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Quasi-Probability Husimi-Distribution Information and Squeezing in a Qubit System Interacting with a Two-Mode Parametric Amplifier Cavity

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Abstract: Squeezing and phase space coherence are investigated for a bimodal cavity accommodating a two-level atom. The two modes of the cavity are initially in the Barut–Girardello coherent states. This system is studied with the $SU(1,1)$ -algebraic model. Quantum effects are analyzed with the Husimi function under the effect of the intrinsic decoherence. Squeezing, quantum mixedness, and the phase information, which are affected by the system parameters, exalt a richer structure dynamic in the presence of the intrinsic decoherence.

Keywords: decoherence; two modes; squeezing; Husimi distribution

1. Introduction

Quantum coherence and correlations are the main resources for more quantum applications [1,2], such as teleportation, cryptography [3–5] and quantum memory [6,7]. Quantum coherence arises from superposition and is a prerequisite for many types of quantum entanglement [8], such as discord, nonlocality and steering [9]. von Neumann entropy [10] and linear entropy [11] are utilized to estimate the amount of entanglement generated by a pure quantum state [1] and a mixed state [12].

Quantum phase information and quantum coherence are quantified by Wehrl density and entropy. These two measures are based on Husimi distribution function (HF) [13]. One of the main advantages of Husimi distribution is the positivity of the distribution [14]. The associated Wehrl entropy of HE gives quantitative and qualitative phase space information about a pure/mixed qubit state [15–17].

Another important quantum phenomenon is the squeezing entropy, that has potential applications in optical communications [18], quantum optics [19], atomic fountain clock, condensed matter and graphene [20–22]. Based on the information entropy and the Heisenberg uncertainty relation variance [23], a special type of atomic squeezing was introduced [24–30].

The interaction of quantum systems addresses an attractive topic for its multiple applications in quantum optics and quantum computing. In particular, three types of interactions have

been extensively studied: field–field [31], field–qubit [32], and qubit–qubit couplings [33,34]. These interactions participated to various phenomena observed in experiments [35].

The two-photon field in quantum systems contains a large amount of entanglement between the photons emanating from the interaction cavity. Several models that achieve two-photon transition have been explored experimentally for two-photon microscopic maser [36].

More attention has been paid to nonlinear interactions between electromagnetic fields and other quantum systems. From these interactions arise phenomena related to physical applications—for example, stimulated and spontaneous emissions of radiation, Raman and Brillouin scattering. The nonlinear interactions are divided into two main types, the first one is a multi-mode type of frequency amplifier while the second is a multi-mode type of frequency converter [37,38]. Two-photon transitions are another resource for non-classicality, presenting high quantum correlations between emitted photons. They were experimentally realized via two-photon micromaser [36].

The decoherence in quantum qubit–cavity systems destroys the quantum effects, which are generated due the unitary qubit–cavity interactions [39–42]. The decoherence have several origins, such as the interaction of the system with the environment without energy relaxation and the intrinsic decoherence (ID) that occurs without the interaction with the surrounding environment. In the ID models, the quantum effects deteriorate when the closed qubit–cavity system evolves [43–45].

In this manuscript, we propose a model containing a two- or four-photon field coupled to a qubit system. Through transformations between the modes, the model is generalized to the $SU(1, 1)$ algebraic system. Then, we analyze the dynamics of quantum coherence (based on the quasi-probability Husimi distribution), and the quantum squeezing when the two-mode parametric amplifier cavity fields start with a Barut–Girardello coherent state.

The rest of this manuscript displays the ID model and its dynamics in Section 2. The study of the phase information via the Husimi function and Quantum coherence via Wehrl entropy is presented in Section 3. The squeezing phenomenon is analyzed in Section 4. Finally, Section 5 is dedicated to the conclusion.

2. Physical Model

Here, the Hamiltonian describes a two-level atom (qubit with upper $|1_A\rangle$ and $|0_A\rangle$ states), interacts with two cavities involving a two parametric amplifier fields. the cavity-qubit interactions are through a two-photon transition [31,46]. The cavity-qubit Hamiltonian of the system is

$$\hat{H} = \frac{\omega_0}{2} \hat{\sigma}_z + \sum_{i=1}^2 \omega_i (\hat{\psi}_i^\dagger \hat{\psi}_i + \frac{1}{2}) + \lambda (\hat{\psi}_1^2 \hat{\psi}_2^2 \hat{\sigma}_+ + \hat{\psi}_1^{2\dagger} \hat{\psi}_2^{2\dagger} \hat{\sigma}_-), \tag{1}$$

where ω_1 and ω_2 design the frequencies of the bimodal cavity fields, which have the annihilation operators $\hat{\psi}_1$ and $\hat{\psi}_2$, respectively. The constant λ represents the coupling between the qubit system and the two-mode parametric amplifier.

