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Coefficient Estimates for a Subclass of Starlike Functions

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Abstract: In this note, we consider a subclass $\mathcal{H}_{3/2}(p)$ of starlike functions f with $f''(0) = p$ for a prescribed $p \in [0,2]$. Usually, in the study of univalent functions, estimates on the Taylor coefficients, Fekete–Szegö functional or Hankel determinats are given. Another coefficient problem which has attracted considerable attention is to estimate the moduli of successive coefficients $|a_{n+1}| - |a_n|$. Recently, the related functional $|a_{n+1} - a_n|$ for the initial successive coefficients has been investigated for several classes of univalent functions. We continue this study and for functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{H}_{3/2}(p)$, we investigate upper bounds of initial coefficients and the difference of moduli of successive coefficients $|a_3 - a_2|$ and $|a_4 - a_3|$. Estimates of the functionals $|a_2 a_4 - a_3^2|$ and $|a_4 - a_2 a_3|$ are also derived. The obtained results expand the scope of the theoretical results related with the functional $|a_{n+1} - a_n|$ for various subclasses of univalent functions.

Keywords: univalent functions; starlike functions; coefficient estimates

MSC: 30C45; 30C50

1. Introduction

As usual, denote by $\mathcal{A}$ the family of all normalized analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ be the subset of univalent functions in $\mathcal{A}$. Let

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > \alpha,\ z \in \mathbb{U}, 0 \le \alpha < 1 \right\} \tag{2}$$

be the class of starlike functions of order α (see [1]). The family $\mathcal{S}^*(0) = \mathcal{S}^*$ is the well-known class of starlike functions in $\mathbb{U}$. Denote by $\mathcal{K}$ the class of convex functions in $\mathbb{U}$, i.e.,

$$\mathcal{K} = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{U} \right\}. \tag{3}$$

In 1997, Silverman [2] investigated the properties of a subclass of $\mathcal{A}$, defined in terms of the quotient $(1 + \frac{zf''(z)}{f'(z)}) / \frac{zf'(z)}{f(z)}$. More precisely, for $0 < b \le 1$, Silverman's class $\mathcal{G}_b$ is defined as follows

$$\mathcal{G}_b = \left\{ f \in \mathcal{A} : \left| \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1 \right| < b,\ z \in \mathbb{U} \right\}. \tag{4}$$

In [2], Silverman proved that all functions in $\mathcal{G}_b$ are starlike of order $2/(1+\sqrt{1+8b})$. Lately, Obradović and Tuneski [3] improved the results of Silverman and obtained new starlike criteria for the class $\mathcal{G}_b$. Among others, they obtained the next result.

Theorem 1 ([3]). *Let $f \in \mathcal{A}$. If*

$$\Re\left(\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right) < \frac{3}{2},\ z \in \mathbb{U},$$

then $f \in \mathcal{S}^*$.

Starting from the above result, we consider the following subclass of $\mathcal{S}^*$:

$$\mathcal{H}_{3/2} = \left\{ f \in \mathcal{A} : \Re\left(\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right) < \frac{3}{2},\ z \in \mathbb{U} \right\}. \tag{5}$$

It is not difficult to show that for $|a| < 1/2$, the function $f(z) = \frac{z}{1+az} \in \mathcal{H}_{3/2}$.

During the years, great attention has been given to the difference of moduli of successive coefficients $||a_{n+1}| - |a_n||$ of a function in $\mathcal{S}^*$. In 1963, Hayman [4] proved that $||a_{n+1}| - |a_n|| \le A$ $(A \ge 1)$ for $f \in \mathcal{S}^*$. Further, Leung [5] proved Pommerenke's [6] conjecture $||a_{n+1}| - |a_n|| \le 1$ for $f \in \mathcal{S}^*$. Estimates of the difference of moduli of successive coefficients, for certain subclasses of $\mathcal{S}^*$, were also obtained by Z. Ye [7,8], and others (see, for example [9]). Moreover, since $||a_{n+1}| - |a_n|| < |a_{n+1} - a_n|$, the study of the functional $|a_{n+1} - a_n|$ has been also considered. For all functions $f \in \mathcal{K}$, Robertson [10] obtained the inequality $|a_{n+1} - a_n| \le \frac{2n+1}{3}|a_2 - 1|$ and proved that the factor $(2n+1)/3$ cannot be replaced by any smaller number independent of f. Recently, Li and Sugawa [11] investigated the problem of maximizing the functionals $|a_3 - a_2|$ and $|a_4 - a_3|$ for a refined subclass of $\mathcal{K}$, $\mathcal{K}(p) = \{f \in \mathcal{K} : f''(0) = p,\ p \in [0,2]\}$. The upper bounds of the same funtionals $|a_3 - a_2|$ and $|a_4 - a_3|$ for various subclasses of univalent functions were obtained by Peng and Obradović [12] and L. Shi et al. [13].

