the feasibility of (43), we can explain it by adopting the similar way as above, and it is ignored here.

Remark 5. *We can apply this regularized method to solve the following classical problem*

$$\begin{cases}\Delta u(y,x) + k^2 u(y,x) = 0, & x \in (0,\pi),\ y \in (0,1), \\ u(0,x) = \varphi(x), & x \in [0,\pi], \\ u_y(0,x) = 0, & x \in [0,\pi], \\ u(y,0) = u(y,\pi) = 0, & y \in [0,1],\end{cases} \tag{48}$$

this is a Cauchy problem for Helmholtz equation in rectangular area. Using the Fourier method, we can express the exact solution as

$$u(y,x)=\sum_{n=1}^{[k]}\cos\left(\sqrt{k^2-n^2}y\right)\varphi_n X_n(x)+\sum_{n=[k]+1}^{\infty}\cosh\left(\sqrt{n^2-k^2}y\right)\varphi_n X_n(x), \tag{49}$$

denote $\langle\cdot,\cdot\rangle$ *as the inner product of* $L^2(0,\pi)$*, then in (49),* $\varphi_n=\langle\varphi(x),X_n\rangle$*, the eigenfunction of* $L^2(0,\pi)$ *is* $X_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx)$.

Since the main factor that leads to the ill-posedness of (48) is the second term of right hand side in (49), similar with the procedure of Section 3.1, we can construct the following regularization solution of (48)

$$u_\alpha^\delta(y,x)=\sum_{n=1}^{[k]}\cos\left(\sqrt{k^2-n^2}y\right)\varphi_n^\delta X_n+\sum_{n=[k]+1}^{\infty}\frac{\cosh(\sqrt{n^2-k^2}y)\varphi_n^\delta}{1+\alpha(n^2-k^2)^\gamma\cosh^q(\sqrt{n^2-k^2}T)}X_n, \tag{50}$$

here $\varphi_n^\delta=\langle\varphi^\delta,X_n\rangle$*,* α *is the regularization parameter. About the convergence estimate, the process is almost the same as one in this paper, we omit it.*

6. Numerical Experiments

In this section, by doing the corresponding numerical experiment we verify the simulation effect of the proposed method. For convenience, we only investigate the case of inhomogeneous Dirichlet data in two dimensions.

Example 1. *Let* $k>0$*, the following problem is investigated*

$$\begin{cases} u_{yy}+u_{xx}-k^2u=0, & 0<x<\pi,\ 0<y<1,\\ u(0,x)=\varphi(x), & 0\le x\le\pi,\\ u_y(0,x)=0, & 0\le x\le\pi,\\ u(y,0)=u(y,\pi)=0, & 0\le y\le 1,\end{cases} \tag{51}$$

here, we take $x\in\Omega=(0,\pi)$*,* $L_x=-\frac{\partial^2}{\partial x^2}+k^2:H^2(\Omega)\bigcap H_0^1(\Omega)\subset L^2(\Omega)\to L^2(\Omega)$*,* $\lambda_n=n^2+k^2$*,* $X_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx)$ *are the eigenvalues and eigenfunction of* L_x*, respectively. This is a classical ill-posed problem—the Cauchy problem for the modified Helmholtz equation. Generally, since the reconstruction of solution at* $y=1$ *is the most difficult, we mainly verify the computational effect of regularization method at this point.*

We know that, if Ω is a more complex domain, generally the analysis form of the exact solution for a considered problem cannot be found easily, which leads to certain difficulty in constructing the exact data $\varphi(x)$, so in order to overcome this problem and possess the representativeness, we consider the forward problem below

$$\begin{cases} u_{yy}+u_{xx}-k^2u=0, & 0<x<\pi, 0<y<1,\\ u(1,x)=f(x), & 0\le x\le\pi,\\ u_y(0,x)=0, & 0\le x\le\pi,\\ u(y,0)=u(y,\pi)=0, & 0\le y\le 1,\end{cases} \tag{52}$$

and use a finite difference method to solve problem (52), here $f(x) = (1 + k^2)x(x - \pi)$, and u_{xx}, u_{yy}, $u_y(0, x)$ are approximated by

$$u_{xx}(y, x) \approx \frac{u(y, x + \Delta x) - 2u(y, x) + u(y, x - \Delta x)}{(\Delta x)^2}, \tag{53}$$

$$u_{yy}(y, x) \approx \frac{u(y + \Delta y, x) - 2u(y, x) + u(y - \Delta y, x)}{(\Delta y)^2}, \tag{54}$$

$$\begin{aligned} u_y(0, x) &\approx 2u_y(\Delta y, x) - u_y(2\Delta y, x) \\ &\approx \frac{u(2\Delta y, x) - u(0, x)}{\Delta y} - \frac{u(3\Delta y, x) - u(\Delta y, x)}{2\Delta y}, \end{aligned} \tag{55}$$

where $\Delta x = \frac{\pi}{\iota}$, $\Delta y = \frac{1}{\kappa}$. Thus, the discrete form of (52) can be expressed as

$$\begin{cases} (\Delta y)^2 u_{p,i-1} - (k^2(\Delta x)^2(\Delta y)^2 + 2(\Delta y)^2 + 2(\Delta x)^2)u_{p,i} + (\Delta y)^2 u_{p,i+1} + (\Delta x)^2 u_{p-1,i} + (\Delta x)^2 u_{p+1,i} = 0, \\ \qquad p = 1, 2, \ldots, \kappa - 1; \quad i = 1, 2, \ldots, \iota - 1, \\ u_{\kappa,i} = f_i, \quad i = 1, 2, \ldots, \iota - 1, \\ -2u_{0,i} + u_{1,i} + 2u_{2,i} - u_{3,i} = 0, \quad i = 1, 2, \ldots, \iota - 1, \\ u_{p,0} = u_{p,\iota} = 0, \quad p = 0, 1, 2, \ldots, \kappa, \end{cases} \tag{56}$$

where $u_{p,i}$ are the approximate values at (y_p, x_i), $x_i = i\Delta x$, $i = 0, 1, 2, \ldots, \iota$, $y_p = p\Delta y$, $p = 0, 1, 2, \ldots, \kappa$.

