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Faber Polynomial Coefficient Estimates for Bi-Univalent Functions Defined by Using Differential Subordination and a Certain Fractional Derivative Operator

Hari M. Srivastava [1,2,3,*], Ahmad Motamednezhad [4] and Ebrahim Analouei Adegani [4]

1 Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada
2 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
3 Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan
4 Faculty of Mathematical Sciences, Shahrood University of Technology, P. O. Box 36155-316, Shahrood 36155-316, Iran; a.motamedne@gmail.com (A.M.); analoey.ebrahim@gmail.com (E.A.A.)
* Correspondence: harimsri@math.uvic.ca

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Abstract: In this article, we introduce a general family of analytic and bi-univalent functions in the open unit disk, which is defined by applying the principle of differential subordination between analytic functions and the Tremblay fractional derivative operator. The upper bounds for the general coefficients $|a_n|$ of functions in this subclass are found by using the Faber polynomial expansion. We have thereby generalized and improved some of the previously published results.

Keywords: analytic functions; univalent functions; bi-univalent functions; coefficient estimates; Taylor-Maclaurin coefficients; Faber polynomial expansion; differential subordination; Tremblay fractional derivative operator

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1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ be a class of functions of the following (normalized) form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are assumed to be analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Further, let $\mathcal{S}$ denote the subclass of functions contained in the class $\mathcal{A}$ of normalized analytic functions in $\mathbb{U}$, which are univalent in $\mathbb{U}$.

We recall the well-established fact that every function $f \in \mathcal{S}$ possesses its inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$\begin{aligned} g(w) = f^{-1}(w) &= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \\ &=: w + \sum_{n=2}^{\infty} A_n w^n. \end{aligned} \tag{2}$$

Given a function $f \in \mathcal{A}$, we say that f bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. We denote by Σ the class of functions $f \in \mathcal{A}$, which are bi-univalent in $\mathbb{U}$ and have the Taylor-Maclaurin series expansion given by (1). In the year 1967, Lewin [1] studied the bi-univalent function class Σ and derived the bound for the second Taylor-Maclaurin coefficient $|a_2|$ in (1).

The interested reader can find a brief historical overview of functions in the class Σ in the work of Srivastava et al. [2], which actually revised the study of the bi-univalent function class Σ, as well as in the references cited therein. Bounds for the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of various subclasses of bi-univalent functions were obtained in a number of sequels to [2] including (among others) [3–12]. As a matter of fact, considering the remarkably huge amount of papers on the subject, the pioneering work by Srivastava et al. [2] appears to have successfully revived the study of analytic and bi-univalent functions in recent years.

The coefficient estimate problem for each of the Taylor-Maclaurin coefficients $|a_n| \quad (n \geqq 4)$ is presumably still an open problem for a number of subclasses of the bi-univalent function class Σ. Nevertheless, in some specific subclasses of the bi-univalent function class Σ, such general coefficient estimate problems were considered by several authors by employing the Faber polynomial expansions under certain conditions (see, for example, [13–33]). Here, in our present investigation of general coefficient expansion problems, we begin by recalling several definitions, lemmas and other preliminaries which are needed in this paper.

Historically, the Faber polynomials were introduced by Georg Faber (1887–1966) (see [34,35]). It has played and it continues to play an important rôle in various areas of mathematical sciences, especially in Geometric Function Theory of Complex Analysis (see, for example, [36]). If we make use of the Faber polynomial expansion of functions $f \in \mathcal{S}$ of the form given by (1), the Taylor-Maclaurin coefficients of its inverse map $g = f^{-1}$ are expressible as follows (see, for details, [37,38]):

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n) w^n, \tag{3}$$

where

$$\begin{aligned} K_{n-1}^{-n} &:= K_{n-1}^{-n}(a_2, a_3, \cdots, a_n) \\ &= \frac{(-n)!}{(-2n+1)!(n-1)!}\, a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!}\, a_2^{n-3}\, a_3 \\ &\quad + \frac{(-n)!}{(-2n+3)!(n-4)!}\, a_2^{n-4} a_4 + \frac{(-n)!}{(2(-n+2))!(n-5)!}\, a_2^{n-5} \\ &\qquad \cdot [a_5 + (-n+2)a_3^2] + \frac{(-n)!}{(-2n+5)!(n-6)!}\, a_2^{n-6}[a_6 + (-2n+5)a_3 a_4] \\ &\quad + \sum_{j \geqq 7} a_2^{n-j} V_j \end{aligned}$$

such that $V_j \quad (7 \leqq j \leqq n)$ is a homogeneous polynomial in the variables $a_2, a_3, \cdots, a_n$ and expressions such as (for example) $(-n)!$ are *symbolically* interpreted as follows:

$$(-n)! \equiv \Gamma(1-n) := (-n)(-n-1)(-n-2)\cdots \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad (\mathbb{N} := \{1, 2, 3, \cdots\})).$$

