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Absence of Non-Trivial Fuzzy Inner Product Spaces and the Cauchy–Schwartz Inequality

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Abstract: First, we show that the non-trivial fuzzy inner product space under the linearity condition does not exist, which means a fuzzy inner product space with linearity produces only a crisp real number for each pair of vectors. If the positive-definiteness is added to the condition, then the Cauchy–Schwartz inequality is also proved.

Keywords: fuzzy inner product space; Cauchy–Schwartz inequality; linearity; positive-definiteness

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1. Introduction

In a real situation, there are many cases when vagueness or ambiguity are observed with numbers, such as, "a few", "several" or "about 10". Zadeh [1] first introduced a fuzzy set to express such numbers which include vagueness. Since then, numerous concepts and theories of fuzzy logic and fuzzy mathematics are introduced by many authors.

Especially, the concept of fuzzy norm on a vector space was first introduced by Katsaras [2]. Since his works, many kinds of norm was made. For example, Felbin [3] introduced an alternative definition of a fuzzy norm(namely, Felbin's fuzzy norm) related to a fuzzy metric of Kaleva-Seikkala's type [4]. For another example, Bag and Samanta defined another fuzzy norm [5].

The concept of a fuzzy inner product was first introduced by Goudarzi et al [6]. Since their works, a study for fuzzy inner product has been progressed actively [6–8]. In 2010, Hasankhani et al defined another fuzzy inner product that arises from Felbin's fuzzy norm. In that paper, they considered a fuzzy inner product under both the linearity condition (IP_1, IP_2) and the positive definite condition (IP_4, IP_5, IP_6) and provided the Cauchy–Schwartz inequality and completion in their contexts. In 2015, Saheli conducted a comparative study of the relationship between Goudarzia's definition and Hasankhani's definition for a fuzzy inner product (see [9–11]). In 2017, Daraby et al. [12] conducted a study about topological properties of fuzzy inner product spaces in Hasankhani's contexts. In 2019, J. M. Kim and K. Y. Lee [13] consider approximation properties in Felbin fuzzy normed spaces. They gave the characterizations of approximation properties in Felbin fuzzy normed spaces. In 2020, the authors in [14] made topological tools to analyze such approximation properties. They gave dual problems for approximation properties.

In order to deal with fuzzy data in applications, the mathematical setting of the vector spaces is very important. Therefore basic definitions and theory for fuzzy inner product spaces are crucial and fundamental for fuzzy applications.

We are motivated by both the linearity and the positive definite condition of fuzzy inner product spaces in Hasankhani's contexts because those conditions may be natural and basic in crisp inner product space. In the fuzzy sense, they are hardly dealt with so far because it is not easy to find a proper example. We show how those two conditions have an effect on the space fuzzy inner products, especially crisp inner products.

In this paper, the fuzzy inner product by Hasankhani's et al is studied much more deeply, which can be naturally considered form the concept of the basic inner product. In fact, it is shown that the linear condition turns any inner product into a crisp one. Furthermore, if the positive-definiteness is added to the condition, then the Cauchy–Schwartz inequality is also satisfied.

2. Preliminaries

Definition 1 ([15])**.** *A mapping $\eta : \mathbb{R} \to [0,1]$ is called a fuzzy real number with α-level set*

$$[\eta]_\alpha = \{t : \eta(t) \geq \alpha\},$$

if it satisfies the following conditions:

(i) *there exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.*
(ii) *for each $\alpha \in (0,1]$, there exist real numbers $\eta_\alpha^- \leq \eta_\alpha^+$ such that the α-level set $[\eta]_\alpha$ is equal to the closed interval $[\eta_\alpha^-, \eta_\alpha^+]$.*

Remark 1. *The condition (ii) of Definition 1 is equivalent to convex and upper semi continuous:*

(1) *a fuzzy real number η is convex if $\eta(t) \geq \min\{\eta(s), \eta(r)\}$ where $s \leq t \leq r$.*
(2) *a fuzzy real number η is called upper semi-continuous if for all $t \in \mathbb{R}$ and $\epsilon > 0$ with $\eta(t) = \alpha$, there is $c > 0$ such that $|s - t| < c = c(t) \Rightarrow \eta(t) < a + \epsilon$, i.e., $\eta^{-1}([0, a + \epsilon))$ for all $a \in [0,1]$ and $\epsilon > 0$ is open in the usual topology of $\mathbb{R}$.*