Denote $u_{\texttt{dif}}$ as the difference solution of (52), then we choose the exact data as

$$\varphi(x) = u(x, 0) \approx u_{\texttt{dif}}(x, 0), \tag{57}$$

the observation data is randomly generated by

$$\varphi^\delta(x) = \varphi(x) + \varepsilon \mathrm{randn}(\mathrm{size}(\varphi(x))). \tag{58}$$

We calculate the bound of observation error δ by

$$\delta := \|\varphi^\delta - \varphi\|_{l_2}. \tag{59}$$

The regularized solution is calculated by (23) for $n = 1, 2, \ldots, M$, the error is computed by

$$\epsilon(u) = \frac{\|u - u_\alpha^\delta\|_{l_2}}{\|u\|_{l_2}}. \tag{60}$$

For $k = 0.5, 1.5$, $\gamma = 2, q = 4$, the exact and regularized solutions for various ε are shown in Figures 1 and 2. We also investigate the influence of q on numerical results by taking $\varepsilon = 0.01$, $\gamma = 3$ to compute the errors for the different q, Table 1 presents the computational results. Meanwhile, we choose $\varepsilon = 0.01$, $q = 5$ to investigate the influence of γ on calculation effect, Table 2 shows the corresponding results. In practice, we cannot acquire the a-priori bound E for exact solution easily, so in the process of computation, the regularized solution is only calculated by the rule of a-posteriori form (41) with $\tau = 1.1$.

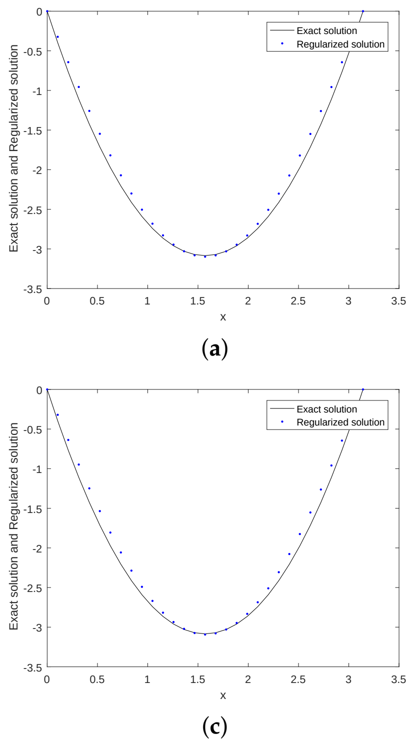

Figure 1. $k = 0.5$, $\gamma = 2$, $q = 4$; Exact and Regularized solutions for various ε. (**a**): $\varepsilon = 0.001$, (**b**): $\varepsilon = 0.01$, (**c**): $\varepsilon = 0.05$, (**d**): $\varepsilon = 0.1$.

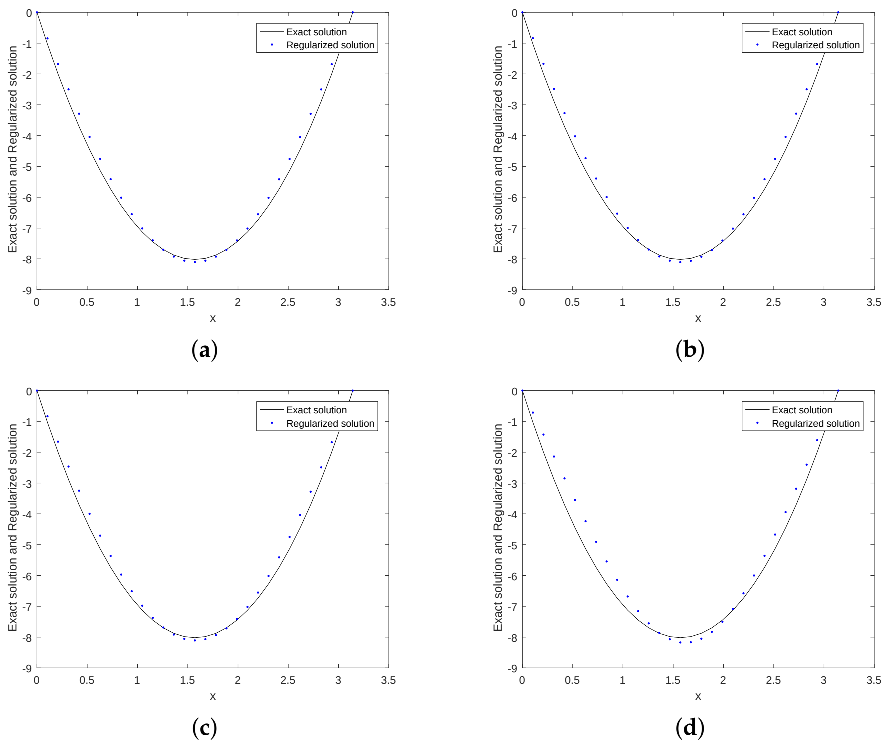

Figure 2. $k = 1.5$, $\gamma = 2$, $q = 4$; Exact and Regularized solutions for various ε. (**a**): $\varepsilon = 0.001$, (**b**): $\varepsilon = 0.01$, (**c**): $\varepsilon = 0.05$, (**d**): $\varepsilon = 0.1$.

Table 1. $k = 0.5, 1.5$, $\gamma = 3$, $\varepsilon = 0.01$, the errors for different q.

q	2	3	4	5	6	7	8
$\epsilon_{k=0.5}(u)$	0.0442	0.0441	0.0439	0.0438	0.0438	0.0438	0.0438
$\epsilon_{k=1.5}(u)$	0.0487	0.0481	0.0480	0.0480	0.0480	0.0480	0.0480

Table 2. $k = 0.5, 1.5$, $q = 5$, $\varepsilon = 0.01$, the errors for different γ.

γ	1	2	3	4	5	6	7
$\epsilon_{k=0.5}(u)$	0.0440	0.0439	0.0438	0.0438	0.0438	0.0438	0.0438
$\epsilon_{k=1.5}(u)$	0.0482	0.0480	0.0480	0.0480	0.0480	0.0480	0.0480

Figures 1 and 2, Tables 1 and 2 show that generalized-fractional Tikhonov-type is stable and feasible. Table 1 indicates that, for the same ε and fixed γ, as q becomes large, the numerical results gradually become steady, and the best value of it should be greater than or equal to 4. The results of Table 2 mean that the errors also become stable as γ increases for the same ε and fixed q, and the best value of γ should be greater than or equal to 2.

