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Review

The General Fractional Derivative and Related Fractional Differential Equations

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Abstract: In this survey paper, we start with a discussion of the general fractional derivative (GFD) introduced by A. Kochubei in his recent publications. In particular, a connection of this derivative to the corresponding fractional integral and the Sonine relation for their kernels are presented. Then we consider some fractional ordinary differential equations (ODEs) with the GFD including the relaxation equation and the growth equation. The main part of the paper is devoted to the fractional partial differential equations (PDEs) with the GFD. We discuss both the Cauchy problems and the initial-boundary-value problems for the time-fractional diffusion equations with the GFD. In the final part of the paper, some results regarding the inverse problems for the differential equations with the GFD are presented.

Keywords: general fractional derivative; general fractional integral; Sonine condition; fractional relaxation equation; fractional diffusion equation; Cauchy problem; initial-boundary-value problem; inverse problem

MSC: 26A33; 35A05; 35B30; 35B50; 35C05; 35E05; 35L05; 35R30; 45K05; 60E99

1. Introduction

In functional analysis, the integral operators with the weakly singular kernels have been an important topic for research for many years. They are defined in the form

$$(T\,f)(t) = \int_{\Omega} K(t,\tau)\, f(\tau)\, d\tau, \tag{1}$$

where Ω is an open subset of $\mathbb{R}^n$ and the kernel $K = K(t,\tau)$ is a real or complex valued continuous function on $\overline{\Omega} \times \overline{\Omega} \backslash D$, $D = \{(t,t) : t \in \overline{\Omega}\}$ being the diagonal of $\overline{\Omega} \times \overline{\Omega}$ that satisfies the condition

$$|K(t,\tau)| \le \frac{c}{|t-\tau|^{\alpha}} \quad \text{with } \alpha < n. \tag{2}$$

In case Ω is a bounded domain, the operator (1) is called the Schur integral operator. It is compact from $C(\overline{\Omega})$ to $C(\overline{\Omega})$ [1]. If, additionally, $\alpha < \frac{n}{q}$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p < +\infty$, then (1) is a Hilbert-Schmidt operator that is compact from $L^p(\Omega)$ to $C(\overline{\Omega})$.

In Fractional Calculus (FC), the operators of type (1) with special weakly singular kernels are studied on both bounded and unbounded domains. For instance, the classical left-hand sided Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}_+$ of a function f on a finite or infinite interval (a, b) is defined as follows:

$$(I^{\alpha}_{a+}f)(t) = \frac{1}{\Gamma(\alpha)} \int\limits_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad t \in (a,b). \tag{3}$$

In the case $\alpha = 0$, this integral is interpreted as the identity operator

$$\left(I^{0}_{a+}f\right)(t) = f(t) \tag{4}$$

because of the relation ([2])

$$\lim_{\alpha \to 0+} (I^{\alpha}_{a+}f)(t) = f(t), \tag{5}$$

that is valid in particular for $f \in L^1(a,b)$ in every Lebesgue point of f, i.e., almost everywhere on (a,b).

Evidently, the Riemann-Liouville fractional integral is a generalization of the well-known formula for the n-fold definite integral

$$(I^{n}_{a+}f)(t) = \int_a^t d\tau \int_a^t d\tau \cdots \int_a^t f(\tau)\, d\tau = \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} f(\tau)\, d\tau,\ n \in \mathbb{N}.$$

In a certain sense, this generalization is unique. In the case of a finite interval (without any restriction of generality, we fix the interval $(0,\ 1)$), the following result was proved in [3]:

Let E be the space $L^p(0,1)$, $1 \le p < +\infty$, or $C[0,1]$. Then there exists precisely one family I_α, $\alpha > 0$ of operators on E satisfying the following conditions:

(CM1) $(I_1 f)(t) = \int_0^t f(\tau)\, d\tau$, $f \in E$ (interpolation condition),
(CM2) $(I_\alpha I_\beta f)(t) = (I_{\alpha+\beta} f)(t)$, $\alpha, \beta > 0$, $f \in E$ (index law),
(CM3) $\alpha \to I_\alpha$ is a continuous map of $\mathbb{R}_+$ into the space $\mathcal{L}(E)$ of the linear bounded operators from E to E for some Hausdorff topology on $\mathcal{L}(E)$, weaker than the norm topology (continuity),
(CM4) $f \in E$ and $f(t) \ge 0$ (a.e. for $E = L^p(0,1)$) $\Rightarrow (I_\alpha f)(t) \ge 0$ (a.e. for $E = L^p(0,1)$) for all $\alpha > 0$ (non-negativity).

That family is given by the Riemann-Liouville Formula (3) with $a = 0$ and $b = 1$. From the present viewpoint ([4]), the conditions (CM1)–(CM4) are very natural for any definition of the fractional integrals defined on a finite interval. As proved in [3], they are also sufficient for uniqueness of the family of the Riemann-Liouville fractional integrals. Thus, in this sense, the Riemann-Liouville fractional integrals are the only "right" one-parameter fractional integrals defined on a finite one-dimensional interval.

The problem regarding the "right" fractional derivatives is more delicate and has no unique solution. Presently, the main approach for introducing the fractional derivatives is to define them as the left-inverse operators to the fractional integrals ([4–6]). However, even for the Riemann-Liouville fractional integral, there exist infinitely many different families of operators that fulfill this property ([6]). In particular, for $0 < \alpha \le 1$, the Riemann-Liouville fractional derivative

$$(D^{\alpha}_{RL} f)(t) = \frac{d}{dt}(I^{1-\alpha}_{0+} f)(t), \tag{6}$$

the Caputo fractional derivative

$$(D^{\alpha}_{C} f)(t) = (I^{1-\alpha}_{0+} \frac{df}{d\tau})(t), \tag{7}$$

and the Hilfer fractional derivative

$$(D^{\alpha}_{H} f)(t) = (I^{\beta(1-\alpha)}_{0+} \frac{d}{d\tau} I^{(1-\alpha)(1-\beta)}_{0+} f)(t),\ 0 \le \beta \le 1 \tag{8}$$

are the left-inverse operators to the Riemann-Liouville fractional integral on the suitable nontrivial spaces of functions including the space of the absolutely continuous functions on $[0,1]$, i.e., the Fundamental Theorem of FC holds true ([6]):

$$(D_X^\alpha \, I_{0+}^\alpha \, f)(t) = f(t), \; t \in [0,\,1], \; X \in \{RL,\, C,\, H\}. \tag{9}$$

Moreover, in [6], infinitely many other families of the fractional derivatives in the sense of Formula (9) called the nth level fractional derivatives were introduced. Let the parameters $\gamma_1,\, \gamma_2, \ldots, \gamma_n \in R$ satisfy the conditions

$$0 \le \gamma_k \text{ and } \alpha + s_k \le k \text{ with } s_k := \sum_{i=1}^{k} \gamma_i, \; k = 1, 2, \ldots, n. \tag{10}$$

The nth level fractional derivative of order α, $0 < \alpha \le 1$ and type $\gamma = (\gamma_1,\, \gamma_2, \ldots, \gamma_n)$ is defined as follows:

$$(D_{nL}^{\alpha,(\gamma)} \, f)(t) = \left(\prod_{k=1}^{n} (I_{0+}^{\gamma_k} \, \frac{d}{d\tau}) \right) \, (I_{0+}^{n-\alpha-s_n} \, f)(t). \tag{11}$$

This derivative satisfies the Fundamental Theorem of FC, i.e., the relation

$$(D_{nL}^{\alpha,(\gamma)} \, I_{0+}^\alpha \, f)(t) = f(t), \;\; t \in [0,\,1] \tag{12}$$

holds true on a nontrivial space of functions (see [6] for details).