The set of all fuzzy real numbers is denoted by $F(\mathbb{R})$. If $\eta \in F(\mathbb{R})$ and $\eta(t) = 0$ whenever $t < 0$, then η is called a non-negative fuzzy real number and $F^*(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers. We note that real number $\eta_\alpha^- \geq 0$ for all $\eta \in F^*(\mathbb{R})$ and all $\alpha \in (0,1]$. Each $r \in \mathbb{R}$ can be considered as the fuzzy real number $\tilde{r} \in F(\mathbb{R})$ denoted by

$$\tilde{r}(t) = \begin{cases} 1, & t = r, \\ 0, & t \neq r, \end{cases}$$

hence it follows that $\mathbb{R}$ can be embedded in $F(\mathbb{R})$.

Definition 2 ([1,16])**.** *The arithmetic operations $\oplus, \ominus, \otimes$ and $\oslash$ on $F(\mathbb{R}) \times F(\mathbb{R})$ are defined by*

$$(\eta \oplus \gamma)(t) = \sup_{t=x+y} (\min(\eta(x), \gamma(y))),$$

$$(\eta \ominus \gamma)(t) = \sup_{t=x-y} (\min(\eta(x), \gamma(y))),$$

$$(\eta \otimes \gamma)(t) = \sup_{t=xy} (\min(\eta(x), \gamma(y))),$$

$$(\eta \oslash \gamma)(t) = \sup_{t=x/y} (\min(\eta(x), \gamma(y))),$$

which are special cases of Zadeh's extension principles.

Definition 3 ([16])**.** *The absolute value $|\eta|$ of $\eta \in F(\mathbb{R})$ is defined by*

$$|\eta|(t) = \begin{cases} \max(\eta(t), \eta(-t)), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Lemma 1 ([1,16])**.** *Let $\eta, \gamma \in F(\mathbb{R})$ and $[\eta]_\alpha = [\eta^-_\alpha, \eta^+_\alpha], [\gamma_\alpha] = [\gamma^-_\alpha, \gamma^+_\alpha]$. Then for all $\alpha \in (0,1]$,*

$$
\begin{aligned}
[\eta \oplus \gamma]_\alpha &= [\eta^-_\alpha + \gamma^-_\alpha, \eta^+_\alpha + \gamma^+_\alpha], \\
[\eta \ominus \gamma]_\alpha &= [\eta^-_\alpha - \gamma^+_\alpha, \eta^+_\alpha - \gamma^-_\alpha], \\
[\eta \otimes \gamma]_\alpha &= [\eta^-_\alpha \gamma^-_\alpha, \eta^+_\alpha \gamma^+_\alpha], \forall \eta, \gamma \in F^*(\mathbb{R}), \\
[\tilde{1} \oslash \eta]_\alpha &= [\frac{1}{\eta^+_\alpha}, \frac{1}{\eta^-_\alpha}], \forall \eta^-_\alpha > 0, \\
[|\eta|]_\alpha &= [\max(0, \eta^-_\alpha, -\eta^+_\alpha), \max(|\eta^-_\alpha|, |\eta^+_\alpha|)].
\end{aligned}
$$

We will use operations in ([17] Page 1405).

Definition 4. *Let X be a vector space over $F(\mathbb{R})$. Assume the mappings $L, R : [0,1] \times [0,1] \to [0,1]$ are symmetric and non-decreasing in both arguments, and that $L(0,0) = 0$ and $R(1,1) = 1$. Let $\|\cdot\| : X \to F^*(\mathbb{R})$. The quadruple $(X, \|\cdot\|, L, R)$ is called a fuzzy normed space ([3]) with the fuzzy norm $\|\cdot\|$, if the following conditions are satisfied:*

(F1) *if $x \neq 0$, then $\inf_{0<\alpha\leq 1} \|x\|^-_\alpha > 0$,*
(F2) *$\|x\| = \tilde{0}$ if and only if $x = 0$,*
(F3) *$\|rx\| = |\tilde{r}|\|x\|$ for $x \in X$ and $r \in \mathbb{R}$,*
(F4) *for all $x, y \in X$,*

(F4L) *$\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$ whenever $s \leq \|x\|^-_1, t \leq \|y\|^-_1$ and $s+t \leq \|x+y\|^-_1$,*
(F4R) *$\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$ whenever $s \geq \|x\|^-_1, t \geq \|y\|^-_1$ and $s+t \geq \|x+y\|^-_1$.*

Here, we fix $L(s,t) = \min(s,t)$ and $R(s,t) = \max(s,t)$ for all $s, t \in [0,1]$ and we write $(X, \|\cdot\|)$.