Here, we consider the case where we set $\omega = \omega_1 = \omega_2$ with the representation of $SU(1, 1)$ Lie algebra generators (\hat{K}_\pm, \hat{K}_z) : $\hat{K}_- = \hat{\psi}_1 \hat{\psi}_2 = (\hat{K}_+)^{\dagger}$ and $\hat{K}_z = \frac{1}{2}(\hat{\psi}_1^\dagger \hat{\psi}_1 + \hat{\psi}_2^\dagger \hat{\psi}_2 + 1)$, which satisfy,

$$[\hat{K}_+, \hat{K}_-] = -2\hat{K}_z, \quad [\hat{K}_z, \hat{K}_\pm] = \pm\hat{K}_\pm, \tag{2}$$

where $\hat{K}^2 = k(k - 1)\hat{I}$ is the Casimir operator and k is the Bargmann number,

$$\hat{K}^2 = \hat{K}_z^2 - \frac{1}{2}(\hat{K}_+ \hat{K}_- + \hat{K}_- \hat{K}_+). \tag{3}$$

Therefore, the Hamiltonian of Equation (1) is then rewritten as:

$$\hat{H} = \frac{\omega_0}{2} \hat{\sigma}_z + \omega \hat{K}_z + \lambda [\hat{K}_-^2 \hat{\sigma}_+ + \hat{K}_+^2 \hat{\sigma}_-], \tag{4}$$

where \hat{K}_+ , \hat{K}_- and \hat{K}_z applied on the eigenstates $|l, k\rangle$ give

$$\begin{aligned} \hat{K}_-|l, k\rangle &= \sqrt{l(l+2k-1)}|l-1, k\rangle, \\ \hat{K}_+|l, k\rangle &= \sqrt{(l+1)(l+2k)}|l+1, k\rangle, \\ \hat{K}_z|l, k\rangle &= (l+k)|l, k\rangle, \quad K^2|l, k\rangle = k(k-1)|l, k\rangle. \end{aligned} \tag{5}$$

It is found that the decoherence [39–41] has crucial effect on quantum squeezing and quantum coherence. Here, we adopt an important type of the decoherence known as “intrinsic decoherence” (ID). The master equation of the intrinsic decoherence can be written as [43]

$$\frac{d}{dt}\hat{\rho}(t) = -i[\hat{H}, \hat{\rho}(t)] - \gamma[\hat{H}, [\hat{H}, \hat{\rho}(t)]], \tag{6}$$

where $\hat{\rho}(t)$ represents the time-dependent SU(1,1)-SU(2) system density operator and γ is the parameter of the decoherence.

We focus on the case where the atom described by the SU(2)-system starts with the upper state—i.e., $\rho^A(0) = |1_A\rangle\langle 1_A|$, while the two-mode parametric amplifier cavity fields start with a Barut–Girardello coherent state (GB-GS) [47], described by

$$\begin{aligned} |\alpha, k\rangle &= \sum_{n=0}^{\infty} \xi_n |n, k\rangle, \\ \xi_n &= \alpha^n \sqrt{\frac{|\alpha|^{2k-1}}{n! I_{2k-1}(2|\alpha|) \Gamma(2k+n)}}, \end{aligned} \tag{7}$$

where $I_\nu(x)$ represents the modified Bessel function.