To keep an overview of these and many other fractional derivatives, it is very natural to consider some general integro-differential operators of convolution type and to clarify the question under what conditions can they be interpreted as a kind of the fractional derivatives. In particular, one expects that for these derivatives and the appropriate defined fractional integrals the Fundamental Theorem of FC holds true. Moreover, for the sake of possible applications, one wants to keep some fundamental properties of solutions to the differential equations with these derivatives. In particular, the property of complete monotonicity of solutions to the appropriate relaxation equation or the positivity of the fundamental solution to the Cauchy problem for the fractional diffusion equation with the time-derivatives of this type.

In the theory of the abstract Volterra integral equations in the Banach spaces, the evolution equations including the integro-differential operators of convolution type

$$(D_k \, u)(t) = \frac{d}{dt} \int_0^t k(t-\tau) u(\tau) \, d\tau, \; t \in [0, T], \; 0 < T \le +\infty \tag{13}$$

have been a subject for research for more than a half century. In particular, in [7] (see also the references therein), the abstract Volterra integral equations including the operators (13) with the completely positive kernels $k \in L^1(0, T)$ have been studied. This class of the kernels can be characterized as follows: A function $k \in L^1(0, T)$ is completely positive on $[0,\, T]$ if and only if there exist $a \ge 0$ and $l \in L^1(0, T)$, non-negative and non-increasing, satisfying the relation

$$a \, k(t) \; + \int_0^t k(t-\tau) l(\tau) \, d\tau \; = 1, \; t \in (0, T]. \tag{14}$$

In particular, the completely monotone kernels are completely positive. The notion of completely positive kernels originated from the "positivity preserving property" that is valid for the corresponding Volterra integral equations in the case of the Banach space $X = \mathbb{R}$ with the usual norm.

In [8,9], the properties of the appropriate defined weak solutions to the linear and quasi-linear evolutionary partial integro-differential equations of second order with the time-operators of type (13) in the form

$$(D_k \, u)(t) = \frac{d}{dt} \int_0^t k(t-\tau)(u(\tau) - u_0) \, d\tau, \; t \in \mathbb{R}_+ \tag{15}$$

were addressed in the case of the kernels $k \in L_{loc}(\mathbb{R}_+)$ that satisfy the following conditions:

(Z1) The kernel k is non-negative and non-increasing on $\mathbb{R}_+$,
(Z2) There exists a kernel $l \in L_{loc}(\mathbb{R}_+)$ such that $\int_0^t k(t-\tau)l(\tau)\,d\tau = 1,\ t \in \mathbb{R}_+$.

An important example of a kernel k that satisfies the conditions (Z1)–(Z2) is the following generalization of the Riemann-Liouville kernel ([8]):

$$k(t) = h_{1-\alpha}(t)\ \exp(-\mu\, t), \mu \ge 0,\ 0 < \alpha < 1, \tag{16}$$

where

$$h_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)},\ t > 0,\ \beta > 0. \tag{17}$$

For this kernel k, the kernel l from the property (Z2) takes the form

$$l(t) = h_\alpha(t)\ \exp(-\mu\, t)\ +\ \mu \int_0^t h_\alpha(\tau)\exp(-\mu\,\tau)\,d\tau,\ t > 0. \tag{18}$$

However, in [8,9] and in earlier publications, the operators of type (13) were not interpreted as a kind of the generalized fractional derivatives. In particular, no construction of the corresponding fractional integral was presented and no conditions that ensure the physically relevant properties of solutions to the time-fractional differential equations including these derivatives were suggested. Both tasks along with a series of other useful properties were addresses in [10–13], where a very nice theory of the general fractional derivative of type (13) was developed and applied for studying properties of the ordinary and partial differential equations with this derivative. In this survey, we present some selected results obtained in these and other related publications.

The rest of the paper is organized as follows. In Section 2, we introduce the GFD and the related fractional integral and discuss some of their basic properties. Section 3 is devoted to the Cauchy problems for the fractional ODEs with the GFD. In particular, the fractional relaxation equation and properties of its solution and the fractional growth equation and long time asymptotic of its solution are considered [10,11,13]. Moreover, existence and uniqueness of solutions to the Cauchy problem for the nonlinear fractional ODEs with the GFD and their continuous dependence on the problem data are also addressed following [14]. In Section 4, we present some results regarding the fractional PDEs with the GFD. We start with the Cauchy problem for the linear fractional diffusion equation and address its well-posedness with the focus on an interpretation of its fundamental solution as a probability density function [10,11]. Then we proceed with a treatment of the initial-boundary-value problems for the time-fractional diffusion equation including the GFD. Based on a suitable estimate for the GFD of a function at its maximum point, a weak maximum principle for the general time-fractional diffusion equation with the GFD is deduced. Then, following [15], the maximum principle is employed to show uniqueness of the strong and the weak solutions to the initial-boundary-value problems for the general time-fractional diffusion equations. Existence of a weak solution in the sense of Vladimirov [16] is also discussed. Finally, some important results from the recent publications [17–19] regarding inverse problems for the fractional differential and integral equations with the GFD are shortly presented.

2. General Fractional Derivative and Integral

Following [10], in this paper we consider the GFD of the Riemann-Liouville type in the form (compare to (13))

$$(\mathbb{D}_k^{RL} f)(t) = \frac{d}{dt}\int_0^t k(t-\tau)f(\tau)\,d\tau \tag{19}$$

and of the Caputo type in the form

$$(\mathbb{D}_k^{C} f)(t) = \int_0^t k(t-\tau)f'(\tau)\,d\tau, \tag{20}$$

where k is a non-negative locally integrable function. For an absolutely continuous function f satisfying $f' \in L^1_{loc}(\mathbb{R}_+)$, the relation (compare to (15))

$$(\mathbb{D}^C_k f)(t) = (\mathbb{D}^{RL}_k (f - f(0)))(t) = (\mathbb{D}^{RL}_k f)(t) - k(t)f(0) = \frac{d}{dt}\int_0^t k(t-\tau)f(\tau)\,d\tau - k(t)f(0) \quad (21)$$

between the Caputo and Riemann-Liouville types of GFDs holds true. In [10], the Caputo type GFD was introduced in form (21) that is well defined for a lager class of functions (in particular, for absolutely continuous functions) compared to the definition (20) that requires the inclusion $f' \in L^1_{loc}(\mathbb{R}_+)$. In what follows, we mainly address the GFD of the Caputo type in the sense of the right-hand side of Formula (21).

The Riemann-Liouville and Caputo fractional derivatives defined by (6) and (7), respectively, are particular cases of the GFDs (19) and (20) with the kernel

$$k(t) = h_{1-\alpha}(t),\ 0 < \alpha < 1, \quad (22)$$

the power function h_β being defined by (17). Other important particular cases of (19) and (20) are the multi-term fractional derivatives and the fractional derivatives of the distributed order. They are generated by (19) and (20) with the kernels

$$k(t) = \sum_{k=1}^{n} a_k\, h_{1-\alpha_k}(t),\ \ 0 < \alpha_1 < \cdots < \alpha_n < 1,\ a_k \in \mathbb{R},\ k = 1, \ldots, n \quad (23)$$

and

$$k(t) = \int_0^1 h_{1-\alpha}(t)\,d\rho(\alpha), \quad (24)$$

where ρ is a Borel measure on $[0, 1]$.

Even if the operators (19) and (20) have been employed in the theory of the abstract Volterra integral equations for many years, the main advantage of the Kochubei's approach was to establish a connection of these operators to FC and to introduce a special class of the kernels that ensures both existence of the corresponding fractional integrals and physically relevant properties of the fractional differential equations with these time-fractional derivatives. Moreover, the results presented in [10] and in the subsequent publications were derived using a completely different technique, namely the theory of the complete Bernstein functions ([20]).