3. Results

Recall the definition of a real-valued fuzzy inner product on a vector space which is linear and positive-definite at the same time [11]:

Definition 5. *Let X be a vector space over $\mathbb{R}$. A fuzzy scalar product on X is a mapping $\langle \cdot, \cdot \rangle : X \times X \to F(\mathbb{R})$ such that for all vectors $x, y, z \in X$ and $r \in \mathbb{R}$, we have*

(IP_1) $\langle x+y, z \rangle = \langle x, z \rangle \oplus \langle y, z \rangle$,
(IP_2) $\langle rx, y \rangle = \widetilde{r} \otimes \langle x, y \rangle$,
(IP_3) $\langle x, y \rangle = \langle y, x \rangle$,

Furthermore, if the following condition $(IP_{4,5,6})$ is added, then $\langle \cdot, \cdot \rangle$ is called a fuzzy inner product:

(IP_4) $\langle x, x \rangle \geq \widetilde{0}$,
(IP_5) $\inf_{0<\alpha\leq 1} \langle x, x \rangle^-_\alpha > 0$, *if $x \neq 0$*,
(IP_6) $\langle x, x \rangle = \widetilde{0}$ *if and only if $x = 0$.*

The vector space X with a real valued fuzzy scalar/inner product is called a fuzzy real scalar/inner product space.

In the sense of linear algebra, a scalar product on a vector space is also called a bilinear symmetric form. The following theorem says that a fuzzy scalar product produces only crisp real numbers.

Theorem 1. *Let X be a vector space over $\mathbb{R}$. A fuzzy scalar product $\langle \cdot, \cdot \rangle : X \times X \to F(\mathbb{R})$, a fuzzy real number valued map satisfying conditions $(IP_{1,2,3})$, produces only crisp real numbers for each pair of vectors.*

Proof. For any $x, y \in X$,

$$\begin{aligned}\tilde{0} &= \tilde{0} \otimes \langle x, y\rangle \\ &= \langle 0 \cdot x, y\rangle \\ &= \langle \vec{0}, y\rangle \\ &= \langle x - x, y\rangle \\ &= \langle x, y\rangle \ominus \langle x, y\rangle.\end{aligned}$$

Denote $\langle x, y\rangle$ by η. Then, for any real number $\alpha \in (0, 1]$, we get

$$\{0\} = [\tilde{0}]_\alpha = [\eta_\alpha^- - \eta_\alpha^+, \eta_\alpha^+ - \eta_\alpha^-],$$

so $\eta_\alpha^- = \eta_\alpha^+$. Thus $\langle x, y\rangle = \eta$ is a crisp real number. □

Theorem 1 shows that, on a given real-valued space, there is one-to-one correspondence between the space of fuzzy scalar products and that of crisp inner products. See the following example:

Example 1. *Let $\langle\ ,\ \rangle_0$ be a given inner product of some Hilbert space X over $\mathbb{R}$. Then let us define a fuzzy inner product $\langle x, y\rangle(t)$ given by*

$$\langle x, y\rangle(t) = \begin{cases} 1, & t = \langle x, y\rangle_0, \\ 0, & o.w. \end{cases}$$

Therefore $\langle x, y\rangle(t)$ satisfies the linearity and positive-definite condition. However, it is clear that the given fuzzy inner product space turns into a crisp inner product space. To show the uniqueness, assume that $\langle\langle\cdot,\cdot\rangle\rangle : X \times X \to F(\mathbb{R})$ is any real-valued fuzzy inner product satisfying $\langle\langle x, y\rangle\rangle(\langle x, y\rangle_0) = 1$. Since $\langle\langle\cdot,\cdot\rangle\rangle$ is also a fuzzy scalar product, we get $\langle\langle x, y\rangle\rangle = \widetilde{\langle x, y\rangle_0}$ from Theorem 1, so $\langle\langle x, y\rangle\rangle = \langle x, y\rangle$.