In particular, the first three terms of K_{n-1}^{-n} are given by

$$K_1^{-2} = -2a_2, \quad K_2^{-3} = 3\left(2a_2^2 - a_3\right) \quad \text{and} \quad K_3^{-4} = -4\left(5a_2^3 - 5a_2a_3 + a_4\right).$$

In general, for any $p \in \mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$, an expansion of K_n^p is given below (see, for details, [36,39]; see also [37,38,40] (p. 349))

$$K_n^p = pa_{n+1} + \frac{p(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!\,3!} D_n^3 + \cdots + \frac{p!}{(p-n)!\,n!} D_n^n,$$

where (see, for details, [30,40])

$$D_n^p = D_n^p(a_2, a_3, \cdots).$$

We also have

$$D_n^m(a_2, a_3, \cdots, a_{n+1}) = \sum \frac{m!(a_2)^{\mu_1} \cdots (a_{n+1})^{\mu_n}}{\mu_1! \cdots \mu_n!}, \tag{4}$$

where the sum is taken over all nonnegative integers $\mu_1, \cdots, \mu_n$ satisfying the following conditions:

$$\begin{cases} \mu_1 + \mu_2 + \cdots + \mu_n = m \\ \\ \mu_1 + 2\mu_2 + \cdots + n\mu_n = n. \end{cases}$$

It is clear that

$$D_n^n(a_2, a_3, \cdots, a_{n+1}) = a_2^n.$$

Definition 1. (see [41]) *For two functions f and g, which are analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$ and write*

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function $\omega(z)$ which, by definition, is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \qquad (z \in \mathbb{U})$$

such that

$$f(z) = g(\omega(z)) \qquad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, then

$$f \prec g \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subseteq g(\mathbb{U}).$$

Ma and Minda [42] unified various subclasses of starlike and convex functions for which either of the quantities

$$\frac{zf'(z)}{f(z)} \qquad \text{and} \qquad 1 + \frac{zf''(z)}{f(z)}$$

is subordinate by a general superordinate function. For this purpose, they considered an analytic function with positive real part in the unit disk $\mathbb{U}$ for which

$$\varphi(0) = 1 \qquad \text{and} \qquad \varphi'(0) > 0$$

and which maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis.

Lemma 1. (see [41]) *Let $u(z)$ be analytic in the unit disk $\mathbb{U}$ with*

$$u(0) = 0 \qquad \text{and} \qquad |u(z)| < 1$$

and suppose that

$$u(z) = \sum_{n=1}^{\infty} p_n z^n \qquad (z \in \mathbb{U}). \tag{5}$$

Then

$$|p_n| \leqq 1 \qquad (n \in \mathbb{N}).$$

Lemma 2. (see [21]) *Let*

$$\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n \in \mathcal{A}$$

be a Schwarz function so that $|\omega(z)| < 1$ *for* $|z| < 1$. *If* $\gamma \geqq 0$, *then*

$$\left|\omega_2 + \gamma\omega_1^2\right| \leqq 1 + (\gamma - 1)\left|\omega_1^2\right|.$$

Definition 2. (see [43,44]) *For a function f, the fractional integral of order γ is defined by*

$$D_z^{-\gamma} f(z) = \frac{1}{\Gamma(\gamma)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\gamma}} d\xi \qquad (\gamma > 0),$$

where $f(z)$ is an analytic function in a simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-\xi)^{\gamma-1}$ is removed by requiring $\log(z-\xi)$ *to be real when $z - \xi > 0$.*

Definition 3. (see [43,44]) *For a function f, the fractional derivative of order γ is defined by*

$$D_z^{\gamma} f(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\gamma}} d\xi \qquad (0 \leqq \gamma < 1),$$

where the function $f(z)$ is constrained, and the multiplicity of $(z-\xi)^{-\gamma}$ is removed, as in Definition 2.