The kernels of the GFDs (19) and (21) considered in [10] satisfy the following conditions:

(K1) The Laplace transform $\tilde{k}$ of k,

$$\tilde{k}(p) = (\mathcal{L}\,k)(p) = \int_0^\infty k(t)\,e^{-pt}\,dt$$

exists for all $p > 0$,

(K2) $\tilde{k}(p)$ is a Stieltjes function,

(K3) $\tilde{k}(p) \to 0$ and $p\tilde{k}(p) \to \infty$ as $p \to \infty$,

(K4) $\tilde{k}(p) \to \infty$ and $p\tilde{k}(p) \to 0$ as $p \to 0$.

In what follows, we denote the set of the kernels that satisfy the conditions (K1)–(K4) by $\mathcal{K}$. As we see, the condition of type (Z2) (Sonine condition) does not belong to the set of the conditions (K1)–(K4). However, it is one of the consequences of these conditions and especially of the strong condition (K2). Roughly speaking, a function defined on $\mathbb{R}_+$ is a Stieltjes function if it can be represented as a restriction of the Laplace transform of a completely monotone function to the real positive semi-axis. Any completely monotone function is non-negative and thus any Stieltjes function is completely monotone as the Laplace transform of a non-negative function. For the strict definition and properties of the Stieltjes functions see e.g., [20,21]. The kernel functions (22) and (23) as well as the function (24) under some suitable conditions on the measure ρ belong to the class $\mathcal{K}$. In [10], another example of a function $k \in \mathcal{K}$ was introduced in terms of its Laplace transform $\tilde{k}(p) = p^{-1}\log(1+p^{\beta}),\ 0 < \beta < 1$.

As shown in [10], for each $k \in \mathcal{K}$, there exists a completely monotone function κ such that the Sonine condition holds true:

$$(k * \kappa)(t) = \int_0^t k(t-\tau)\kappa(\tau)\, d\tau \ = \ 1. \tag{25}$$

Henceforth by

$$(\mathbb{I}_k f)(t) = \int_0^t \kappa(t-\tau) f(\tau)\, d\tau \tag{26}$$

we denote a general fractional integral (GFI) with the kernel κ associated with the kernel k of the GFD by means of the relation (25).

The notion of the GFI is justified by the following Fundamental Theorem of FC:

Theorem 1 ([10]). *If f is a locally bounded measurable function on $\mathbb{R}_+$, then*

$$(\mathbb{D}_k^C\, \mathbb{I}_k\, f)(t) = f(t). \tag{27}$$

If f is absolutely continuous on $[0, +\infty)$, then

$$(\mathbb{I}_k\, \mathbb{D}_k^C\, f)(t) = f(t) - f(0). \tag{28}$$

The Formula (27) and the relation (21) between the GFDs of the Caputo and the Riemann-Liouville types lead to the identity

$$(\mathbb{D}_k^{RL}\, \mathbb{I}_k\, f)(t) = f(t), \tag{29}$$

i.e., the Riemann-Liouville GFD is also a left inverse operator to the GFI defined by (26).

In the case of the Riemann-Liouville and the Caputo fractional derivatives that are particular cases of the GFDs (19) and (20), respectively, with the power function kernel k defined by (22), the kernel κ in the GFI (25) is also the power function $\kappa(t) = h_\alpha(t)$ and thus in this case the GFI (26) is nothing else as the conventional Riemann-Liouville fractional integral.

As shown in [22], the functions that satisfy the Sonine condition (25) cannot be continuous at the point $t = 0$ and thus the "new fractional derivatives" with the continuous kernels introduced recently in the FC literature do not belong to the class of the GFDs that are discussed in this paper.

In the next sections, we consider other physically relevant properties of the GFD including complete monotonicity of the solutions to the fractional relaxation equation with this derivative, positivity of the fundamental solution to the Cauchy problem for the fractional diffusion equation with the time-derivative in form of the GFD, and a maximum principle for the initial-boundary-value problems for the fractional diffusion equation with this derivative.

3. Fractional ODEs with the GFD

3.1. Fractional Relaxation Equation

In this subsection, we consider the fractional relaxation equation

$$(\mathbb{D}_k^C u)(t) = -\lambda u(t),\ \lambda > 0,\ t > 0 \tag{30}$$

subject to the initial condition

$$u(0) = 1. \tag{31}$$

As discussed in [23] (see also references therein), in the framework of the linear viscoelasticity models, the solutions to the relaxation equations are expected to be completely monotone. Only in this case the relaxation processes can be interpreted as superpositions of (infinitely many) elementary, i.e., exponential, relaxation processes. For the fractional relaxation equation with the GFD of the Caputo type, the following result holds true:

Theorem 2 ([10]). *Let the kernel k of the Caputo type GFD belong to the class $\mathcal{K}$.*

Then the Cauchy problem (30), (31) has a unique solution $u_\lambda = u_\lambda(t)$, continuous on $[0, +\infty)$, infinitely differentiable and completely monotone on $\mathbb{R}_+$.

We remind the readers that a function $u : \mathbb{R}_+ \to \mathbb{R}$ is called completely monotone if the conditions

$$(-1)^n u^{(n)}(t) \geq 0, \;\; t > 0, \; n = 0, 1, 2 \dots \tag{32}$$

hold true.

In the case of the Cauchy problem (30), (31) with the Caputo fractional derivative (7), the solution can be expressed in terms of the Mittag-Leffler function

$$u_\lambda(t) = E_{\alpha,1}(-\lambda t^\alpha),$$

where $E_{\alpha,\beta}$ stands for the two-parameters Mittag-Leffler function that is defined by the following convergent series:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha\, k + \beta)}, \; \alpha > 0, \; \beta, z \in \mathbb{C}. \tag{33}$$

It is worth mentioning that the fractional relaxation equations with the Riemann-Liouville fractional derivative (6) and the Hilfer derivative (8) have the solutions

$$u_\lambda(t) = t^{\alpha-1}\, E_{\alpha,\alpha}(-\lambda t^\alpha),$$

and

$$u_\lambda(t) = t^{\alpha+\gamma_1-1}\, E_{\alpha,\alpha+\gamma_1}(-\lambda t^\alpha),$$

respectively, provided that we set suitable initial conditions. These solutions are continuous, infinitely differentiable and completely monotone on $\mathbb{R}_+$, but have an integrable singularity at the point $t = 0$. As to the relaxation equation with the nth level fractional derivative (11), its solution is given by the following theorem:

Theorem 3 ([24]). *The fractional relaxation equation*

$$(D_{nL}^{\alpha,(\gamma)}\, u)(t) = -\lambda\, u(t), \; \lambda > 0, \; t > 0 \tag{34}$$

subject to the initial conditions

$$\left(\prod_{i=k+1}^{n} \left(I_{0+}^{\gamma_i} \frac{d}{dt} \right) I_{0+}^{n-\alpha-s_n}\, u \right)(0) \;=\; u_k, \; k = 1, \dots, n \tag{35}$$

has a unique solution, continuous and infinitely differentiable on $\mathbb{R}_+$, given by the formula

$$u(t) = \sum_{k=1}^{n} u_k\, t^{\alpha+s_k-k}\, E_{\alpha,\alpha+s_k-k+1}(-\lambda t^\alpha), \; s_k = \sum_{i=1}^{k} \gamma_i. \tag{36}$$

If the initial conditions are non-negative ($u_k \geq 0,\ k = 1, \ldots, n$ in (35)) and the inequalities

$$k - 1 \leq s_k = \sum_{i=1}^{k} \gamma_i,\ k = 1, \ldots, n \tag{37}$$

hold true, the solution (36) is completely monotone on $\mathbb{R}_+$.