Motivated by the results given in [11–13], in the present paper we obtain upper bounds of the initial coefficients and upper bounds of $|a_3 - a_2|$ and $|a_4 - a_3|$ for a refined subclass of $\mathcal{H}_{3/2}$, defined by

$$\mathcal{H}_{3/2}(p) = \left\{ f \in \mathcal{H}_{3/2} : f''(0) = p \right\}, \tag{6}$$

where p is a given number satisfying $-2 \le p \le 2$.

Moreover, upper bounds for functionals $|a_2a_4 - a_3^2|$ and $|a_4 - a_2a_3|$ for the same subclass $\mathcal{H}_{3/2}(p)$ are also derived. The first functional is known as the second Hankel determinant, studied in many papers (see [14–17]). The second functional is a particular case of the generalized Zalcman functional, investigated by Ma [18], Efraimidis and Vukotić [19] and many others (see [20–23]).

2. Preliminary Results

Let $\mathcal{P}$ be the class of analytic functions p with a positive real part in $\mathbb{U}$, satisfying the condition $p(0) = 1$. A member $p \in \mathcal{P}$ is called a Carathéodory function and has the Taylor series expansion

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{7}$$

It is known that $|p_n| \le 2$ for $p \in \mathcal{P}$ and $n = 1, 2, \ldots$ (see [1]).

In order to prove our main results, the following two lemmas will be used. The first is due to Libera and Złotkiewicz [24,25].

Lemma 1. *Let $-2 \le p_1 \le 2$ and $p_2, p_3 \in \mathbb{C}$. Then there exists a function $p \in \mathcal{P}$ of the form (7) such that*

$$2p_2 = p_1^2 + x(4 - p_1^2) \tag{8}$$

and

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - (4 - p_1^2)p_1 x^2 + 2(4 - p_1^2)(1 - |x|^2)y \tag{9}$$

for some $x, y \in \mathbb{C}$ with $|x| \le 1$ and $|y| \le 1$.

The second lemma is a special case of a more general result due to Ohno and Sugawa [26] (see also [11]).

Lemma 2. *For some given real numbers a, b, c, let*

$$Y(a,b,c) = \max_{z \in \overline{\mathbb{U}}} \left(|a + bz + cz^2| + 1 - |z|^2 \right). \tag{10}$$

If $ac \ge 0$, then

$$Y(a,b,c) = \begin{cases} |a| + |b| + |c|\,, & |b| \ge 2(1 - |c|) \\ 1 + |a| + \dfrac{b^2}{4(1 - |c|)}\,, & |b| < 2(1 - |c|). \end{cases} \tag{11}$$

If $ac < 0$, then

$$Y(a,b,c) = \begin{cases} 1 - |a| + \dfrac{b^2}{4(1 - |c|)}\,, & -4ac(c^{-2} - 1) \le b^2 \;\text{ and }\; |b| < 2(1 - |c|) \\ 1 + |a| + \dfrac{b^2}{4(1 + |c|)}\,, & b^2 < \min\left\{4(1 + |c|)^2, -4ac(c^{-2} - 1)\right\} \\ R(a,b,c)\,, & \text{otherwise} \end{cases} \tag{12}$$

where

$$R(a,b,c) = \begin{cases} |a| + |b| - |c|\,, & |c|(|b| + 4|a|) \le |ab| \\ -|a| + |b| + |c|\,, & |ab| \le |c|(|b| - 4|a|) \\ (|c| + |a|)\sqrt{1 - \dfrac{b^2}{4ac}}\,, & \text{otherwise.} \end{cases} \tag{13}$$

3. Main Results

We begin this section by finding the absolute values of the first three initial coefficients in the function class $\mathcal{H}_{3/2}(p)$.