Definition 4. (see [43,44]) *Under the hypotheses of Definition 3, the fractional derivative of order $n+\gamma$ is defined by*

$$D_z^{n+\gamma} f(z) = \frac{d^n}{dz^n} \left\{ D_z^{\gamma} f(z) \right\} \qquad (0 \leqq \gamma < 1;\ n \in \mathbb{N}_0).$$

As consequences of Definitions 2–4, we note that

$$D_z^{-\gamma} z^n = \frac{\Gamma(n+1)}{\Gamma(n+\gamma+1)}\, z^{n+\gamma} \qquad (n \in \mathbb{N};\ \gamma > 0)$$

and

$$D_z^{\gamma} z^n = \frac{\Gamma(n+1)}{\Gamma(n-\gamma+1)}\, z^{n-\gamma} \qquad (n \in \mathbb{N};\ 0 \leqq \gamma < 1).$$

Definition 5. (see [45]) *The Tremblay fractional derivative operator $\mathfrak{T}_z^{\mu,\gamma}$ of a function $f \in \mathcal{A}$ is defined, for all $z \in \mathbb{U}$, by*

$$\mathfrak{T}_z^{\mu,\gamma} f(z) = \frac{\Gamma(\gamma)}{\Gamma(\mu)}\, z^{1-\gamma} D_z^{\mu-\gamma} z^{\mu-1} f(z) \qquad (0 < \gamma;\ \mu \leqq 1;\ \mu > \gamma;\ 0 < \mu - \gamma < 1).$$

It is clear from Definition 5 that, for $\mu = \gamma = 1$, we have $\mathfrak{T}_z^{1,1} f(z) = f(z)$ and we can easily see that

$$\mathfrak{T}_z^{\mu,\gamma} f(z) = \frac{\mu}{\gamma}\, z + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma)\Gamma(n+\mu)}{\Gamma(\mu)\Gamma(n+\gamma)}\, a_n z^n.$$

The purpose of our study is to make use of the Faber polynomial expansion in order to obtain the upper bounds for the general Taylor-Maclaurin coefficients $|a_n|$ of functions in a new subclass of Σ, which is defined by the principle of differential subordination between analytic functions in the open unit disk $\mathbb{U}$. We also show that our main results and their corollaries and consequences would generalize and improve some of the previously published results. Moreover, with a view to potentially motivate the interested reader, we choose to include a citation of a very recent survey-cum-expository article [46], which also provides a review of many other related recent works in Geometric Function Theory of Complex Analysis.

2. A Set of Main Results

We begin this section by assuming that φ is an analytic function with positive real part in the unit disk $\mathbb{U}$, which satisfies the following conditions:

$$\varphi(0) = 1 \qquad \text{and} \qquad \varphi'(0) > 0$$

and is so constrained that $\varphi(\mathbb{U})$ is symmetric with respect to the real axis. Such a function has series expansion of the form:

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \qquad (B_1 > 0).$$

We now introduce the general subclass $\mathcal{A}_\Sigma(\lambda, \gamma, \mu, \varphi)$.

Definition 6. *For $0 \leqq \lambda \leqq 1$, $0 < \gamma, \mu \leqq 1$, $\mu > \gamma$ and $0 < \mu - \gamma < 1$, a function $f \in \Sigma$ is said to be in the subclass $\mathcal{A}_\Sigma(\lambda, \gamma, \mu; \varphi)$ if the following subordination conditions hold true*:

$$(1-\lambda)\,\frac{\gamma \mathfrak{T}_z^{\mu,\gamma} f(z)}{\mu z} + \lambda\,\frac{\gamma\left(\mathfrak{T}_z^{\mu,\gamma} f(z)\right)'}{\mu} \prec \varphi(z) \qquad (z \in \mathbb{U})$$

and

$$(1-\lambda)\frac{\gamma \mathfrak{T}_w^{\mu,\gamma} g(w)}{\mu w} + \lambda\frac{\gamma\left(\mathfrak{T}_w^{\mu,\gamma} g(w)\right)'}{\mu} \prec \varphi(w) \qquad (w \in \mathbb{U}),$$

where $g = f^{-1}$ is given by (2).

Theorem 1 below gives an upper bound for the coefficients $|a_n|$ of functions in the subclass $\mathcal{A}_\Sigma(\lambda, \gamma, \mu; \varphi)$.