In [10], an important probabilistic interpretation of Theorem 2 has been provided. Let $D(t)$ be a subordinator of the Lévy process with the Laplace exponent $\Psi = \Psi(s)$ ([25]):

$$\mathbf{E}\left[e^{-sD(t)}\right] = e^{-t\Psi(s)},$$

where Ψ is a Bernstein function ([20]) with the representation

$$\Psi(s) = bs + \int_0^{+\infty} (1 - e^{-s\tau})\Phi(d\tau),$$

where $b \geq 0$ is the drift coefficient and Φ is the Lévy measure, such that either $b > 0$, or $\Phi(\mathbb{R}_+) = +\infty$, or both.

Because the process D is strictly increasing, it possesses an inverse function

$$E(t) = \inf\{r > 0 :\ D(r) > t\}.$$

Now we consider a Poisson process $N(t)$ with the intensity λ. It is known that $N(E(t))$ is a renewal process with the waiting times J_n and

$$\mathbf{P}[J_n > t] = \mathbf{E}\left[e^{-\lambda E(t)}\right]. \tag{38}$$

As shown in [10], if the restriction $\Psi = \Psi(p),\ p > 0$ of the Laplace exponent $\Psi = \Psi(s)$ to the real semi-axes $\mathbb{R}_+$ is a complete Bernstein function that satisfies the conditions (K3) and (K4), then the right-hand side of Formula (38) can be interpreted as the solution to the Cauchy problem (30), (31). According to Theorem 2, it is continuous on $[0, +\infty)$ and completely monotone.

3.2. Fractional Growth Equation

In this subsection, some of the results derived in [13] are shortly addressed. We consider the fractional growth equation with the GFD of Caputo type

$$(\mathbb{D}_k^C u)(t) = \lambda u(t),\ \lambda > 0,\ t > 0 \tag{39}$$

subject to the initial condition

$$u(0) = 1. \tag{40}$$

In the case of the conventional growth equation, the solution is $u_\lambda(t) = \exp(\lambda\, t)$. The problem (39), (40) with the Caputo fractional derivative (7) is solved by the function $u_\lambda(t) = E_{\alpha,1}(\lambda t^\alpha)$ that is known to be of exponential growth as $t \to +\infty$ ([26]). It turns out that the solution to the problem (39), (40) with the GFD with a kernel $k \in \mathcal{K}$ (k fulfills the conditions (K1)–(K4)) is also of exponential growth as $t \to +\infty$.

For formulation of the corresponding result, the notation

$$\Phi(p) = p\,\tilde{k}(p)$$

is introduced, $\tilde{k}$ being the Laplace transform of k. Because $\tilde{k}$ is a Stieltjes function, Φ is a Bernstein function and Φ' is completely monotone ([20]). The made assumptions ensure that Φ is strictly monotone and thus for every $\lambda > 0$ there exists a unique $p_0 = p_0(\lambda)$ such that $\Phi(p_0) = \lambda$. Then we have the following result regarding the asymptotic of the unique solution u_λ of the Cauchy problem (39), (40) as $t \to +\infty$.

Theorem 4 ([13]). *Let $k \in \mathcal{K}$ and the condition*

$$\int_1^{+\infty} \frac{dp}{p\,\Phi(p)} < +\infty \tag{41}$$

hold true. Then the solution u_λ of the Cauchy problem (39), (40) has the following exponential asymptotic behavior:

$$u_\lambda(t) = \frac{\lambda}{\Phi'(p_0(\lambda))p_0(\lambda)}\, e^{p_0(\lambda)\,t} + o\left(e^{p_0(\lambda)\,t}\right),\ t \to +\infty. \tag{42}$$

In the case of the Cauchy problem (39), (40) with the Caputo derivative (7), the function Φ takes the form $\Phi(p) = p^\alpha$ and $p_0(\lambda) = \lambda^{1/\alpha}$. The condition (41) is evidently fulfilled and Formula (42) takes the form

$$u_\lambda(t) = \frac{1}{\alpha}\, e^{\lambda^{\frac{1}{\alpha}}\,t} + o\left(e^{\lambda^{\frac{1}{\alpha}}\,t}\right),$$

that corresponds to the main term of the asymptotic of the Mittag-Leffler function $u_\lambda(t) = E_{\alpha,1}(\lambda t^\alpha)$ as $t \to +\infty$ (see e.g., [26]).

Another important particular case is the Cauchy problem (39), (40) with the distributed order derivative (the GFD with the kernel (24)). It turns out that under some standard assumptions Theorem 4 is applicable also in this case (see [13] for details).

3.3. The Cauchy Problem for a Nonlinear Fractional ODE

Following [14], in this subsection, we address a nonlinear fractional differential equation with the GFD of the Caputo type

$$(\mathbb{D}_k^C u)(t) = f(t, u(t)),\ t > 0 \tag{43}$$

subject to the initial condition

$$u(0) = u_0 \in \mathbb{R}. \tag{44}$$

In what follows, we again assume that the kernel k of $\mathbb{D}_k^C$ defined by (21) belongs to the class $\mathcal{K}$, i.e., that the conditions (K1)–(K4) are fulfilled.

For derivation of results regarding existence and uniqueness of the solution to the Cauchy problem (43), (44), we first transform it into an integral equation by applying the GFI (26). Let $L > 0$ and $f : [0, L] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the Cauchy problem (43), (44) is equivalent to the integral equation

$$u(t) = u_0 + \int_0^t \kappa(t-\tau)\, f(\tau, u(\tau))\, d\tau,\ t > 0, \tag{45}$$

where the kernel function κ is determined by the relation (25).

The sufficient conditions for existence of the local and global solutions to the integral Equation (45) and thus to the Cauchy problem (43), (44) are provided in the following two theorems.

Theorem 5 ([14]). *Let $L,\ Q > 0$, f be a continuous function on the closed domain $G = \{(t,\tau) : 0 \le t \le L,\ |\tau - u_0| \le Q\}$, and $l \in (0, L]$ satisfy the inequality*

$$\max_{(t,\tau)\in G} |f(t,\tau)| \int_0^l \kappa(\tau)\, d\tau \le Q.$$

Then the Cauchy problem (43), (44) has a solution absolutely continuous on the interval $[0, l]$.

Theorem 6 ([14])**.** *Let $L > 0$ and $f : [0, L] \times \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies the inequality*

$$|f(t,\tau)| \leq b_0 + b_1|\tau|^p, \; 0 \leq t \leq L, \; \tau \in \mathbb{R}$$

with $b_0, b_1 > 0$ and $0 < p \leq 1$.

Then the Cauchy problem (43), (44) *has a solution absolutely continuous on the interval $[0, L]$.*

As a corollary from Theorem 6, we get the following result: Let $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies the inequality

$$|f(t,\tau)| \leq b_0 + b_1|\tau|^p, \; 0 \leq t < +\infty, \; \tau \in \mathbb{R}$$

with $b_0, b_1 > 0$ and $0 < p \leq 1$. Then the Cauchy problem (43), (44) has a global solution absolutely continuous on $[0, +\infty)$.

As to uniqueness of the solution, it was proved under some stronger conditions compared to the ones required for its existence.

Theorem 7 ([14])**.** *Let $L > 0$ and $f : [0, L] \times \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies the Lipschitz condition with respect to its second variable*

$$|f(t,s) - f(t,\tau)| \leq b|s-\tau|, \; 0 \leq t \leq L, \; s,\tau \in \mathbb{R}, \; b > 0. \tag{46}$$

Then the Cauchy problem (43), (44) *has a unique absolutely continuous solution on the interval $[0, L]$.*

As before, the result formulated in Theorem 7 can be extended to the case of the infinite interval $[0, +\infty)$: Let $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies the Lipschitz condition

$$|f(t,s) - f(t,\tau)| \leq b|s-\tau|, \; 0 \leq t < +\infty, \; s,\tau \in \mathbb{R}, \; b > 0.$$

Then the Cauchy problem (43), (44) has a unique absolutely continuous solution on the interval $[0, +\infty)$.