Based on the considered initial states, and the Hamiltonian eigenstates of Equation (4), the analytical solution of the Equation (6) in the space state $\{|1_A, i, k\rangle, |0_A, i+2, k\rangle\}$ can be expressed as

$$\begin{aligned} \rho(t) &= \sum_{i,j=0} \frac{1}{4} \xi_i \xi_j^* \{ [\chi_{ij}^{++} + \chi_{ij}^{+-} + \chi_{ij}^{-+} + \chi_{ij}^{--}] \\ &\times |1_A, i, k\rangle\langle 1_A, j, k| + [\chi_{ij}^{++} - \chi_{ij}^{+-} + \chi_{ij}^{-+} - \chi_{ij}^{--}] \\ &\times |1_A, i, k\rangle\langle 0_A, j+2, k| + [\chi_{ij}^{++} + U_{ij}^{+-} - \chi_{ij}^{-+} - \chi_{ij}^{--}] \\ &\times |0_A, i+2, k\rangle\langle 1_A, j, k| + [\chi_{ij}^{++} - \chi_{ij}^{+-} - \chi_{ij}^{-+} + \chi_{ij}^{--}] \\ &\times |0_A, i+2, k\rangle\langle 0_A, j+2, k| \}, \end{aligned} \tag{8}$$

with

$$\chi_{ij}^{\pm\pm} = e^{-i(V_i^\pm - V_j^\pm)t - \gamma(V_i^\pm - V_j^\pm)^2 t}, \tag{9}$$

while V_i^\pm represent the Hamiltonian eigenvalues of Equation (4),

$$\begin{aligned} V_i^\pm &= \omega(i+k+1) \\ &\pm \sqrt{\delta^2 + \lambda^2(i+1)(i+2)(i+2k)(i+2k+1)}, \end{aligned} \tag{10}$$

where $\delta = (\omega_0 - 2\omega)/2$ is the detuning between the frequencies of the qubits and the two-mode parametric amplifier cavity fields. Before exploring the quantum effects generated in this system, let us define the atomic reduced density matrix

$$\rho^A(t) = \text{Tr}_F\{\rho(t)\}, \tag{11}$$

where Tr_F represents the operation of the tracing out the state of the two-mode parametric amplifier cavity.

3. Husimi Distribution (HD)

Here, we consider the quasi-probability Husimi distributions that depend on the reduce density matrix elements of the qubit system. Quantum effects of the HD and their associated as Wehrl entropy, phase space information and the mixedness, will be analyzed.

3.1. Husimi Function

For, angular momentum j , the j -spin coherent states $|\theta, \phi\rangle$ [48,49], can be written as

$$|\theta, \phi\rangle = \sum_{n=-j}^{n=j} \frac{e^{i(j-n)\phi}}{2^j} \sqrt{\frac{(2j)!}{(j-n)!(j+n)!}} \sin^j \theta \cot^n \frac{\theta}{2} |j, n\rangle. \tag{12}$$

For a qubit state, spin- $\frac{1}{2}$ ($j = \frac{1}{2}$), the Bloch coherent state, $|\mu\rangle$, is defined by

$$|\mu\rangle = \cos \frac{\theta}{2} |1_A\rangle + \sin \frac{\theta}{2} e^{i\phi} |0_A\rangle. \tag{13}$$

The SU(2)-system is identified in the phase space by θ , and ϕ , where $d\mu = \sin \theta d\theta d\phi$. Therefore, the H-function (HF) is defined by [13]

$$H(\mu, t) = \frac{1}{2\pi} \langle \mu | \rho^A(t) | \mu \rangle. \tag{14}$$

Due to the fact that the $H(\mu, t)$ depends on the angles of the phase space distribution, it is used as a measure of the information loss for the SU(2)-system.

Figure 1 illustrates the effects of the detuning and the intrinsic decoherence on the behavior of the Husimi distribution $H(\mu, t) = H(\theta, \phi, t)$. From Figure 1a, the H-function has distributed regularity with 2π -period with respect to the angles θ and ϕ . In general the H-function has periodical-peaks with a Gaussian distribution, and the distribution is centered at $(\frac{\theta}{\pi}, \frac{\phi}{\pi}) = (\frac{2n+1}{2}, \frac{2n+1}{2})$, ($n = 0, 1, 2, 3, \dots$). In the absence of ID and detuning, the maximum values of the peaks increase with enhancement of ϕ as can be observed in Figure 1a. The location of the peaks does not change but reduces with the increase in ϕ after considering the detuning. Both peaks and bottoms are squeezed and almost vanish after adding the decays, as seen in Figure 1c.

In the resonance case, and neglecting the intrinsic decoherence, the $H(t)$ function oscillates chaotically; the amplitude of the oscillations range from 0.02 to 0.13. We also note that the function reaches its smallest values at $\frac{n\pi}{2}$ ($n = 1, 2, 3, \dots$) as can be observed in Figure 2.