Theorem 1. *For $0 \leqq \lambda \leqq 1$, $0 < \gamma, \mu \leqq 1$, $\mu > \gamma$ and $0 < \mu - \gamma < 1$, let the function $f \in \mathcal{A}_\Sigma(\lambda, \gamma, \mu; \varphi)$ be given by* (1). *If $a_k = 0$ for $2 \leqq k \leqq n-1$, then*

$$|a_n| \leqq \frac{B_1 \Gamma(\mu+1)\Gamma(n+\gamma)}{[1+\lambda(n-1)]\Gamma(\gamma+1)\Gamma(n+\mu)} \qquad (n \geqq 3)\,. \tag{6}$$

Proof. For f given by (1), we have

$$\begin{aligned}(1-\lambda)\,&\frac{\gamma \mathfrak{T}_z^{\mu,\gamma} f(z)}{\mu z} + \lambda\,\frac{\gamma\left(\mathfrak{T}_z^{\mu,\gamma} f(z)\right)'}{\mu} \\ &= 1 + \sum_{n=2}^{\infty} [1+\lambda(n-1)]\,\frac{\Gamma(\gamma+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\gamma)}\,a_n z^{n-1}.\end{aligned} \tag{7}$$

Thus, by using the equation (3), we find for the inverse map $g = f^{-1}$ given by (2) that

$$
\begin{aligned}
&(1-\lambda)\ \frac{\gamma \mathfrak{T}_w^{\mu,\gamma} g(w)}{\mu w} + \lambda\ \frac{\gamma\big(\mathfrak{T}_w^{\mu,\gamma} g(w)\big)'}{\mu} \\
&\quad = 1 + \sum_{n=2}^{\infty} [1+\lambda(n-1)]\ \frac{\Gamma(\gamma+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\gamma)}\ b_n w^{n-1} \\
&\quad = 1 + \sum_{n=2}^{\infty} [1+\lambda(n-1)]\ \frac{\Gamma(\gamma+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\gamma)}\ \frac{1}{n}\ K_{n-1}^{-n}(a_2, a_3, \cdots, a_n) w^{n-1}.
\end{aligned} \tag{8}
$$

Furthermore, since $f \in \mathcal{A}_\Sigma(\lambda,\gamma,\mu,\varphi)$, there are two Schwarz functions (see Definition 1) $u, v : \mathbb{U} \to \mathbb{U}$ with

$$
u(0) = v(0) = 0 \qquad \text{and} \qquad u(z) = \sum_{n=1}^{\infty} p_n z^n \qquad \text{and} \qquad v(z) = \sum_{n=1}^{\infty} q_n z^n,
$$

so that

$$
(1-\lambda)\ \frac{\gamma \mathfrak{T}_z^{\mu,\gamma} f(z)}{\mu z} + \lambda\ \frac{\gamma\big(\mathfrak{T}_z^{\mu,\gamma} f(z)\big)'}{\mu} = \varphi\big(u(z)\big), \tag{9}
$$

and

$$
(1-\lambda)\ \frac{\gamma \mathfrak{T}_w^{\mu,\gamma} g(w)}{\mu w} + \lambda\ \frac{\gamma\big(\mathfrak{T}_w^{\mu,\gamma} g(w)\big)'}{\mu} = \varphi\big(v(w)\big). \tag{10}
$$

In addition, by applying (4), we have

$$
\begin{aligned}
\varphi\big(u(z)\big) &= 1 + B_1 p_1 z + (B_1 p_2 + B_2 p_1^2) z^2 + \cdots \\
&= 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} B_k D_n^k(p_1, p_2, \cdots, p_n) z^n,
\end{aligned} \tag{11}
$$

and

$$
\begin{aligned}
\varphi\big(v(w)\big) &= 1 + B_1 q_1 w + (B_1 q_2 + B_2 q_1^2) w^2 + \cdots \\
&= 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} B_k D_n^k(q_1, q_2, \cdots, q_n) w^n.
\end{aligned} \tag{12}
$$

By comparing the corresponding coefficients in (7) and (9), and then using (11), we get

$$
[1+\lambda(n-1)]\ \frac{\Gamma(\gamma+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\gamma)}\ a_n = \sum_{k=1}^{n-1} B_k D_{n-1}^k(p_1, p_2, \cdots, p_{n-1}). \tag{13}
$$

Similarly, from (8) and (10), by using (12), we have

$$
\begin{aligned}
&[1+\lambda(n-1)]\ \frac{\Gamma(\gamma+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\gamma)}\ \frac{1}{n}\ K_{n-1}^{-n}(a_2, a_3, \cdots, a_n) \\
&\qquad = \sum_{k=1}^{n-1} B_k D_{n-1}^k(q_1, q_2, \cdots, q_{n-1}).
\end{aligned} \tag{14}
$$