To address a continuous dependence of solutions to the Cauchy problem (43), (44) on the problem data, in [14], a Gronwall-type inequality was derived. It is important by itself and we formulate it below.

Lemma 1 ([14])**.** *Let $l, v_0, b > 0$ and $v \in C[0,l]$. If the inequality*

$$v(t) \leq v_0 + b\int_0^t \kappa(t-\tau)\, v(\tau)\, d\tau \tag{47}$$

holds true for $t \in [0, l]$, then

$$v(t) \leq u(t), \; t \in [0, l], \tag{48}$$

where $u = u(t)$ is the solution to the fractional relaxation equation

$$(\mathbb{D}_k^C u)(t) = -b\, u(t) \tag{49}$$

subject to the initial condition

$$u(0) = v_0. \tag{50}$$

Based on Lemma 1, in [14], the continuous dependence of the solution to the Cauchy problem (43), (44) on the problem data was proved.

Theorem 8 ([14]). *Let the conditions of Theorem 7 be fulfilled, u be the solution to the Cauchy problem (43), (44), and u_k be the solution to the Cauchy problem (43), (44) with the initial condition $u(0) = u_{0k}$. Then*

$$\|u - u_k\|_{C[0,L]} \to 0 \ \text{as} \ u_{0k} \to u_0.$$

Theorem 9 ([14]). *Let the conditions of Theorem 7 be fulfilled for the functions f and g, u be the solution to the Cauchy problem (43), (44), and v be the solution to the Cauchy problem (43), (44) with the right-hand side g and with the same initial condition u_0. Then*

$$\|u - v\|_{C[0,L]} \to 0 \ \text{as} \ \max_{(t,\tau)\in[0,L]\times[-\Omega,\Omega]} |f(t,\tau) - g(t,\tau)| \to 0,$$

where Ω is the upper bound of the supremum norm of the solution u.

4. Time-Fractional PDEs with the GFD

In this section, we present some important results concerning the direct and inverse problems for the time-fractional PDEs with the GFD of the Caputo type.

4.1. Cauchy Problem for the Time-Fractional Diffusion Equation

Following [10], in this subsection we address the properties of solutions to the Cauchy problem for the general time-fractional diffusion equation in the form

$$(\mathbb{D}_k^C u(x,\cdot))(t) = \Delta u(x,t) \ \ t > 0, \ x \in \mathbb{R}^n, \ u(x,0) = u_0(x), \tag{51}$$

where u_0 is a bounded globally Hölder continuous function on $\mathbb{R}^n$.

In [10], solutions to the Cauchy problem (51) were understood in the following sense: Applying formally the Laplace transform in the variable t to the equation in (51), we arrive at the equation

$$p\tilde{k}(p)\tilde{u}(x,p) - \tilde{k}(p)u_0(x) = \Delta\tilde{u}(x,p), \ p > 0, \ x \in \mathbb{R}^n \tag{52}$$

for the Laplace transform $\tilde{u}(x,p)$ of a solution to the Cauchy problem (51). A bounded function $u = u(x,t)$ is called an LT-solution of (51), if u is continuous in t on $[0, +\infty)$ uniformly with respect to $x \in \mathbb{R}^n$, satisfies the initial condition $u(x,0) = u_0(x)$, while its Laplace transform $\tilde{u}(x,p)$ is twice continuously differentiable in x, for each $p > 0$, and satisfies Equation (52).

Theorem 10 ([10]). *Let the kernel k of the GFD of the Caputo type belong to the class $\mathcal{K}$.*

Then there exist a non-negative function $Z = Z(x,t)$, $t > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, locally integrable in t and infinitely differentiable in $x \neq 0$ that satisfies the relation

$$\int_{\mathbb{R}^n} Z(x,t)\,dx = 1, \ t > 0, \tag{53}$$

and for any bounded globally Hölder continuous u_0, the function

$$u(x,t) = \int_{\mathbb{R}^n} Z(x-\zeta,t)\,u_0(\zeta)\,d\zeta \tag{54}$$

is an LT-solution to the Cauchy problem (51).

The function $Z = Z(x,t)$ is what is usually called the fundamental solution to the Cauchy problem (51), i.e., the one that formally corresponds to the initial condition $u(x,0) = u_0(x) = \delta(x)$, δ being the Dirac delta function. As stated in Theorem 10, the fundamental solution to the Cauchy problem (51) can be interpreted as a probability density function for each $t > 0$ and thus the time-fractional diffusion Equation (51) with the GFD could be potentially useful for modeling of the anomalous diffusion processes.

The explicit form of the fundamental solution is as follows ([10]):

$$Z(x,t) = \int_0^{+\infty} (4\pi s)^{-n/2} e^{-\frac{|x|^2}{4s}} G(s,t)\, ds,\ x \neq 0,$$

where

$$G(s,t) = \int_0^t k(t-\tau)\mu_s(d\tau)$$

with the probability measure $\mu_s(d\tau)$ that satisfies the relation

$$e^{-s p \tilde{k}(p)} = \int_0^{+\infty} e^{-p\tau} \mu_s(d\tau).$$

Under the conditions of Theorem 10, the measure $\mu_s(d\tau)$ always exists because the function $p \to e^{-sp\tilde{k}(p)}$ is completely monotone on $\mathbb{R}_+$ for each $s \geq 0$.

For the validity of Theorem 10, the condition $k \in \mathcal{K}$ is essential. However, it is worth mentioning that for the well-posedness of the Cauchy problem for the equations with the operators of type $\mathbb{D}_k^C$ much weaker conditions than (K1)–(K4) are sufficient (see e.g., [27]). Here we formulate the uniqueness result for the Cauchy problem (51) in the class of the LT-solutions proved in [10].

Theorem 11 ([10])**.** *Let the kernel k of the GFD of the Caputo type be non-negative, locally integrable, nonzero on a set of positive measure, and its Laplace transform $\tilde{k} = \tilde{k}(p)$ exist for all $p > 0$.*

If $u = u(x,t)$ is a polynomially bounded LT-solution to the Cauchy problem (51) with $u_0(x) \equiv 0$, then $u(x,t) \equiv 0$.

In the definition of the LT-solutions, their boundedness was required. However, this definition makes sense also for the polynomially bounded solutions, i.e., such solutions $u = u(x,t)$ that satisfy the inequality $|u(x,t)| \leq P_u(|x|)$, where P_u are some polynomials independent on t.

4.2. Initial-Boundary-Value Problems for the Time-Fractional Diffusion Equation

In this subsection, we present some results regarding the initial-boundary-value problems for the time-fractional diffusion equation with the GFD of the Caputo type in the form

$$(\mathbb{D}_k^C u(x,\cdot))(t) = D_2(u) + D_1(u) - q(x)u(x,t) + F(x,t),\ \ (x,t) \in \Omega \times (0,T], \tag{55}$$

subject to the initial condition

$$u(x,t)\big|_{t=0} = u_0(x),\ x \in \bar{\Omega} \tag{56}$$

and the boundary condition

$$u(x,t)\big|_{(x,t)\in\partial\Omega\times(0,T]} = v(x,t),\ (x,t) \in \partial\Omega \times (0,T]. \tag{57}$$

In Equations (55)–(57), Ω is a bounded open domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $q \in C(\bar{\Omega})$, $q(x) \geq 0$ for $x \in \bar{\Omega}$, and

$$D_1(u) = \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i},\ D_2(u) = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}. \tag{58}$$

Moreover we assume that D_2 is a uniformly elliptic differential operator.

A function $u \in S(\Omega, T)$ is called a strong solution to the initial-boundary-value problem (55)–(57) if it satisfies both Equation (55) and the initial and boundary conditions (56) and (57), respectively.