7. Conclusions and Discussion

We establish the conditional stabilities by placing the a-priori restrictions on exact solutions of (4), (5). Based on the generalized Tikhonov and fractional Tikhonov method, we construct a generalized-fractional Tikhonov-type regularized solution to recover the stability of the considered problem, and the convergence estimates of a-priori and an a-posteriori forms for this method are obtained. Ultimately, by doing the related numerical simulations, we verify that the proposed method is efficient and acceptable.

In fact, our regularized method can be seen as a variational method. Recently, we note that there are some new works in which the variational regularized methods are researched, such as [26–30], and so on. Meanwhile, this method is based on the eigenvalues and eigenfunctions of the operator involved. This means it is of limited applicability since often these are not explicitly known. For instance, many problems in science and engineering usually can be transformed into the form of operator equations (10) or (11). For each fixed y, (10) and (11) both can be expressed as the Fredholm integral operator with the first kind $(AU)(x) := \int_\Omega K(x,\xi)U(\xi)d\xi = \Phi(x)$. In order to overcome the ill-posedness of this integral equation, we often discretize it to obtain the operator equation of discrete form $\mathbf{AU} = \mathbf{\Phi}$, and then restore the numerical stability by imposing one regularization method (such as the Tikhonov method) on the operator equation of discrete form. Note that if the numerical approximations of eigenfunctions or eigenvalues of the operator L_x are known, we can conveniently compute the integral kernel $K(x,\xi)$, coefficient matrix $\mathbf{A}$, and construct the regularization solution. Then, if we can obtain the numerical approximations of eigenfunctions or eigenvalues of the operator L_x, this method can be extended to some broader application areas. In the future, we need make further consideration in this respect and study some practical problems in science and engineering.

We should point out that, in the procedure of the computation, we need to choose suitable parameters which include the regularization parameter α, the number of truncated term and positive numbers γ, q. We choose the number of truncated term, γ and q by using the a-priori method, but the a-posteriori method is not investigated. It is necessary to study the a-posteriori selection method for the number of truncated term, γ and q in the next works. Finally, we point out that this method also can be used to solve some other inverse problems of partial differential equations, such as the elliptic problem of quasi-linear case, inverse initial value problem of heat equation (also called final value problem of heat equation, or parabolic problem backward in time), and so on.

Author Contributions: Investigation, Writing—original draft preparation, H.Z.; Investigation, Writing—review and editing, X.Z. All authors have read and agreed to the published version of the manuscript.

Funding: The work was supported by the NSF of China (11761004), NSF of Ningxia (2019AAC03128) and SRP(2017SXKY05) at North Minzu University.

Acknowledgments: The authors would like to thank the reviewers for their constructive comments and valuable suggestions that improve the quality of our paper.

Conflicts of Interest: The authors declare that they have no conflict of interest.

Appendix A

Proof of Theorem 1. We know that, as $0 < y \le T, n \ge 1$, it holds that $e^{\sqrt{\lambda_n}y}/2 \le \cosh(\sqrt{\lambda_n}y) \le e^{\sqrt{\lambda_n}y}$, $\lambda_n \ge \lambda_1$, from Hölder inequality and (8), (12), it can be derived that

$$\begin{aligned}
\|u(y,x)\|^2_{L^2(\Omega)} &\le \sum_{n=1}^{\infty} \cosh^2(\sqrt{\lambda_n}y)\varphi_n^2 = \sum_{n=1}^{\infty} \cosh^2(\sqrt{\lambda_n}y)\varphi_n^{\frac{2y}{qT}}\varphi_n^{2-\frac{2y}{qT}} \\
&\le \left(\sum_{n=1}^{\infty} (\cosh(\sqrt{\lambda_n}y))^{\frac{2qT}{y}}\varphi_n^2\right)^{\frac{y}{qT}} \left(\sum_{n=1}^{\infty}\varphi_n^2\right)^{1-\frac{y}{qT}} \le \left(\sum_{n=1}^{\infty} (e^{\sqrt{\lambda_n}y})^{\frac{2qT}{y}}\varphi_n^2\right)^{\frac{y}{qT}} \left(\sum_{n=1}^{\infty}\varphi_n^2\right)^{1-\frac{y}{qT}} \\
&= \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{2\gamma}\cosh^2(\sqrt{\lambda_n}T)} \cdot \lambda_n^{2\gamma} e^{2qT\sqrt{\lambda_n}}\cosh^2(\sqrt{\lambda_n}T)\varphi_n^2\right)^{\frac{y}{qT}} \left(\sum_{n=1}^{\infty}\varphi_n^2\right)^{1-\frac{y}{qT}} \\
&\le \left(\frac{4}{\lambda_1^{2\gamma}e^{2\sqrt{\lambda_1}T}}\right)^{\frac{y}{qT}} \left(\sum_{n=1}^{\infty}\lambda_n^{2\gamma}e^{2qT\sqrt{\lambda_n}}|<u(T,x),X_n(x)>|^2\right)^{\frac{y}{qT}} \left(\sum_{n=1}^{\infty}\varphi_n^2\right)^{1-\frac{y}{qT}} \\
&\le \left(\frac{2}{\lambda_1^{\gamma}e^{\sqrt{\lambda_1}T}}\right)^{\frac{2y}{qT}} E^{\frac{2y}{qT}}\|\varphi\|^{2-\frac{2y}{qT}}_{L^2(\Omega)}.
\end{aligned}$$