If positive-definite condition $(IP_{4,5,6})$ is added to the hypothesis of Theorem 1, i.e., if X is a real-valued fuzzy inner product space, then it can be shown that the Cauchy–Schwartz inequality holds in a more strong sense.

To do it, consider the following lemma which is easily checked:

Lemma 2. *Given a fuzzy real number $x \in F(\mathbb{R})$,*

(i) if $|x|_\alpha^+ = |x_\alpha^-|$, then $|x|_\alpha^+ = -x_\alpha^-$ and $\left(\widetilde{|x|_\alpha^+} \otimes x\right)_\alpha^- = -(|x|_\alpha^+)^2$,
(ii) if $|x|_\alpha^+ = |x_\alpha^+|$, then $|x|_\alpha^+ = x_\alpha^+$ and $\left(-\widetilde{|x|_\alpha^+} \otimes x\right)_\alpha^- = -(|x|_\alpha^+)^2$.

Recall the Cauchy–Schwartz inequality with respect to a complex fuzzy inner product [18]. The one with respect to a real fuzzy inner product can also be shown as follows:

Theorem 2 (Cauchy–Schwartz inequality)**.** *For vectors v, w, and for each $\alpha \in (0, 1]$, we have*

$$|\langle v, w\rangle|_\alpha^+ \le \|v\|_\alpha^- \, \|w\|_\alpha^-.$$

Hence, it holds that $|\langle v, w\rangle| \le \|v\| \otimes \|w\|$.

Proof. Since all of $|\langle v, w\rangle|$, $\|v\|$ and $\|w\|$ are fuzzy real numbers, it suffices to show that the inequality $|\langle v, w\rangle|_\alpha^+ \le \|v\|_\alpha^- \|w\|_\alpha^-$ holds for each $\alpha \in (0, 1]$. If w is a zero vector $\vec{0}$, then

$$|\langle v, w\rangle| = |\langle v, \vec{0}\rangle| = |\langle v, 0 \cdot v\rangle| = |\tilde{0} \otimes \langle v, v\rangle| = \tilde{0},$$

which implies $|\langle v,w\rangle|_\alpha^+ = 0$ and so the theorem holds. Assume that w is not a zero vector. Then from the definition 5(IP_5), $\|w\|_\alpha^- \neq 0$. Denote $\langle v,w\rangle = x$ by a fuzzy real number x. Let

$$a = \begin{cases} \frac{|x|_\alpha^+}{(\|w\|_\alpha^-)^2}, & \text{if } |x|_\alpha^+ = |x_\alpha^-|, \\ \frac{-|x|_\alpha^+}{(\|w\|_\alpha^-)^2}, & \text{if } |x|_\alpha^+ = |x_\alpha^+|. \end{cases}$$

Consider $a \in \mathbb{R}$. Then, the inequality

$$\begin{aligned} \tilde{0} &\leq |\langle v+aw, v+aw\rangle| = \langle v+aw, v+aw\rangle \\ &= \|v\|^2 \oplus \widetilde{2} \otimes (\tilde{a} \otimes \langle v,w\rangle) \oplus |\tilde{a}|^2 \otimes \|w\|^2 \end{aligned}$$

holds. Thus we can rewrite the equation as follows:

$$\langle v+aw, v+aw\rangle = \|v\|^2 \oplus \left(\tilde{2} \otimes \tilde{a} \otimes x \oplus \widetilde{a^2} \otimes \|w\|^2\right).$$

From Lemma 2,

$$\begin{aligned} \left(\tilde{2} \otimes \tilde{a} \otimes x \oplus \widetilde{a^2} \otimes \|w\|^2\right)_\alpha^- &= 2\,(\tilde{a} \otimes x)_\alpha^- + a^2\,(\|w\|_\alpha^-)^2 \\ &= \frac{-2\,(|x|_\alpha^+)^2}{(\|w\|_\alpha^-)^2} + \frac{(|x|_\alpha^+)^2}{(\|w\|_\alpha^-)^2} \\ &= \frac{-(|x|_\alpha^+)^2}{(\|w\|_\alpha^-)^2}. \end{aligned}$$