By $S(\Omega, T)$, we denoted the space of functions $u = u(x,t),\ (x,t) \in \bar{\Omega} \times [0,T]$ that satisfy the inclusions $u \in C(\bar{\Omega} \times [0,T])$, $u(\cdot,t) \in C^2(\Omega)$ for any $t > 0$, and $\partial_t u(x,\cdot) \in C(0,T] \cap L^1(0,T)$ for any $x \in \Omega$.

First we discuss a maximum principle for the time-fractional diffusion Equation (55). For its validity, we assume that the following conditions for the kernel k of $\mathbb{D}_k^C$ are satisfied:

(LY1) $k \in C^1(\mathbb{R}_+) \cap L^1_{loc}(\mathbb{R}_+)$,

(LY2) $k(\tau) > 0$ and $k'(\tau) < 0$ for $\tau > 0$,

(LY3) $k(\tau) = o(\tau^{-1}),\ \tau \to 0$.

Let us note that the Kochubei's conditions (K1)–(K4) are not needed for validity of the maximum principle for the general diffusion Equation (55). However, if the condition (K3) holds true, then it follows from the Feller-Karamata Tauberian theorem for the Laplace transform ([21]) that the condition (LY3) is also satisfied.

The maximum principle for the general diffusion Equation (55) is based on an appropriate estimate of the GFD of a function f at its maximum point. It is given in the following theorem.

Theorem 12 ([28])**.** *Let the conditions (LY1)–(LY3) be fulfilled, a function $f \in C[0,T]$ attain its maximum over the interval $[0,T]$ at the point $t_0,\ t_0 \in (0,T]$, and $f' \in C(0,T] \cap L^1(0,T)$.*

Then the inequality

$$(\mathbb{D}_k^C f)(t_0) \geq k(t_0)(f(t_0) - f(0)) \geq 0 \tag{59}$$

holds true.

In the case of the Caputo fractional derivative, the inequality (59) takes the known form

$$(D_C^\alpha f)(t_0) \geq \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)}(f(t_0) - f(0)) \geq 0. \tag{60}$$

In what follows, we use the notation

$$\mathbb{P}_k(u) := (\mathbb{D}_k^C u)(t) - D_2(u) - D_1(u) + q(x)u(x,t). \tag{61}$$

Theorem 13 ([28])**.** *Let the conditions (LY1)–(LY3) be fulfilled and a function $u \in S(\Omega, T)$ satisfy the inequality*

$$\mathbb{P}_k(u) \leq 0,\ (x,t) \in \Omega \times (0,T]. \tag{62}$$

Then the following maximum principle holds true:

$$\max_{(x,t)\in\,\bar{\Omega}\times[0,T]} u(x,t) \leq \max\{\max_{x\in\bar{\Omega}} u(x,0),\ \max_{(x,t)\in\,\partial\Omega\times[0,T]} u(x,t),\ 0\}. \tag{63}$$

The maximum principle formulated in Theorem 13 can be applied, among other things, for derivation of some a priori estimates for the strong solutions of the initial-boundary-value problem (55)–(57).

Theorem 14 ([28])**.** *Let the conditions (K1)–(K4) and (LY1)–(LY3) be fulfilled and u be a strong solution to the initial-boundary-value problem (55)–(57).*

Then

$$\|u\|_{C(\bar{\Omega}\times[0,T])} \leq \max\{M_0,\ M_1\} + M\,f(T), \tag{64}$$

where

$$M_0 = \|u_0\|_{C(\bar{\Omega})},\ M_1 = \|v\|_{C(\partial\Omega\times[0,T])},\ M = \|F\|_{C(\Omega\times[0,T])}, \tag{65}$$

and

$$f(t) = \int_0^t \kappa(\tau)\,d\tau, \tag{66}$$

where the function κ is the kernel of the GFI defined by (25).

The uniqueness of the strong solution to the initial-boundary-value problem (55)–(57) and its continuous dependence on problem data easily follow from the solution norm estimate (64).

Theorem 15 ([28])**.** *The initial-boundary-value problem (55)–(57) possesses at most one strong solution.*

This solution—if it exists—continuously depends on the problem data in the sense that if u and $\tilde{u}$ are strong solutions to the problems with the sources functions F and $\tilde{F}$ and the initial and boundary conditions u_0 and $\tilde{u}_0$ and v and $\tilde{v}$, respectively, and

$$\|F-\tilde{F}\|_{C(\bar{\Omega}\times[0,T])} \le \epsilon,$$

$$\|u_0-\tilde{u}_0\|_{C(\bar{\Omega})} \le \epsilon_0,\ \|v-\tilde{v}\|_{C(\partial\Omega\times[0,T])} \le \epsilon_1,$$

then the norm estimate

$$\|u-\tilde{u}\|_{C(\bar{\Omega}\times[0,T])} \le \max\{\epsilon_0,\epsilon_1\} + \epsilon\, f(T) \tag{67}$$

holds true, where the function f is defined by (66).

In the rest of this subsection, we address uniqueness and existence of a weak solution to the initial-boundary-value problem (55)–(57) defined in the sense of Vladimirov [16]: We call $u \in C(\bar{\Omega}\times[0,T])$ a weak solution to the initial-boundary-value problem (55)–(57) in the sense of Vladimirov, if there exist $F_k \in C(\bar{\Omega}\times[0,T])$, $u_{0k} \in C(\bar{\Omega})$ and $v_k \in C(\partial\Omega\times[0,T])$, $k = 1,2,\ldots$ satisfying (V1) and (V2) below such that

$$\|u_k-u\|_{C(\bar{\Omega}\times[0,T])} \to 0 \ \text{ as } \ k\to+\infty. \tag{68}$$

(V1) There exist the functions F, u_0, and v, such that

$$\|F_k-F\|_{C(\bar{\Omega}\times[0,T])} \to 0 \ \text{ as } \ k\to+\infty, \tag{69}$$

$$\|u_{0k}-u_0\|_{C(\bar{\Omega})} \to 0 \ \text{ as } \ k\to+\infty, \tag{70}$$

$$\|v_k-v\|_{C(\partial\Omega\times[0,T])} \to 0 \ \text{ as } \ k\to+\infty. \tag{71}$$

(V2) For each $k = 1,2,\ldots$ there exists a strong solution $u_k = u_k(x,t)$ to the general time-fractional diffusion equation

$$(\mathbb{D}_k^C u_k(x,\cdot))(t) = D_2(u_k) + D_1(u_k) - q(x)u_k(x,t) + F_k(x,t),\ (x,t)\in\Omega\times(0,T]. \tag{72}$$

subject to the initial condition

$$u_k\big|_{t=0} = u_{0k}(x),\ x\in\bar{\Omega} \tag{73}$$

and the boundary condition

$$u_k\big|_{\partial\Omega\times(0,T]} = v_k(x,t),\ (x,t)\in\partial\Omega\times(0,T]. \tag{74}$$

In [28], the correctness of the definition of a weak solution was shown. A weak solution to the problem (55)–(57) in the sense of Vladimirov is a continuous function, not a distribution. However, the weak solutions are not required to be smooth.

Any strong solution to the problem (55)–(57) is evidently also its weak solution. If the problem (55)–(57) possesses a weak solution, then the functions F, u_0 and v from the problem formulation have to belong to the spaces $C(\bar{\Omega}\times[0,T])$, $C(\bar{\Omega})$ and $C(\partial\Omega\times[0,T])$, respectively, as the limits of the sequences of the continuous functions in the uniform norm.

The estimate (64) for the strong solutions holds true also for the weak solutions. To show this, we just let $k \to +\infty$ in the inequality

$$\|u_k\|_{C(\bar{\Omega}_T)} \le \max\{M_{0k}, M_{1k}\} + M_k f(T),\ k = 1, 2 \dots \tag{75}$$

with

$$M_{0k} := \|u_{0k}\|_{C(\bar{\Omega})},\ M_{1k} := \|v_k\|_{C(\partial\Omega\times[0,T])},\ M_k := \|F_k\|_{C(\bar{\Omega}\times[0,T])}.$$

The estimate (64) for the weak solutions is employed to prove the following uniqueness result.