The proof is completed. □

Proof of Theorem 2. For $0 < y \le T$, $n \ge 1$, we know that $\sinh(\sqrt{\lambda_n}y) \le e^{\sqrt{\lambda_n}y}$, and $\lambda_n \ge \lambda_1$, $\sinh(\sqrt{\lambda_n}y) \ge e^{\sqrt{\lambda_1}y}(1-e^{-2\sqrt{\lambda_1}y})/2$, then from (9), (14) and Hölder inequality, we have

$$\begin{aligned}
\|v(y,x)\|^2_{L^2(\Omega)} &\le \left(\sum_{n=1}^{\infty}\left(\frac{\sinh(\sqrt{\lambda_n}y)}{\sqrt{\lambda_n}}\right)^{\frac{2qT}{y}}\psi_n^2\right)^{\frac{y}{qT}} \left(\sum_{n=1}^{\infty}\psi_n^2\right)^{1-\frac{y}{qT}} \\
&\le \left(\sum_{n=1}^{\infty}\left(\frac{e^{\sqrt{\lambda_n}y}}{\sqrt{\lambda_n}}\right)^{\frac{2qT}{y}}\psi_n^2\right)^{\frac{y}{qT}} \left(\sum_{n=1}^{\infty}\psi_n^2\right)^{1-\frac{y}{qT}} = \left(\sum_{n=1}^{\infty} e^{2qT\sqrt{\lambda_n}}\psi_n^2\left(\frac{1}{\sqrt{\lambda_n}}\right)^{\frac{2qT}{y}}\right)^{\frac{y}{qT}} \left(\sum_{n=1}^{\infty}\psi_n^2\right)^{1-\frac{y}{qT}} \\
&= \left(\sum_{n=1}^{\infty}\frac{\left(\sqrt{\lambda_n}\right)^2}{\lambda_n^{2\gamma}\sinh^2(\sqrt{\lambda_n}T)} \cdot \lambda_n^{2\gamma}e^{2qT\sqrt{\lambda_n}}\frac{\sinh^2(\sqrt{\lambda_n}T)}{(\sqrt{\lambda_n})^2}\psi_n^2\left(\frac{1}{\sqrt{\lambda_n}}\right)^{\frac{2qT}{y}}\right)^{\frac{y}{qT}} \left(\sum_{n=1}^{\infty}\psi_n^2\right)^{1-\frac{y}{qT}} \\
&\le \left(\sum_{n=1}^{\infty}\frac{\left(\sqrt{\lambda_1}\right)^{2-\frac{2qT}{y}}}{\lambda_1^{2\gamma}} \cdot \frac{4}{e^{2T\sqrt{\lambda_1}}\left(1-e^{-2T\sqrt{\lambda_1}}\right)^2}\lambda_n^{2\gamma}e^{2qT\sqrt{\lambda_n}}|<v(T,x),X_n(x)>|^2\right)^{\frac{y}{qT}} \left(\sum_{n=1}^{\infty}\psi_n^2\right)^{1-\frac{y}{qT}} \\
&\le \left(\frac{2\left(\sqrt{\lambda_1}\right)^{1-\frac{qT}{y}}}{\lambda_1^{\gamma}e^{T\sqrt{\lambda_1}}\left(1-e^{-2T\sqrt{\lambda_1}}\right)}\right)^{\frac{2y}{qT}} \left(\sum_{n=1}^{\infty}\lambda_n^{2\gamma}e^{2qT\sqrt{\lambda_n}}|<v(T,x),X_n(x)>|^2\right)^{\frac{y}{qT}} \left(\sum_{n=1}^{\infty}\psi_n^2\right)^{1-\frac{y}{qT}} \\
&\le \left(\frac{2\left(\sqrt{\lambda_1}\right)^{1-\frac{qT}{y}}}{\lambda_1^{\gamma}e^{T\sqrt{\lambda_1}}\left(1-e^{-2T\sqrt{\lambda_1}}\right)}\right)^{\frac{2y}{qT}} E^{\frac{2y}{qT}}\|\psi\|^{2-\frac{2y}{qT}}_{L^2(\Omega)}.
\end{aligned}$$

This finishes the proof of the result in (15). □

Proof of Theorem 3. We select $\nu = \frac{\alpha\lambda_n^\gamma}{2^q}$, $r = (qT-y)\sqrt{\lambda_n}$, $s = qT\sqrt{\lambda_n}$, use Lemma 1, and $H(\eta) \le 1$, then

$$G_1(\lambda_n) = \frac{e^{-(qT-y)\sqrt{\lambda_n}}}{\frac{\alpha}{2^q}\lambda_n^\gamma + e^{-qT\sqrt{\lambda_n}}} = \frac{1}{\frac{\alpha}{2^q}\lambda_n^\gamma}\frac{\frac{\alpha}{2^q}\lambda_n^\gamma \cdot e^{-(qT-y)\sqrt{\lambda_n}}}{\frac{\alpha}{2^q}\lambda_n^\gamma + e^{-qT\sqrt{\lambda_n}}}$$
$$\le \left(\frac{\alpha\lambda_n^\gamma}{2^q}\right)^{-1} \cdot H\left(\frac{qT-y}{qT}\right)\left(\frac{\alpha\lambda_n^\gamma}{2^q}\right)^{\frac{qT-y}{qT}} = \left(1-\frac{y}{qT}\right)^{1-\frac{y}{qT}}\left(\frac{y}{qT}\right)^{\frac{y}{qT}}\left(\frac{\alpha\lambda_n^\gamma}{2^q}\right)^{-\frac{y}{qT}}$$
$$= 2^{\frac{y}{T}}(\lambda_n^\gamma)^{-\frac{y}{qT}}\left(1-\frac{y}{qT}\right)^{1-\frac{y}{qT}}\left(\frac{y}{qT}\right)^{\frac{y}{qT}}\alpha^{-\frac{y}{qT}} \le 2(\lambda_n^\gamma)^{-\frac{y}{qT}}\alpha^{-\frac{y}{qT}}.$$

Note that, $(\lambda_n^\gamma)^{-\frac{y}{qT}} \le (\lambda_1^\gamma)^{-\frac{y}{qT}}$. If $\lambda_1 \ge 1$, then $(\lambda_1^\gamma)^{-\frac{y}{qT}} \le 1$; as $0 < \lambda_1 < 1$, from $y < qT$, we get that $(\lambda_1^\gamma)^{-\frac{y}{qT}} \le (\lambda_1^\gamma)^{-1}$. Setting $C_1 = \max\{1, (\lambda_1^\gamma)^{-1}\}$, one can derive that, $(\lambda_n^\gamma)^{-\frac{y}{qT}} \le (\lambda_1^\gamma)^{-\frac{y}{qT}} \le C_1$, and thus $G_1(\lambda_n) \le 2C_1\alpha^{-\frac{y}{qT}}$. □