Theorem 2. *Let $0 \le p \le 2$ and let f, given by (1), be in the class $\mathcal{H}_{3/2}(p)$. Then*

$$|a_2| \le 1 \tag{14}$$

$$|a_3| \le \frac{1}{8}(p^2 + 2) \tag{15}$$

$$|a_4| \le \frac{135p^3 + 210p^2 + 512}{4608}\,. \tag{16}$$

Proof. Let $f \in \mathcal{H}_{3/2}(p)$. Then

$$\Re\left(\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \right) < \frac{3}{2},\ z \in \mathbb{U}$$

or equivalently

$$\Re\left(3-2\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right)>0,\ z\in\mathbb{U}.$$

Therefore, there exists a function $p \in \mathcal{P}$, given by (7), such that

$$3\frac{zf'(z)}{f(z)}-2\left(1+\frac{zf''(z)}{f'(z)}\right)=p(z)\frac{zf'(z)}{f(z)}. \tag{17}$$

Making use of the Taylor series representations for functions f and p and equating the coefficients of z^n $(n = 1, 2, 3)$ on both sides of (17), we obtain

$$a_2=-\frac{p_1}{2} \qquad a_3=\frac{1}{8}(-p_2-p_1a_2+6a_2^2) \tag{18}$$

$$a_4=\frac{1}{18}(p_3+2p_1a_3+p_2a_2-p_1a_2^2-30a_2a_3+14a_2^3). \tag{19}$$

Since $f \in \mathcal{H}_{3/2}(p)$ we have $2a_2 = f''(0) = p$ and then, by (18), we get $p_1 = -2a_2 = -p$. In view of the last equality and Lemma 1, we obtain

$$2p_2=p^2+(4-p^2)x \tag{20}$$

$$4p_3=-p^3-2(4-p^2)px+(4-p^2)px^2+2(4-p^2)(1-|x|^2)y, \tag{21}$$

where $x, y \in \mathbb{C}$ with $|x| \le 1$ and $|y| \le 1$. Making use of (18)–(21), elementary calculations yield to

$$a_2=\frac{p}{2} \qquad a_3=\frac{1}{16}[3p^2-(4-p^2)x] \tag{22}$$

$$a_4=\frac{1}{18}\left[\frac{19}{16}p^3-\frac{13}{16}(4-p^2)px-\frac{1}{4}(4-p^2)px^2-\frac{1}{2}(4-p^2)(1-|x|^2)y\right]. \tag{23}$$

Since $p \in [0, 2]$, we get $|a_2| \le 1$. We have

$$|a_3|\le\frac{1}{16}|3p^2-(4-p^2)x|\le\frac{1}{16}(3p^2+4-p^2)=\frac{1}{8}(p^2+2).$$

For the estimate of $|a_4|$, we obtain

$$|a_4|\le\frac{4-p^2}{36}\left(1-|x|^2+\left|-\frac{19p^3}{8(4-p^2)}+\frac{13p}{8}x+\frac{p}{2}x^2\right|\right)\le\frac{4-p^2}{36}Y(a,b,c),$$

where $Y(a, b, c)$ is given by (10) and

$$a=-\frac{19p^3}{8(4-p^2)},\quad b=\frac{13p}{8},\quad c=\frac{p}{2}.$$

Since $p \in [0, 2]$, it is easy to verify that $ac < 0$ and $b^2 < \min\left\{4(1+|c|)^2, -4ac(c^{-2}-1)\right\}$. In view of Lemma 2, we have

$$Y(a,b,c)=\frac{135p^3+210p^2+512}{128(4-p^2)}$$

and thus

$$|a_4|\le\frac{4-p^2}{36}Y(a,b,c)=\frac{135p^3+210p^2+512}{4608}.$$

Denote by

$$\mathcal{H}_{3/2}(+) = \bigcup_{0\le p\le 2} \mathcal{H}_{3/2}(p). \tag{24}$$

Then, by using (15) and (16), a simple computation shows that

$$\sup_{f\in\mathcal{H}_{3/2}(+)} |a_3(f)| = \frac{3}{4}$$

and

$$\sup_{f\in\mathcal{H}_{3/2}(+)} |a_4(f)| = \frac{19}{36},$$

where $a_3(f)$ and $a_4(f)$ are the coefficients of f. □

The upper bounds for the difference of the initial coefficients for the class $\mathcal{H}_{3/2}(p)$ are given in the next result.