The chaotic oscillations become regular for the off-resonant case and the maximum values are enhanced. Note that the maximum values are periodically achieved and are following the pattern (0, 0.4, 0.8, 0.12, ...) as shown in Figure 2. After considering the ID, the previous fluctuations will completely disappear over time, and the Husimi function tends to the value 0.8, as shown in Figure 2.

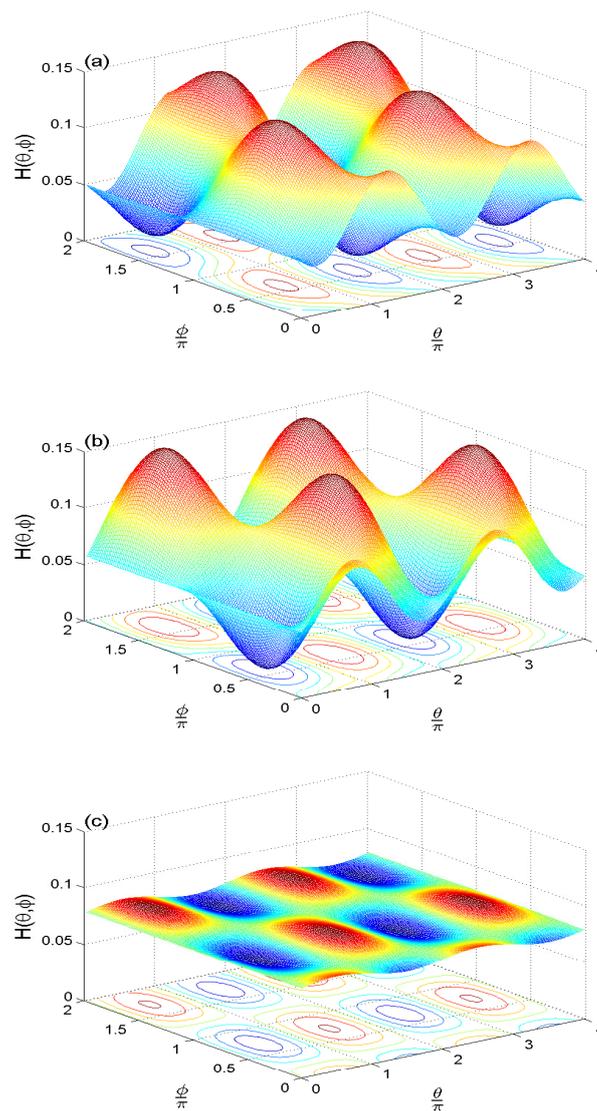


Figure 1. The dependence of the HD on the atomic distribution angles θ and ϕ at the time $\lambda t = 2.112\pi$ is shown for the initial coherent intensity $\alpha = 2$ and $k = \frac{1}{2}$, and also for different detuning and decoherence values: $(\delta, \gamma) = (0, 0)$ in (a), $(\delta, \gamma) = (30\lambda, 0)$ in (b) and $(\delta, \gamma) = (0, 0.3\lambda)$ in (c).

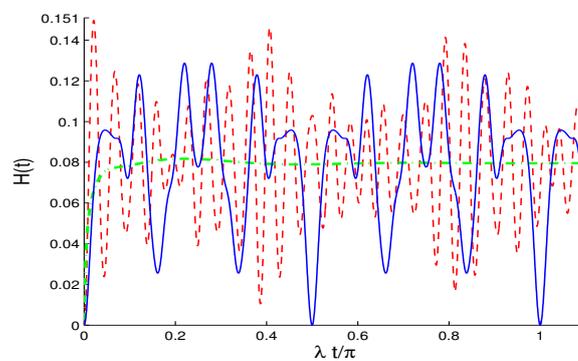


Figure 2. Time evolution of Husimi distribution is shown for the initial coherent intensity $\alpha = 2$, with detuning and decoherence values: $(\delta, \gamma) = (0, 0)$ (solid curve), $(\delta, \gamma) = (20\lambda, 0)$ (dashed curve) and $(\delta, \gamma) = (0, 0.1\lambda)$ (dashed-dotted curve).