Proof of Theorem 4. Selecting $\nu = \beta\lambda_n^{\gamma-1}\left(\frac{1-e^{-2\sqrt{\lambda_1}T}}{2}\right)^q$, $r = (qT-y)\sqrt{\lambda_n}$, $s = qT\sqrt{\lambda_n}$ in Lemma 1, and combining with $H(\eta) \le 1$, one can obtain the inequality (37). □

Proof of Theorem 5. In (23), we take the exact data φ, and denote the corresponding solution as u_α, then

$$\|u_\alpha^\delta - u\| \le \|u_\alpha^\delta - u_\alpha\| + \|u_\alpha - u\|, \tag{A1}$$

For $0 < y \le T$, $e^{\sqrt{\lambda_n}y}/2 \le \cosh(\sqrt{\lambda_n}y) \le e^{\sqrt{\lambda_n}y}$, from (23), (36), (19), we note that

$$\|u_\alpha^\delta(y,\cdot) - u_\alpha(y,\cdot)\|^2 \le \sum_{n=1}^\infty \left(\frac{\cosh(\sqrt{\lambda_n}y)}{1+\alpha\lambda_n^\gamma\cosh^q(\sqrt{\lambda_n}T)}\right)^2\left(\varphi_n^\delta - \varphi_n\right)^2 \tag{A2}$$
$$\le \sum_{n=1}^\infty\left(\frac{e^{-(qT-y)\sqrt{\lambda_n}}}{\frac{\alpha\lambda_n^\gamma}{2^q} + e^{-qT\sqrt{\lambda_n}}}\right)^2\left(\varphi_n^\delta - \varphi_n\right)^2 \le 4C_1^2\delta^2\alpha^{-\frac{2y}{qT}}.$$

On the other hand, by (8), (23), (36), (38), we can derive that

$$\|u_\alpha(y,\cdot) - u(y,\cdot)\|^2 = \left\|\sum_{n=1}^\infty \frac{\alpha\lambda_n^\gamma\cosh^q(\sqrt{\lambda_n}T)}{1+\alpha\lambda_n^\gamma\cosh^q(\sqrt{\lambda_n}T)}\cosh(\sqrt{\lambda_n}y)\varphi_n X_n\right\|^2$$
$$\le \sum_{n=1}^\infty\left(\frac{\alpha\lambda_n^\gamma\cosh^q(\sqrt{\lambda_n}T)}{1+\alpha\lambda_n^\gamma\cosh^q(\sqrt{\lambda_n}T)}\right)^2\left(\cosh(\sqrt{\lambda_n}T)\varphi_n\right)^2$$
$$\le \alpha^2\sum_{n=1}^\infty\left(\frac{e^{-(qT-y)\sqrt{\lambda_n}}}{\frac{\alpha\lambda_n^\gamma}{2^q} + e^{-qT\sqrt{\lambda_n}}}\right)^2\lambda_n^{2\gamma}e^{2\sqrt{\lambda_n}(qT-y)}|\langle u(T,\cdot), X_n\rangle|^2 \tag{A3}$$
$$\le \alpha^2\sum_{n=1}^\infty\left(\frac{e^{-(qT-y)\sqrt{\lambda_n}}}{\frac{\alpha\lambda_n^\gamma}{2^q} + e^{-qT\sqrt{\lambda_n}}}\right)^2\lambda_n^{2\gamma}e^{2qT\sqrt{\lambda_n}}|\langle u(T,\cdot), X_n\rangle|^2$$
$$\le 4C_1^2\alpha^{2-\frac{2y}{qT}}E^2.$$

From (A1), (A2), (A3), (39), and (19), one can obtain the estimate in (40). □

Proof of Lemma 2. By setting

$$\rho(\alpha) = \left(\sum_{n=1}^{\infty}\left(\frac{\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}{1+\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}\right)^2\left(\varphi_n^{\delta}\right)^2\right)^{1/2}, \tag{A4}$$

one can prove it easily, here we omit it. □

Proof of Lemma 3. According to (41), we can obtain that

$$\begin{aligned}\tau\delta &= \left\|\sum_{n=1}^{\infty}\frac{\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}{1+\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}\varphi_n^{\delta}X_n(x)\right\| \\ &\le \left\|\sum_{n=1}^{\infty}\frac{\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}{1+\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}(\varphi_n^{\delta}-\varphi_n)X_n(x)\right\| + \left\|\sum_{n=1}^{\infty}\frac{\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}{1+\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}\varphi_nX_n(x)\right\| \\ &\le \delta + \left\|\sum_{n=1}^{\infty}\frac{\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}{1+\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}\varphi_nX_n(x)\right\|,\end{aligned} \tag{A5}$$

and

$$\begin{aligned}&\left\|\sum_{n=1}^{\infty}\frac{\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}{1+\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}\varphi_nX_n(x)\right\| \le \left(\sum_{n=1}^{\infty}\left(\frac{\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}{1+\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}\right)^2\varphi_n^2\right)^{1/2} \\ &\le \left(\sum_{n=1}^{\infty}\alpha^2\lambda_n^{2\gamma}\cosh^{2q}(\sqrt{\lambda_n}T)\varphi_n^2\right)^{1/2} \le \left(\sum_{n=1}^{\infty}\frac{\alpha^2}{\cosh^2(\sqrt{\lambda_n}T)}\cdot\lambda_n^{2\gamma}e^{2qT\sqrt{\lambda_n}}\cosh^2(\sqrt{\lambda_n}T)\varphi_n^2\right)^{1/2} \\ &\le \left(\sum_{n=1}^{\infty}\frac{4\alpha^2}{e^{2\sqrt{\lambda_n}T}}\cdot\lambda_n^{2\gamma}e^{2qT\sqrt{\lambda_n}}|\langle u(T,\cdot),X_n\rangle|^2\right)^{1/2} \le (2/e^{\sqrt{\lambda_1}T})\alpha E,\end{aligned} \tag{A6}$$

by (A5), (A6), $(\tau-1)\delta \le (2/e^{\sqrt{\lambda_1}T})\alpha E$. We finish its proof. □

Proof of Theorem 6. Similar to (A1), by the trigonometric inequality, then

$$\|u_\alpha^{\delta}(y,\cdot)-u(y,\cdot)\| \le \|u_\alpha^{\delta}(y,\cdot)-u_\alpha(y,\cdot)\| + \|u_\alpha(y,\cdot)-u(y,\cdot)\|. \tag{A7}$$