This gives

$$0 \leq \left(\langle v+aw, v+aw\rangle\right)_\alpha^- = (\|v\|_\alpha^-)^2 - \frac{(|x|_\alpha^+)^2}{(\|w\|_\alpha^-)^2} = (\|v\|_\alpha^-)^2 - \frac{(|\langle v,w\rangle|_\alpha^+)^2}{(\|w\|_\alpha^-)^2}$$

and

$$|\langle v,w\rangle|_\alpha^+ \leq \|v\|_\alpha^-\,\|w\|_\alpha^-.$$

□

Remark 2 indicates that a fuzzy scalar product is different from a crisp inner product because a crisp inner product always satisfies Cauchy–Schwartz inequality.

Remark 2. *Without the positive-definite condition, a fuzzy scalar product may not satisfy the Cauchy–Schwartz inequality: consider the real vector space $X = \mathbb{R} \times \mathbb{R}$ and a fuzzy scalar product $\langle \cdot,\cdot\rangle : X \times X \to F(\mathbb{R})$, defined by*

$$\langle\, (a,b), (c,d)\,\rangle = -\widetilde{ac + bd}.$$

Then, $\langle(-1,1),(1,1)\rangle = \tilde{2}$ and $||(1,1)|| = \tilde{0}$, which implies that $\langle\cdot,\cdot\rangle$ does not satisfy the Cauchy–Schwartz inequality even though it is non-degenerate: if $\langle\,(a,b),(x,y)\,\rangle = \tilde{0}$ for any $(x,y) \in X$, we get $\tilde{0} = \langle\,(a,b),(-a,b)\,\rangle = \widetilde{a^2 + b^2}$, so $(a,b) = (0,0)$.

Recall Theorem 1. Its result may be shown through the Cauchy–Schwartz inequality for a real-valued fuzzy inner product space, too:

Corollary 1. *Given $v, w \in X$, $\langle v,w\rangle = \tilde{a}$ for some $a \in \mathbb{R}$.*

Proof. From the Cauchy–Schwartz inequality (see Theorem 2), for any $\alpha \in (0,1]$

$$(\|v\|^2)_\alpha^+ = |\langle v,v\rangle|_\alpha^+ \leq \|v\|_\alpha^-\,\|v\|_\alpha^- = |\langle v,v\rangle|_\alpha^-,$$

which implies $\|v\|^2 = |\langle v, v\rangle| = \widetilde{b}$ for some $b \in \mathbb{R}$. Then for any $\alpha \in (0, 1]$,

$$\begin{aligned}
(\langle v, w\rangle)^+_\alpha &= \frac{1}{2}\{(\|v\|^2)^+_\alpha + (\|w\|^2)^+_\alpha - (\|v-w\|^2)^-_\alpha\} \\
&= \frac{1}{2}\{(\|v\|^2)^-_\alpha + (\|w\|^2)^-_\alpha - (\|v-w\|^2)^+_\alpha\} \\
&= (\langle v, w\rangle)^-_\alpha
\end{aligned}$$

which gives $\langle v, w\rangle = \widetilde{a}$ for some $a \in \mathbb{R}$. □

Corollary 1 shows that, on a given real-valued space, there is one-to-one correspondence between the space of fuzzy inner products and that of crisp inner products with positive-definiteness. Furthermore, they are subspaces of a space of fuzzy number satisfying the Cauchy–Schwartz inequality.

Remark 3. *Corollary 1 might not hold if both conditions (IP_1, IP_2) and (IP_4, IP_5, IP_6) in Definition 5 do not hold. In fact, M. Saheli and S. K. Gelousalar show that the modified linearity condition can produce a nontrivial fuzzy inner product space (see [9] Example 2.9).*

4. Conclusions

In this paper, we defined a fuzzy scalar product space and a fuzzy inner product space. We proved that there is no meaningful fuzzy inner product space under linearity condition, which turns them into a crisp inner product one. In addition, we proved that the Cauchy–Schwartz inequality holds under the positive-definite condition. In future study, the fuzzy sub-linearity and sub-bilinearity conditions will be defined and the Cauchy–Schwartz inequality will be dealt with under those conditions.

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