3.2. Wehrl Entropy

For the case where the intrinsic decoherence is existing, the atomic Wehrl entropy [50] is used to quantify qubit mixedness, which measures the entanglement in closed systems $\gamma = 0$ [51,52]. It is defined by

$$E(t) = \int_0^{2\pi} \int_0^\pi D(\mu, t) d\mu, \tag{15}$$

where the Wehrl density of the Husimi function $D(\mu, t)$ is represented by

$$D(\mu, t) = -H(\mu, t) \ln[H(\mu, t)]. \tag{16}$$

When the the SU(2)-system is initially in the excited state $|1_A\rangle$, the Wehrl entropy is given by

$$\begin{aligned} E(0) &= \frac{-1}{2} \int_0^\pi (2 \sin 2\theta + \sin 4\theta) \ln[\cos^2(\theta)/2\pi] d\theta \\ &= 2.3379. \end{aligned} \tag{17}$$

Therefore, it satisfies [53],

$$E(0) \leq E(t) \leq \ln(4\pi).$$

The initial value of $E(0)$ is for a pure state, and its maximal value $\ln(4\pi)$ indicates that the qubit is a maximally mixed state. The Wehrl entropy is a good quantifier that provides a measure about the degree of the mixedness of the considered qubit state.

In the absence of the decoherence, the dynamics of the Wehrl entropy, $E(t)$ is shown (in Figure 3a) to investigate the generated partial and maximal entanglement between the atomic SU(2)-system and the SU(1, 1)-system of the two-mode cavity. We note that the generated entanglement grows with a quasi-regular oscillatory dynamic.

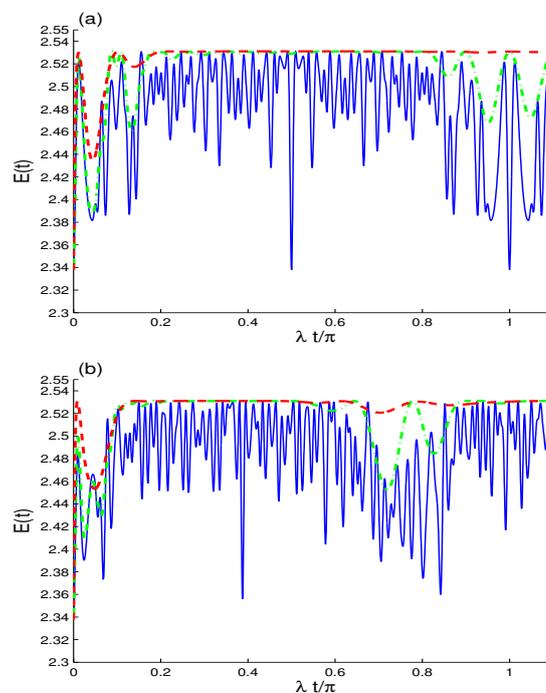


Figure 3. Dynamics of the Wehrl entropy are shown for the initial coherent intensity $\alpha = 4$ and $k = \frac{1}{2}$ with different decoherence values $\gamma = 0$ (solid plots), $\gamma = 0.01\lambda$ (dashed plots) and $\gamma = 0.001\lambda$ (dashed-dotted plots). The case $\delta = 0$ in (a) and $\delta = 20\lambda$ in (b).

In resonance and by neglecting the decoherence, we see that the Wehrl entropy function $E(t)$ oscillates regularly. We also find that the function reaches the smallest values periodically at points $\frac{n\pi}{2} (n = 1, 2, 3, \dots)$, which is completely consistent with the observations mentioned in the previous section, see the Figures 2 and 3a).

By considering the intrinsic decoherence, the Wherl entropy $E(t)$ starts from its initial value $E(0)$ of the pure qubit state $|1_A\rangle$ and quickly reaches the maximum value, without oscillations, as can be observed in the Figure 3a).

For the non-resonant case, without ID, the function $E(t)$ fluctuates more than the previous case. When the intrinsic decoherence enters into play, gradually the oscillations are eliminated and the maximum values are quickly attained.