By (A2) and the inequality given in Lemma 3, we can derived that

$$\|u_\alpha^{\delta}(y,\cdot)-u_\alpha(y,\cdot)\| \le 2C_1\delta\alpha^{-\frac{y}{qT}} \le 2C_1\left((\tau-1)e^{\sqrt{\lambda_1}T}/2\right)^{-\frac{y}{qT}}E^{\frac{y}{qT}}\delta^{1-\frac{y}{qT}}. \tag{A8}$$

Now we estimate $\|u_\alpha(y,\cdot)-u(y,\cdot)\|$ of (A7). It is noticed that

$$\begin{aligned}A_1(y)\,(u_\alpha(y,\cdot)-u(y,\cdot)) &= \sum_{n=1}^{\infty}\frac{-\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}{1+\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}\varphi_nX_n(x) \\ &= \sum_{n=1}^{\infty}\frac{\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}{1+\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}(\varphi_n^{\delta}-\varphi_n)X_n(x) + \sum_{n=1}^{\infty}\frac{-\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}{1+\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}\varphi_n^{\delta}X_n(x),\end{aligned} \tag{A9}$$

using (19), (41), (A9), we can obtain

$$\|A_1(y)\,(u_\alpha(y,\cdot)-u(y,\cdot))\| \le \delta+\tau\delta = (\tau+1)\delta. \tag{A10}$$

Furthermore, in the light of (7) and (38), we get that

$$\|u_\alpha(y,\cdot)-u(y,\cdot)\|_{\mathcal{D}^{u_\alpha-u}_{\gamma,q}} = \left(\sum_{n=1}^{\infty}\lambda_n^{2\gamma}e^{2qT\sqrt{\lambda_n}}\left(\frac{\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}{1+\alpha\lambda_n^{\gamma}\cosh^q(\sqrt{\lambda_n}T)}\right)^2\cosh^2(\sqrt{\lambda_n}y)\varphi_n^2\right)^{\frac{1}{2}}$$
$$\le\left(\sum_{n=1}^{\infty}\lambda_n^{2\gamma}e^{2qT\sqrt{\lambda_n}}\cosh^2(\sqrt{\lambda_n}T)\varphi_n^2\right)^{\frac{1}{2}}\le E, \tag{A11}$$

subsequently, using the result of conditional stability in (13), we have

$$\|u_\alpha(y,\cdot)-u(y,\cdot)\|\le 2^{\frac{y}{qT}}\left(\lambda_1^{\gamma}e^{\sqrt{\lambda_1}T}\right)^{-\frac{y}{qT}}(\tau+1)^{1-\frac{y}{qT}}E^{\frac{y}{qT}}\delta^{1-\frac{y}{qT}}. \tag{A12}$$

Ultimately, from (A8), (A12), the estimate of convergence (42) can be proven. □

Proof of Theorem 7. In (31), Take the exact data ψ denote the corresponding solution as v_β, then

$$\|v_\beta^\delta-v\|\le\|v_\beta^\delta-v_\beta\|+\|v_\beta-v\|. \tag{A13}$$

For $0<y\le T$, $n\ge 1$, we know that $\sinh(\sqrt{\lambda_n}y)\le e^{\sqrt{\lambda_n}y}$, and $\lambda_n\ge\lambda_1$, $\sinh(\sqrt{\lambda_n}y)\ge e^{\sqrt{\lambda_n}y}(1-e^{-2\sqrt{\lambda_1}y})/2$, from (31), (37), (27), we note that

$$\|v_\beta^\delta(y,\cdot)-v_\beta(y,\cdot)\|^2\le\sum_{n=1}^{\infty}\left(\frac{\sinh(\sqrt{\lambda_n}y)}{\sqrt{\lambda_n}\left(1+\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)\right)}\right)^2\left(\psi_n^\delta-\psi_n\right)^2 \tag{A14}$$
$$\le\sum_{n=1}^{\infty}\left(\frac{e^{-(qT-y)\sqrt{\lambda_n}}}{\sqrt{\lambda_1}\left(\beta\lambda_n^{\gamma-1}\left(\frac{1-e^{-2\sqrt{\lambda_1}T}}{2}\right)^q+e^{-qT\sqrt{\lambda_n}}\right)}\right)^2\left(\psi_n^\delta-\psi_n\right)^2\le 4D_1^2\delta^2\beta^{-\frac{2y}{qT}}.$$

On the other hand, by (9), (31), (37), (43), one gets that

$$\|v_\beta(y,\cdot)-v(y,\cdot)\|^2=\left\|\sum_{n=1}^{\infty}\frac{\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}{\sqrt{\lambda_n}\left(1+\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)\right)}\sinh(\sqrt{\lambda_n}y)\psi_nX_n\right\|^2$$
$$\le\sum_{n=1}^{\infty}\left(\frac{\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}{1+\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}\right)^2\left(\frac{\sinh(\sqrt{\lambda_n}T)}{\sqrt{\lambda_n}}\psi_n\right)^2$$
$$\le\beta^2\sum_{n=1}^{\infty}\left(\frac{e^{\sqrt{\lambda_n}y}}{\lambda_ne^{\sqrt{\lambda_n}y}\left(1+\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)\right)}\right)^2\lambda_n^{2\gamma}e^{2qT\sqrt{\lambda_n}}\left(\frac{\sinh(\sqrt{\lambda_n}T)}{\sqrt{\lambda_n}}\psi_n\right)^2 \tag{A15}$$
$$\le\frac{\beta^2}{\lambda_1e^{2\sqrt{\lambda_1}y}}\sum_{n=1}^{\infty}\left(\frac{e^{-(qT-y)\sqrt{\lambda_n}}}{\sqrt{\lambda_1}\left(\beta\lambda_n^{\gamma-1}\left(\frac{1-e^{-2\sqrt{\lambda_1}T}}{2}\right)^q+e^{-qT\sqrt{\lambda_n}}\right)}\right)^2\lambda_n^{2\gamma}e^{2qT\sqrt{\lambda_n}}|\langle v(T,\cdot),X_n\rangle|^2$$
$$\le\frac{4}{\lambda_1e^{2\sqrt{\lambda_1}y}}D_1^2\beta^{2-\frac{2y}{qT}}E^2.$$