Now, in view of the assumption that $a_k = 0\ \ (2 \leqq k \leqq n-1)$, the coefficients b_n corresponding to $D_{n-1}^k(q_1, q_2, \cdots, q_{n-1})$ equals $-a_n$, so we have

$$
[1+\lambda(n-1)]\frac{\Gamma(\gamma+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\gamma)}\ a_n = B_1 p_{n-1} \tag{15}
$$

and

$$-\left[1+\lambda(n-1)\right]\frac{\Gamma(\gamma+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\gamma)}\,a_n=B_1q_{n-1}. \tag{16}$$

Since

$$|p_{n-1}|\leqq 1 \qquad \text{and} \qquad |q_{n-1}|\leqq 1,$$

by taking the absolute values of either of the above two equations, we obtain (6). This completes the proof of Theorem 1. $\square$

Theorem 2. *For $0\leqq\lambda\leqq 1, 0<\gamma,\mu\leqq 1,\ \mu>\gamma$ and $0<\mu-\gamma<1$, let the function $f\in\mathcal{A}_\Sigma(\lambda,\gamma,\mu;\varphi)$ be given by (1). Also let*

$$B_2=\alpha B_1 \qquad (0<\alpha\leqq 1).$$

Then the following coefficient inequalities hold true:

$$|a_2|\leqq\begin{cases}\frac{B_1(\gamma+1)}{(1+\lambda)(\mu+1)} & \left(B_1\geqq\frac{(\gamma+2)(1+\lambda)^2(\mu+1)}{(1+2\lambda)(\mu+2)(\gamma+1)}\right)\\[2ex] \sqrt{\frac{B_1(\gamma+2)(\gamma+1)}{(1+2\lambda)(\mu+2)(\mu+1)}} & \left(0<B_1\leqq\frac{(\gamma+2)(1+\lambda)^2(\mu+1)}{(1+2\lambda)(\mu+2)(\gamma+1)}\right)\end{cases} \tag{17}$$

and

$$|a_3|\leqq\frac{B_1(\gamma+2)(\gamma+1)}{(1+2\lambda)(\mu+2)(\mu+1)}. \tag{18}$$

Proof. If we set $n=2$ and $n=3$ in (13) and (14), respectively, we obtain

$$\frac{(1+\lambda)(\mu+1)}{\gamma+1}\,a_2=B_1p_1, \tag{19}$$

$$\frac{(1+2\lambda)(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)}\,a_3=B_1p_2+\alpha B_1p_1^2, \tag{20}$$

$$-\frac{(1+\lambda)(\mu+1)}{\gamma+1}\,a_2=B_1q_1 \tag{21}$$

and

$$\frac{(1+2\lambda)(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)}\left(2a_2^2-a_3\right)=B_1q_2+\alpha B_1q_1^2. \tag{22}$$

From (19) or (21), by taking absolute values, we get

$$|a_2|\leqq\frac{B_1(\gamma+1)}{(1+\lambda)(\mu+1)}. \tag{23}$$

Furthermore, by adding (20) and (22), we find that

$$\frac{2(1+2\lambda)(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)}\,a_2^2=B_1\left[(p_2+\alpha p_1^2)+(q_2+\alpha q_1^2)\right],$$

which, upon taking the moduli of both sides, yields

$$\frac{2(1+2\lambda)(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)}\,|a_2|^2\leqq B_1\left[\left|p_2+\alpha p_1^2\right|+\left|q_2+\alpha q_1^2\right|\right].$$

Thus, by using Lemma 2, we obtain

$$\frac{2(1+2\lambda)(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)}\,|a_2|^2 \leqq B_1\left[1+(\alpha-1)|p_1|^2+1+(\alpha-1)|q_1|^2\right] \leqq 2B_1.$$

Therefore, we have

$$|a_2| \leqq \sqrt{\frac{B_1(\gamma+2)(\gamma+1)}{(1+2\lambda)(\mu+2)(\mu+1)}}. \tag{24}$$

Equation (23) in conjunction with (24) would readily yield (17).
We next solve (20) for a_3, take the absolute values and apply Lemma 2. We thus obtain

$$|a_3| \leqq \frac{B_1(\gamma+2)(\gamma+1)}{(1+2\lambda)(\mu+2)(\mu+1)}\left[1+(\alpha-1)|p_m|^2\right] \leqq \frac{B_1(\gamma+2)(\gamma+1)}{(1+2\lambda)(\mu+2)(\mu+1)}.$$

Hence we obtain the desired estimate on $|a_3|$ given in (18). This completes the proof of Theorem 2. □

3. Concluding Remarks and Observations

In this concluding section, we give several remarks and observations which related to the developments resented in this paper.