Theorem 16 ([28])**.** *The initial-boundary-value problem* (55)–(57) *possesses at most one weak solution. The weak solution—if it exists—continuously depends on the data given in the problem in the sense of the estimate* (67)*.*

In the rest of this subsection, we address the question of existence of a weak solution to the initial-boundary-value problem (55)–(57) in the case of the homogeneous Equation (55) without the first order spatial differential operator D_1 subject to the initial condition (56) and the homogeneous boundary condition (57), i.e., we consider the initial-boundary-value problem

$$(\mathbb{D}_k^C u(x, \cdot))(t) = D_2(u) - q(x)u(x,t), \quad (x,t) \in \Omega \times (0, T], \tag{76}$$

$$u(x,t)\big|_{t=0} = u_0(x),\ x \in \bar{\Omega}, \tag{77}$$

$$u(x,t)\big|_{(x,t)\in\partial\Omega\times(0,T]} = 0,\ (x,t) \in \partial\Omega \times (0, T] \tag{78}$$

under the same conditions on the coefficients of the operator D_2 and the function q that we assumed at the beginning of the subsection. Moreover, we also assume that the kernel k of $\mathbb{D}_k^C$ is from $\mathcal{K}$, i.e., the conditions (K1)–(K4) are satisfied.

First, a formal solution to the initial-boundary-value problem (76)–(78) is constructed in form of the Fourier series

$$u(x,t) = \sum_{k=1}^{\infty} (u_0, X_k) U_k(t)\, X_k(x), \tag{79}$$

where X_k, $k = 1, 2, \dots$ are the eigenfunctions corresponding to the eigenvalues λ_k of the eigenvalue problem

$$L(X(x)) = \lambda\, X(x),\ x \in \Omega, \tag{80}$$

$$X(x)\big|_{x\in\partial\Omega} = 0 \tag{81}$$

for the operator L, $L(x) = -D_2(X) + q(x)X(x)$. Because of the conditions posed on the operator D_2 and the function q, the differential operator L is positive definite and self-adjoint. Thus the eigenvalue problem (80)–(81) has a countable number of the positive eigenvalues $0 < \lambda_1 \le \lambda_2 \le \dots$ with the finite multiplicities and—if the boundary $\partial\Omega$ of Ω is smooth—any function $f \in \mathcal{M}_L$ can be represented through its Fourier series in the form

$$f(x) = \sum_{k=1}^{\infty} (f, X_k)\, X_k(x), \tag{82}$$

where $X_k \in \mathcal{M}_L$ are the eigenfunctions corresponding to the eigenvalues λ_k:

$$L(X_k) = \lambda_k\, X_k,\ k = 1, 2, \dots. \tag{83}$$

By $\mathcal{M}_L$, the space of the functions f that satisfy the boundary condition (81) and the inclusions $f \in C^1(\bar{\Omega}) \cap C^2(\Omega)$, $L(f) \in L^2(\Omega)$ is denoted.

As to the functions $U_k = U_k(t)$, they are solutions to the fractional relaxation equations

$$(\mathbb{D}_k^C U_k)(t) = -\lambda_k U_k(t),\ t > 0,\ k = 1, 2, \ldots \tag{84}$$

subject to the initial conditions

$$U_k(0) = 1,\ k = 1, 2, \ldots. \tag{85}$$

According to Theorem 2 from Section 3, for any $\lambda = \lambda_k > 0$, $k = 1, 2, \ldots$ this initial-value problem has a unique solution $U_k = U_k(t)$ that belongs to the class $C^\infty(\mathbb{R}_+)$ and is a completely monotone function. In particular, any U_k is non-negative and non-increasing and thus the inequalities

$$0 \le U_k(t) \le U_k(0) = 1 \tag{86}$$

hold true. Let us mention that in the case of the single-term time-fractional diffusion equation with the Caputo fractional derivative ($k(\tau) = \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)}$, $0 < \alpha < 1$), the solution to the initial-value problem (84), (85) with $\lambda = \lambda_k$, $k = 1, 2, \ldots$ has the form ([29])

$$U_k(t) = E_{\alpha,1}(-\lambda_k t^\alpha). \tag{87}$$

Under some standard assumptions, the formal solution (79) is a weak solution to the initial-boundary-value problem (76)–(78) in the sense of Vladimirov.

Theorem 17 ([15])**.** *Let the function u_0 in the initial condition (77) be from the space $\mathcal{M}_L$. Then the formal solution (79) of the problem (76)–(78) is its weak solution in the sense of Vladimirov.*

For a survey of other results regarding the maximum principles for the time-fractional PDEs of different types see the recent publication [30].

4.3. Inverse Problems Involving GFD

The starting point of this subsection is a reconstruction problem for a function based on its values and the values of its GFD in a neighborhood of the final time ([19]):

IP1. Let $0 < t_0 < T < +\infty$. Given ϕ, $g : (t_0, T) \to \mathbb{R}$, find a function $u : (0, T) \to \mathbb{R}$ such that

$$u(t)|_{(t_0,T)} = \phi(t), \text{ and } (\mathbb{D}_k^C\, u)(t)|_{(t_0,T)} = g(t). \tag{88}$$

The inverse problems of type IP1 are potentially useful for applications. For instance, in the framework of the Scott-Blair model of viscoelasticity, the stress is proportional to a time-fractional derivative of the strain ([23]). In this context, the IP1 means a reconstruction of the strain history based on the measurements of strain and stress starting from a certain time t_0.

In [19], a uniqueness result for the IP1 was proved under the following conditions on the kernel k of $\mathbb{D}_k^C$:

(KJ1) $\exists \mu \in \mathbb{R} : \int_0^{+\infty} e^{-\mu t} |k(t)|\, dt < +\infty$,
(KJ2) k is real analytic on $\mathbb{R}_+$,
(KJ3) the Laplace transform $\tilde{k}$ of k cannot be meromorphically extended to the whole complex plane $\mathbb{C}$.

Theorem 18 ([19])**.** *Let the kernel k of $\mathbb{D}_k^C$ fulfill the conditions (KJ1)–(KJ3). Then the following uniqueness results for the IP1 hold true:*

(i) If $u \in L^1(0, T)$, $k * u \in W^{1,1}(0, T)$, and $u(t)|_{(t_0,T)} = (\mathbb{D}_k^C\, u)(t)|_{(t_0,T)} = 0$, then $u = 0$,
(ii) If $u \in W^{1,1}(0, T)$ and $u(t)|_{(t_0,T)} = (\mathbb{D}_k^C\, u)(t)|_{(t_0,T)} = 0$, then $u = 0$.

The results formulated in Theorem 18 were employed in [19] for studying uniqueness of solution to the following source reconstruction problem for the fractional PDEs with the GFD of Caputo type:

IP2. Let $0 < t_0 < T < +\infty$ and $\Omega \subseteq \mathbb{R}^n$. Given $\phi, \Phi : \Omega \times (t_0, T) \to \mathbb{R}$, find the functions $u,\ F : \Omega \times (0, T) \to \mathbb{R}$ such that they fulfill the equation

$$(\mathbb{D}_k^C B(u))(t) + D^l(u) - A(u) = F(x,t),\ \ x \in \Omega,\ t \in (0,T) \tag{89}$$

and the relations

$$u(x,t)|_{\Omega\times(t_0,T)} = \phi,\ \text{ and }\ F|_{\Omega\times(t_0,T)} = \Phi \tag{90}$$

hold true.