Theorem 3. *Let $0 \le p \le 2$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{H}_{3/2}(p)$. Then,*

$$|a_3 - a_2| \le \frac{1}{4}(-p^2 + 2p + 1) \tag{25}$$

and

$$|a_4 - a_3| \le \begin{cases} \dfrac{-5p^3 + 18p^2 - 18p + 36}{144}, & p \in \left[0, \dfrac{2}{5}\right] \\[2ex] \dfrac{-135p^3 + 606p^2 - 612p + 1160}{4608}, & p \in \left[\dfrac{2}{5}, \dfrac{34}{21}\right] \\[2ex] \dfrac{-9p^3 + 18p^2 + 17p - 18}{72}, & p \in \left[\dfrac{34}{21}, 2\right]. \end{cases} \tag{26}$$

Proof. Proceeding as in the proof of Theorem 2 and making use of (22), we obtain

$$|a_3 - a_2| = \frac{1}{16}|3p^2 - 8p - (4-p^2)x| \le \frac{1}{4}(-p^2 + 2p + 1).$$

Now, we shall find the estimate of $|a_4 - a_3|$. For this, using (22) and (23), we have

$$\begin{aligned} |a_4 - a_3| &= \frac{1}{144}\left|\frac{19p^3 - 54p^2}{2} + \frac{18 - 13p}{2}(4-p^2)x - 2(4-p^2)px^2 - 4(4-p^2)(1-|x|^2)y\right| \\ &\le \frac{4-p^2}{36}\left[1 - |x|^2 + \left|\frac{p^2(54-19p)}{8(4-p^2)} - \frac{18-13p}{8}x + \frac{p}{2}x^2\right|\right] \le \frac{4-p^2}{36}Y(a,b,c), \end{aligned}$$

where $Y(a,b,c)$ is given by (10) and

$$a = \frac{p^2(54-19p)}{8(4-p^2)}, \quad b = -\frac{18-13p}{8} \quad \text{and} \quad c = \frac{p}{2}.$$

Since $0 \le p \le 2$, we have $a > 0$. Note also that for $p \in [0,2]$ the inequality $|b| \ge 2(1 - |c|)$ is equivalent to

$$p \in \left[0, \frac{2}{5}\right] \cup \left[\frac{34}{21}, 2\right].$$

Making use of Lemma 2, a computation gives

$$Y(a,b,c)=\begin{cases}\dfrac{p^2(54-19p)}{8(4-p^2)}+\dfrac{18-13p}{8}+\dfrac{p}{2}\,, & p\in\left[0,\dfrac{2}{5}\right]\\[2ex] 1+\dfrac{p^2(54-19p)}{8(4-p^2)}+\dfrac{(18-13p)^2}{128(2-p)}\,, & p\in\left[\dfrac{2}{5},\dfrac{34}{21}\right]\\[2ex] \dfrac{p^2(54-19p)}{8(4-p^2)}-\dfrac{18-13p}{8}+\dfrac{p}{2}\,, & p\in\left[\dfrac{34}{21},2\right].\end{cases}$$

Therefore, we get

$$|a_4-a_3|\le\begin{cases}\dfrac{-5p^3+18p^2-18p+36}{144}\,, & p\in\left[0,\dfrac{2}{5}\right]\\[2ex] \dfrac{-135p^3+606p^2-612p+1160}{4608}\,, & p\in\left[\dfrac{2}{5},\dfrac{34}{21}\right]\\[2ex] \dfrac{-9p^3+18p^2+17p-18}{72}\,, & p\in\left[\dfrac{34}{21},2\right].\end{cases}$$

In view of the estimates (25) and (26), we deduce that

$$\sup_{f\in\mathcal{H}_{3/2}(+)}|a_3(f)-a_2(f)|=\frac{1}{2}$$

and

$$\sup_{f\in\mathcal{H}_{3/2}(+)}|a_4(f)-a_3(f)|=\frac{58\sqrt{87}-36}{1944}\approx 0.259\ldots$$

where $\mathcal{H}_{3/2}(+)$ is given by (24) and $a_2(f),\ a_3(f),\ 4_2(f)$ are the coefficients of f. □

In the next result, we obtain the estimates of the functionals $|a_2a_4-a_3^2|$ and $|a_4-a_2a_3|$.