Remark 1. *By letting $\mu = \gamma = \lambda = 1$ in Theorem 1, we obtain estimates on the general coefficients $|a_n|$ $(n \geqq 3)$ for subclass defined by Ali et al. [47] (Theorem 2.1), which are not obtained until now.*

Remark 2. *By setting*

$$\varphi(z) = \frac{1+(1-2\beta)z}{1-z} \qquad (0 \leqq \beta < 1),$$

in Theorem 1, we get the results which were obtained by Srivastava et al. [44] (Theorem 1).

Remark 3. *By taking*

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \qquad (0 < \alpha \leqq 1)$$

in Theorem 1, we get an upper bound for the coefficients $|a_n|$ of functions in a subclass which is defined by argument in the following corollary, which is presumably new.

Corollary. *For $0 \leqq \lambda \leqq 1$, $0 < \alpha, \gamma, \mu \leqq 1$, $\mu > \gamma$, $0 \leqq \beta < 1$ and $0 < \mu - \gamma < 1$, let the function*

$$f \in \mathcal{A}_{\Sigma}\left(\lambda, \gamma, \mu; \left(\frac{1+z}{1-z}\right)^{\alpha}\right)$$

be given by (1). If $a_k = 0$ $(2 \leqq k \leqq n-1)$, then

$$|a_n| \leqq \frac{2\alpha\Gamma(\mu+1)\Gamma(n+\gamma)}{[1+\lambda(n-1)]\Gamma(\gamma+1)\Gamma(n+\mu)} \qquad (n \geqq 3).$$

Remark 4. *By setting*

$$\varphi(z) = \frac{1+(1-2\beta)z}{1-z} \qquad (0 \leqq \beta < 1)$$

in Theorem 2, we get the results which were obtained by Srivastava et al. [44] (Theorem 2).

Remark 5. *By taking*

$$\mu = \gamma = 1 \quad \text{and} \quad \varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \qquad (0 < \alpha \leqq 1)$$

in Theorem 2, we can improve the estimates which were given by Frasin and Aouf [4] (Theorem 2.2). Also, by setting

$$\mu = \gamma = 1 \quad \text{and} \quad \varphi(z) = \frac{1+(1-2\beta)z}{1-z} \qquad (0 \leqq \beta < 1)$$

in Theorem 2, we can improve the estimates which were given by Frasin and Aouf [4] (Theorem 3.2).

Remark 6. *By setting*

$$\mu = \gamma = \lambda = 1 \quad \text{and} \quad \varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \qquad (0 < \alpha \leqq 1)$$

in Theorem 2, we obtain an improvement of the estimates which were given by Srivastava et al. [2] (Theorem 1). Moreover, by setting

$$\mu = \gamma = \lambda = 1 \quad \text{and} \quad \varphi(z) = \frac{1+(1-2\beta)z}{1-z} \qquad (0 \leqq \beta < 1)$$

in Theorem 2, we obtain an improvement of the estimates which were given by Srivastava et al. [2] (Theorem 2).

Remark 7. *By taking*

$$\mu = \gamma = \lambda = 1 \quad \text{and} \quad \varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \qquad (0 < \alpha \leqq 1)$$

in Theorem 2, we get an improvement of the estimates which were given by Zaprawa [48] (Corollary 3). Also, by taking

$$\mu = \gamma = \lambda = 1 \quad \text{and} \quad \varphi(z) = \frac{1+(1-2\beta)z}{1-z} \qquad (0 \leqq \beta < 1)$$

in Theorem 2, we obtain an improvement of the estimates which were given by Zaprawa [48] (Corollary 4).

Remark 8. *By letting*

$$\mu = \gamma = \lambda = 1 \quad \text{and} \quad B_2 = \alpha B_1 \qquad (0 < \alpha \leqq 1)$$

in Theorem 2, we obtain an improvement of the estimates which were given by Ali et al. [47] (Theorem 2.1).

We conclude our present investigation by observing that the interested reader will find several related recent developments concerning Geometric Function Theory of Complex Analysis (see, for example, [46,49–51]) to be potentially useful for motivating further researches in this subject and on other related topics.

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