From (44), (A13), (A14), (A15), and (27), we can derive the estimate (45). □

Proof of Lemma 4. One can prove this Lemma easily by writing

$$\varrho(\beta)=\left(\sum_{n=1}^{\infty}\left(\frac{\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}{1+\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}\right)^2\left(\psi_n^\delta\right)^2\right)^{1/2}, \tag{A16}$$

here we also skip it. □

Proof of Lemma 5. According to (46), it can be known that

$$\begin{aligned}\tau\delta&=\left\|\sum_{n=1}^{\infty}\frac{\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}{1+\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}\psi_n^\delta X_n(x)\right\|\\&\le\left\|\sum_{n=1}^{\infty}\frac{\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}{1+\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}(\psi_n^\delta-\psi_n)X_n(x)\right\|+\left\|\sum_{n=1}^{\infty}\frac{\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}{1+\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}\psi_n X_n(x)\right\|\\&\le\delta+\left\|\sum_{n=1}^{\infty}\frac{\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}{1+\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}\psi_n X_n(x)\right\|,\end{aligned}\tag{A17}$$

and

$$\begin{aligned}&\left\|\sum_{n=1}^{\infty}\frac{\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}{1+\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}\psi_n X_n(x)\right\|\le\left(\sum_{n=1}^{\infty}\left(\frac{\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}{1+\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}\right)^2\psi_n^2\right)^{1/2}\\&\le\left(\sum_{n=1}^{\infty}\beta^2\lambda_n^{2\gamma-2}\sinh^{2q}(\sqrt{\lambda_n}T)\psi_n^2\right)^{1/2}\le\left(\sum_{n=1}^{\infty}\frac{\beta^2}{\lambda_n\sinh^2(\sqrt{\lambda_n}T)}\cdot\lambda_n^{2\gamma}e^{2qT\sqrt{\lambda_n}}\frac{\sinh^2(\sqrt{\lambda_n}T)}{(\sqrt{\lambda_n})^2}\psi_n^2\right)^{1/2}\\&\le\left(\sum_{n=1}^{\infty}\frac{\beta^2}{\lambda_1\sinh^2(\sqrt{\lambda_1}T)}\cdot\lambda_n^{2\gamma}e^{2qT\sqrt{\lambda_n}}|\langle v(T,\cdot),X_n\rangle|^2\right)^{1/2}\le(1/(\sqrt{\lambda_1}\sinh(\sqrt{\lambda_1}T)))\beta E,\end{aligned}\tag{A18}$$

by (A17), (A18), $(\tau-1)\delta\le(1/(\sqrt{\lambda_1}\sinh(\sqrt{\lambda_1}T)))\beta E$. We finish the proof. □

Proof of Theorem 8. From (A13), we have

$$\|v_\beta^\delta(y,\cdot)-v(y,\cdot)\|\le\|v_\beta^\delta(y,\cdot)-v_\beta(y,\cdot)\|+\|v_\beta(y,\cdot)-v(y,\cdot)\|. \tag{A19}$$

Using (A14) and the result in Lemma 5, it can be obtained that

$$\|v_\beta^\delta(y,\cdot)-v_\beta(y,\cdot)\|\le 2D_1\delta\beta^{-\frac{y}{qT}}\le 2D_1\left(\sqrt{\lambda_1}\sinh(\sqrt{\lambda_1}T)(\tau-1)\right)^{-\frac{y}{qT}}E^{\frac{y}{qT}}\delta^{1-\frac{y}{qT}}. \tag{A20}$$

In the following, let us estimate $\|v_\beta(y,\cdot)-v(y,\cdot)\|$ of (A19). It is noted that

$$\begin{aligned}A_2(y)\left(v_\beta(y,\cdot)-v(y,\cdot)\right)&=\sum_{n=1}^{\infty}\frac{-\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}{1+\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}\psi_n X_n(x)\\&=\sum_{n=1}^{\infty}\frac{\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}{1+\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}(\psi_n^\delta-\psi_n)X_n(x)+\sum_{n=1}^{\infty}\frac{-\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}{1+\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}\psi_n^\delta X_n(x),\end{aligned}\tag{A21}$$

using (27), (46), (A21), we can obtain

$$\|A_2(y)\left(v_\beta(y,\cdot)-v(y,\cdot)\right)\|\le\delta+\tau\delta=(\tau+1)\delta. \tag{A22}$$

Furthermore, in the light of (7) and (43), one derives that

$$\begin{aligned} \|v_\beta(y,\cdot)-v(y,\cdot)\|_{\mathcal{D}^{v_\beta - v}_{\gamma,q}} &= \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma} e^{2qT\sqrt{\lambda_n}} \left(\frac{\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)}{1+\beta\lambda_n^{\gamma-1}\sinh^q(\sqrt{\lambda_n}T)} \right)^2 \left(\frac{\sinh(\sqrt{\lambda_n}y)}{\sqrt{\lambda_n}} \right)^2 \psi_n^2 \right)^{\frac{1}{2}} \\ &\le \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma} e^{2qT\sqrt{\lambda_n}} \left(\frac{\sinh(\sqrt{\lambda_n}T)}{\sqrt{\lambda_n}} \right)^2 \psi_n^2 \right)^{\frac{1}{2}} \le E, \end{aligned} \tag{A23}$$

subsequently, using the result of conditional stability in (15), there holds that

$$\|v_\beta(y,\cdot)-v(y,\cdot)\| \le 2^{\frac{y}{qT}} \lambda_1^{\left(\frac{1}{2}-\gamma\right)-\frac{qT}{2y}} \left(e^{\sqrt{\lambda_1}T} \left(1-e^{-2\sqrt{\lambda_1}T} \right) \right)^{-\frac{y}{qT}} (\tau+1)^{1-\frac{y}{qT}} E^{\frac{y}{qT}} \delta^{1-\frac{y}{qT}}. \tag{A24}$$

Ultimately, we can prove the result of convergence (47) by combing (A20) with (A24). □