In the formulation of IP2, $D^l = \sum_{j=1}^{l} q_j \frac{\partial^j}{\partial t^j}$ is a differential operator of order l with respect to the time variable t and with $q_j \in \mathbb{R}$ and A and B are some operators that act with respect to the spatial variable x. Moreover, we assume that $\mathcal{D}(A) \subseteq C(\Omega) \to C(\Omega)$, $\mathcal{D}(B) \subseteq C(\Omega) \to C(\Omega)$ and the operator B is invertible.

In particular, the time-fractional PDE (89) includes the time-fractional diffusion Equations (51) and (55) that were considered in the previous subsections of this section.

As shown in [19], IP2 can be reduced to IP1. Indeed, let the pair of functions (u, F) solve the IP2. The Equation (89) restricted to $\Omega \times (t_0, T)$ has the form $(\mathbb{D}_k^C B(u))(t) + D^l(\phi) - A(\phi) = \Phi(x,t)$ and thus the function Bu is a solution to the following inverse problem of IP1 type:

$$Bu|_{\Omega\times(t_0,T)} = B\phi,\ \text{ and }\ \mathbb{D}_k^C\, Bu|_{\Omega\times(t_0,T)} = g, \tag{91}$$

where

$$g(x,t) = \Phi(x,t) + A(\phi) - D^l(\phi),\ x \in \Omega,\ t \in (t_0,T).$$

The solution (u, F) of IP2 can be explicitly expressed in terms of the solution Bu of the IP1 formulated above as follows: $u = B^{-1} Bu,\ F = \mathbb{D}_k^C\, Bu + D^l(u) - A(u)$. Accordingly, a uniqueness result for the IP2 immediately follows from Theorem 18 (see [19] for details).

It is worth mentioning that the inverse problems IP1 and IP2 are severely ill-posed ([19]) and thus appropriate regularization methods are needed for their numerical treatment.

Next, we consider the following evolutionary integral equation:

$$u(x,t) = \int_0^t \kappa(t-\tau)\Delta u(x,\tau)\, d\tau\ +\ f(x,t),\ x \in \mathbb{R}^n,\ t \geq 0. \tag{92}$$

Please note that the Cauchy problem (compare to (51))

$$(\mathbb{D}_k^C u(x,\cdot))(t) = \Delta u(x,t)\ +\ F(x,t)\ \ t > 0,\ x \in \mathbb{R}^n,\ u(x,0) = u_0(x) \tag{93}$$

can be reduced to an evolutionary integral equation of type (92) by applying the GFI (26) to both sides of this equation and by using Formula (28) from Theorem 1:

$$u(x,t) = \int_0^t \kappa(t-\tau)\Delta u(x,\tau)\, d\tau + (\mathbb{I}_k^C\, F(x,\cdot))(t) + u_0(x),\ x \in \mathbb{R}^n,\ t \geq 0, \tag{94}$$

where the kernel κ is connected with the kernel k of $\mathbb{D}_k^C$ by means of the relation (25).

In [17], an important inverse problem of kernel identification in the boundary value problems for an equation associated with the evolutionary integral Equation (92) was addressed. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. The direct boundary value problem formulated in [17] is as follows:

$$\begin{cases} u(x,t) = \int_0^t \kappa(t-\tau)\Delta u(x,\tau)\,d\tau + f(x,t), & x \in \Omega,\ t \in [0,T], \\ \mathcal{B}u(x,t) = 0, & x \in \partial\Omega,\ t \in [0,T], \end{cases} \tag{95}$$

where $\mathcal{B}$ is a boundary operator of the Dirichlet, Neumann, or Robin type, respectively:

$$\mathcal{B}\nu(x) = \nu(x),\ \mathcal{B}\nu(x) = n(x)\cdot\nabla\nu(x),\ \mathcal{B}\nu(x) = n(x)\cdot\nabla\nu(x) + \theta\nu(x),\ \theta \geq 0,$$

$n(x)$ being the unit outer normal of $\partial\Omega$ at the point $x \in \Omega$.

The inverse problem addressed in [17] is formulated via the so called observation functional Φ that maps the functions defined on $\overline{\Omega}$ onto $\mathbb{R}$. Usually, the functional Φ is defined in one of the following ways:

$$\Phi[\nu] = \nu(x_0),\ \ \Phi[\nu] = n(x_0)\cdot\nabla\nu(x_0),\ \ \Phi[\nu] = \int_\Omega \mu(x)\,\nu(x)\,dx,$$

where $x_0 \in \overline{\Omega}$ and $\mu : \Omega \to \mathbb{R}$ are given. In the case $x_0 \in \partial\Omega$, the observation functional has to be different from the boundary operator, i.e., $\Phi[\nu] \neq \mathcal{B}\nu(x_0)$.

The inverse problem considered in [17] is as follows:

IP3. Given $h : (0,T) \to \mathbb{R}$ find a kernel κ such that the solution u of the boundary value problem (95) satisfies the condition

$$\Phi[u(t,\cdot)] = h(t),\ t \in (0,T). \tag{96}$$

In [17], existence, uniqueness, and stability of solutions to the IP3 were studied for a certain class of kernels (see [17] for details).

Finally, we mention that in [18] two other inverse problems for a time-fractional PDE with the GFD of Caputo type were addressed. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial\Omega$. The direct initial-boundary-value problem is formulated as follows:

$$\begin{cases} \frac{d}{dt}(k*(U-\Phi))(x,t) = L_x U(x,t) + H(x,t),\ x \in \Omega,\ t \in (0,T), \\ U(x,0) = \Phi(x),\ x \in \Omega, \\ \mathcal{B}(U-b)(x,t) = 0,\ x \in \partial\Omega,\ t \in (0,T), \end{cases} \tag{97}$$

where Φ and b are given functions, the operator L_x is a linear second order differential operator with respect to the variable x in the form

$$L_x U(x,t) = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 U}{\partial x_i \partial x_j} + \sum_{j=1}^n a_j(x)\frac{\partial U}{\partial x_j} + r(x)U(x,t),$$

and $\mathcal{B}$ is a boundary operator of the Dirichlet or Neumann type, respectively:

$$\mathcal{B}v(x) = v(x)\ \text{ or }\ \mathcal{B}v(x) = \omega(x)\cdot\nabla v(x)$$

with $\omega \cdot n > 0$, n being the unite outer normal of $\partial\Omega$ at the point $x \in \Omega$.

The inverse problems considered in [18] are formulated in terms of the given observation function Φ at the final time T in the form

$$U(x,T) = \Phi(x),\ x \in \Omega. \tag{98}$$

In [18], the following inverse problems were addressed:

IP4. (inverse source problem). Let

$$H(x,t) = g(x,t)f(x) + h_0(x,t), \tag{99}$$

where the components gf and h_0 may correspond to different sources or sinks. The factor f is unknown and has to be reconstructed by means of the data given by the observation function Φ from (98). The inverse problem consists in determination of a pair of functions (f, U) that satisfies (97), (98), and (99).

Another inverse problem considered in [18] is determination of the coefficient r in the operator $L_x\, U$:

IP5. Determine a pair of functions (r, U) that satisfies (97) and (98).

In [18], existence, uniqueness, and stability of solutions to IP4 and to IP5 were shown under some additional conditions posed on the problem data (see [18] for details).

For the surveys of the recent results concerning the inverse problems for the fractional PDEs including different kinds of the conventional fractional derivatives we refer the readers to [31–33].

Finally we mention that the theory of the fractional PDEs with the GFDs is still far away from being completed. In particular, the regularity of their solutions is not yet investigated in detail. Another interesting problem for further research would be to address the abstract fractional evolution equation in the form

$$(\mathbb{D}_k^C u)(t) = Au(t) \tag{100}$$

subject to the initial condition $u(0) = x$. In (100), A stands for a linear closed unbounded operator densely defined in a Banach space X and the initial condition x belongs to the space X. This problem will be considered elsewhere.

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