4. Squeezing Phenomenon

Based on the Pauli operators $\hat{\sigma}_r^j$ of a qubit system, the atomic SU(2)-system information entropy (AIE) is defined by [10,11]

$$S_r = -\frac{1}{2} \sum_{n=1,2} [1 + (-1)^n \langle \hat{\sigma}_r \rangle] \ln \frac{1}{2} [1 + (-1)^n \langle \hat{\sigma}_r \rangle]. \tag{18}$$

The AIE is used as a general criterion for the the atomic SU(2)-system squeezing. For $\delta S_r \equiv \exp[S_r]$, the entropy uncertainty relation is

$$\delta S_x \delta S_y \delta S_z \geq 4, \tag{19}$$

therefore, these information entropies satisfy the entropy uncertainty relation [54]

$$S_x + S_y + S_z \geq 2 \ln 2. \tag{20}$$

Based on the AIEs, the entropy squeezing is given by

$$E_r(t) = \delta S_r - \frac{2}{\sqrt{\delta S_z}}. \tag{21}$$

The fluctuations in the atomic SU(2)-system components $\hat{\sigma}_r (r = x, y, z)$ are indicators of “squeezing phenomenon” if the condition $E_r(t) < 0$ is occurred. It is found that the component $\hat{\sigma}_x$ is not squeezed—i.e., $E_x(t) > 0$. Consequently, the dynamics of the entropy squeezing of the component $\hat{\sigma}_y$ is investigated only.

In Figure 4, the intrinsic decoherence effect on the dynamics of the entropy squeezing $E_y(t)$ is shown for the resonant and non-resonant cases. Solid curve of Figure 4a, representing the case where $\delta = 0$, illustrates that the squeezing phenomenon of the entropy squeezing appears during several time windows, periodically with the π -period.

In the case where zero-detuning takes place and in the absence of the intrinsic decoherence, we notice that the squeezing intervals appears periodically around the points $\lambda t = n\pi (n = 0, 1, 2, \dots)$, due to the unitary interaction. These squeezing intervals are decreased and vanished after we consider the ID decay, as seen in Figure 4a.

Figure 4b illustrates that the non-resonant case leads to the destruction of the squeezing hefting the minima $E_y(t)$. The squeezing disappears completely after the intrinsic decoherence is considered.

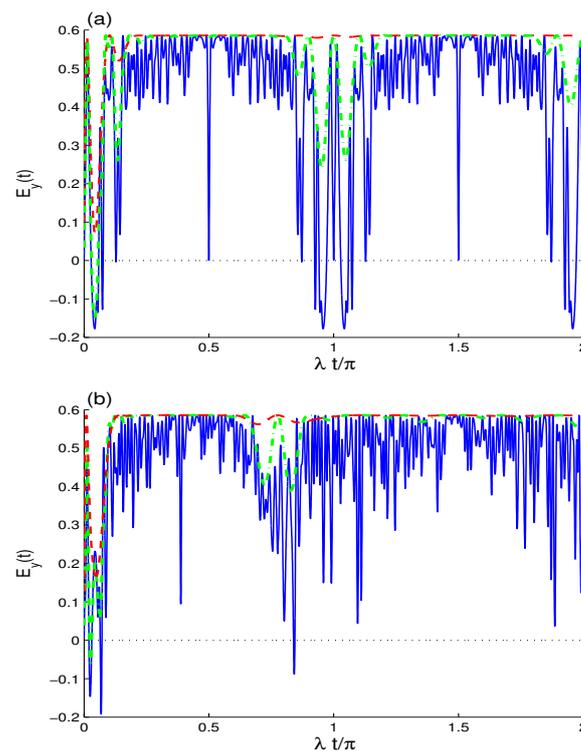


Figure 4. Dynamics of the entropy squeezing are shown for $k = \frac{1}{2}$ and $\alpha = 4$ with different decoherence values $\gamma = 0$ (solid plots), $\gamma = 0.01\lambda$ (dashed plots) and $\gamma = 0.001\lambda$ (dashed-dotted plots). The case $\delta = 0$ in (a) and $\delta = 20\lambda$ in (b).

5. Conclusions

We have explored here, the dynamics of bimode cavity fields incorporating a qubit system by applying the $SU(1,1)$ -algebraic representation. The dynamics of the Husimi distribution and its associated Wehrl entropy, as well as the entropy squeezing are discussed. The non-resonance amplifies the fluctuations of those quantities. The intrinsic decoherence reduces the squeezing intervals. The detuning leads to the enhancement of the generated Wehrl mixedness entropy, and delays the appearance of the stationary mixedness. For the off-resonant case, the squeezing decreases significantly. It is found that the phase space Husimi distribution information, the quantum coherence (qubit–cavity entanglement and atomic mixedness) and the quantum atomic squeezing are very sensitive to the nonlinear qubit–cavity couplings.

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