Theorem 4. *Let $0\le p\le 2$ and let f, given by (1), be in the function class $\mathcal{H}_{3/2}(p)$. Then, the following estimates hold*

$$|a_2a_4-a_3^2|\le\frac{(6-p^2)(6+p^2)}{576} \tag{27}$$

$$|a_4-a_2a_3|\le\begin{cases}\dfrac{9p^3-6p^2+32}{288}\,, & p\in\left[0,\dfrac{4}{3}\right]\\[2ex] \dfrac{p}{9}\,, & p\in\left[\dfrac{4}{3},2\right].\end{cases} \tag{28}$$

Proof. Proceeding again as in the proof of Theorem 2 and making use of (22) and (23), we have

$$|a_2a_4-a_3^2|=\frac{1}{2304}\left|-5p^2+2(4-p^2)p^2x-(7p^2+36)(4-p^2)x^2-32p(4-p^2)(1-|x|^2)y\right|$$

$$\le\frac{p(4-p^2)}{72}\left[1-|x|^2+\left|\frac{5p^3}{32(4-p^2)}-\frac{p}{16}x+\frac{7p^2+36}{32p}x^2\right|\right]=\frac{p(4-p^2)}{72}Y(a,b,c),$$

where $Y(a,b,c)$ is given by (10) and

$$a = \frac{5p^3}{32(4-p^2)}, \quad b = -\frac{p}{16} \quad \text{and} \quad c = \frac{7p^2+36}{32p}.$$

The inequality $|b| \geq 2(1-|c|)$ holds true for all $p \in [0,2]$ and therefore, from Lemma 2, we deduce that

$$Y(a,b,c) = \frac{5p^3}{32(4-p^2)} + \frac{p}{16} + \frac{7p^2+36}{32p} = \frac{-p^4+36}{8p(4-p^2)}.$$

It follows that

$$|a_2a_4 - a_3^2| \leq \frac{(6-p^2)(6+p^2)}{576}.$$

To find the upper bound of $|a_4 - a_2a_3|$ we use once more (23) and (24) and obtain

$$|a_4 - a_2a_3| = \frac{1}{288}\left|-8p^3 - 4(4-p^2)px - 4(4-p^2)px^2 - 8(4-p^2)(1-|x|^2)y\right|$$

$$\leq \frac{4-p^2}{36}\left[1-|x|^2 + \left|\frac{p^3}{4-p^2} + \frac{p}{2}x + \frac{p}{2}x^2\right|\right] = \frac{4-p^2}{36}Y(a,b,c),$$

where $Y(a,b,c)$ is given by (10) and

$$a = \frac{p^3}{4-p^2} \quad \text{and} \quad b = c = \frac{p}{2}.$$

It easy to show that $|b| < 2(1-|c|)$ for $p \in [0,4/3]$. An application of Lemma 2 yields

$$Y(a,b,c) = \begin{cases} 1 + \dfrac{p^3}{4-p^2} + \dfrac{p^2}{8(2-p)}, & p \in \left[0, \dfrac{4}{3}\right] \\[2ex] \dfrac{p^3}{4-p^2} + p, & p \in \left[\dfrac{4}{3}, 2\right]. \end{cases}$$

Hence, inequality (28) holds true.

Finally, using the estimates (27) and (28) we get

$$\sup_{f \in \mathcal{H}_{3/2}(+)} |a_2(f)a_4(f) - a_3^2(f)| = \frac{1}{16}$$

and

$$\sup_{f \in \mathcal{H}_{3/2}(+)} |a_4(f) - a_2(f)a_3(f)| = \frac{2}{9},$$

where $\mathcal{H}_{3/2}(+)$ is given by (24) and $a_2(f)$, $a_3(f)$, $4_2(f)$ are the coefficients of f. $\square$

4. Conclusions

In this paper, we first considered a presumably new subclass $\mathcal{H}_{3/2}$ of starlike functions in the open unit disk. For a refined family $\mathcal{H}_{3/2}(p)$ $(0 \leq p \leq 2)$ of $\mathcal{H}_{3/2}$, we investigated the upper bounds of the initial coefficients and the moduli of the initial successive coefficients. Moreover, upper bounds for functionals $|a_2a_4 - a_3^2|$ and $|a_4 - a_2a_3|$ for the same subclass $\mathcal{H}_{3/2}(p)$ were derived. The results obtained in this note could be a subject of further investigation related to Fekete–Szegö type functionals or Hankel determinants for the functions class $\mathcal{H}_{3/2}$.

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