References

1. Kaup, P.G.; Santosa, F. Nondestructive evaluation of corrosion damage using electrostatic measurements. *J. Nondestruct. Eval.* **1995**, *14*, 127–136. [CrossRef]
2. McIver, M. An inverse problem in electromagnetic crack detection. *IMA J. Appl. Math.* **1991**, *47*, 127–145. [CrossRef]
3. Stefanesco, S.; Schlumberger, C.; Schlumberger, M. Sur la distribution *é*lectrique potentielle autour d'uneprise de terre ponctuelle dans un terrain *à* couches horizontales, homog*è*nes et isotropes. *J. Phys. Radium* **1930**, *1*, 132–140. [CrossRef]
4. Colli Franzone, P.; Guerri, L.; Tentoni, S.; Viganotti, C.; Baruffi, S.; Spaggiari, S.; Taccardi, B. A mathematical procedure for solving the inverse potential problem of electrocardiography. Analysis of the time-space accuracy from in vitro experimental data. *Math. Biosci.* **1985**, *77*, 353–396. [CrossRef]
5. Alessandrini, G.; Rondi, L.; Rosset, E.; Vessella, S. The stability for the Cauchy problem for elliptic equations. *Inverse Probl.* **2009**, *25*, 123004. [CrossRef]
6. Belgacem, F.B. Why is the Cauchy problem severely ill-posed? *Inverse Probl.* **2007**, *23*, 823. [CrossRef]
7. Cheng, H.; Feng, X.L.; Fu, C.L. A mollification regularization method for the Cauchy problem of an elliptic equation in a multi-dimensional case. *Inverse Probl. Sci. Eng.* **2010**, *18*, 971–982. [CrossRef]
8. Bouzitouna, A.; Boussetila, N.; Rebbani, F. Two regularization methods for a class of inverse boundary value problems of elliptic type. *Bound. Value Probl.* **2013**, *2013*, 178. [CrossRef]
9. Eld*é*n, L.; Berntsson, F. A stability estimate for a Cauchy problem for an elliptic partial differential equation. *Inverse Probl.* **2009**, *21*, 1643–1653.
10. Eld*é*n, L.; Simoncini, V. A numerical solution of a Cauchy problem for an elliptic equation by Krylov subspaces. *Inverse Probl.* **2009**, *25*, 065002.
11. Feng, X.L.; Eld*é*n, L.; Fu, C.L. A quasi-boundary-value method for the Cauchy problem for elliptic equations with nonhomogeneous neumann data. *J. Inverse -Ill-Posed Probl.* **2010**, *18*, 617–645. [CrossRef]
12. Feng, X.L.; Ning, W.T.; Qian, Z. A quasi-boundary-value method for a Cauchy problem of an elliptic equation in multiple dimensions. *Inverse Probl. Sci. Eng.* **2014**, *22*, 1045–1061. [CrossRef]
13. H*à*o, D.N.; Duc, N.V.; Lesnic, D. A non-local boundary value problem method for the Cauchy problem for elliptic equations. *Inverse Probl.* **2009**, *25*, 055002.
14. Tautenhahn, U. Optimal stable solution of Cauchy problems of elliptic equations. *J. Anal. Its Appl.* **1996**, *15*, 961–984. [CrossRef]
15. Tautenhahn, U.; H*ä*marik, U.; Hofmann, B.; Shao, Y. Conditional stability estimates for ill-posed pdeproblems by using interpolation. *Numer. Funct. Anal. Optim.* **2013**, *34*, 1370–1417. [CrossRef]
16. Tuan, N.H.; Trong, D.D.; Quan, P.H. A note on a Cauchy problem for the Laplace equation: Regularization and error estimates. *Appl. Math. Comput.* **2010**, *217*, 2913–2922. [CrossRef]
17. Zhao, J.J.; Liu, S.S.; Liu, T. A new regularization method for cauchy problem of elliptic equation. *Complex Var. Elliptic Equations* **2014**, *59*, 1302–1314. [CrossRef]

18. Hochstenbach, M.E.; Reichel, L. Fractional Tikhonov regularization for linear discrete ill-posed problems. *BIT Numer. Math.* **2011**, *51*, 197–215. [CrossRef]
19. Cheng, J.; Yamamoto, M. One new strategy for a priori choice of regularizing parameters in Tikhonov's regularization. *Inverse Probl.* **2000**, *16*, L31–L38. [CrossRef]
20. Payne, L.E. Bounds in the Cauchy problem for the Laplace equation. *Arch. Ration. Mech. Anal.* **1960**, *5*, 35–45. [CrossRef]
21. Engl, H.W.; Hanke, M.; Neubauer, A. *Regularization of Inverse Problems, Volume 375 of Mathematics and Its Applications*; Kluwer Academic Publishers Group: Dordrecht, The Netherlands, 1996.
22. Kirsch, A. *An Introduction to the Mathematical Theory of Inverse Problems, Volume 120 of Applied Mathematical Sciences*; Springer: New York, NY, USA, 1996.
23. Feng, X.L.; Fu, C.L.; Cheng, H. A regularization method for solving the Cauchy problem for the Helmholtz equation. *Appl. Math. Model.* **2011**, *35*, 3301–3315. [CrossRef]
24. Qin, H.H.; Wei, T. Two regularization methods for the Cauchy problems of the Helmholtz equation. *Appl. Math. Model.* **2010**, *34*, 947–967. [CrossRef]
25. Morozov, V.A.; Nashed, Z.; Aries, A.B. *Methods for Solving Incorrectly Posed Problems*; Springer: New York, NY, USA, 1984.
26. Egger, H.; Hofmann, B. Tikhonov regularization in Hilbert scales under conditional stability assumptions. *Inverse Probl.* **2018**, *34*, 115015. [CrossRef]
27. Hofmann, B.; Mathe, P. Tikhonov regularization with oversmoothing penalty for non-linear ill-posed problems in Hilbert scales. *Inverse Probl.* **2017**, *34*, 015007. [CrossRef]
28. Hofmann, B.; Kaltenbacher, B.; Resmerita, E. Lavrentiev's regularization method in Hilbert spaces revisited. *Inverse Probl. Imaging* **2017**, *10*, 741–764. [CrossRef]
29. Plato, R.; Mathé, P.; Hofmann, B. Optimal rates for lavrentiev regularization with adjoint source conditions. *Math. Comput.* **2017**, *87*, 785–801. [CrossRef]
30. Chen, D.H.; Hofmann, B.; Zou, J. Regularization and convergence for ill-posed backward evolution equations in Banach spaces. *J. Differ. Equations* **2018**, *265*, 3533–3566. [CrossRef]