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Statistical Deferred Nörlund Summability and Korovkin-Type Approximation Theorem

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Abstract: The concept of the deferred Nörlund equi-statistical convergence was introduced and studied by Srivastava et al. [Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. (RACSAM) 112 (2018), 1487–1501]. In the present paper, we have studied the notion of the deferred Nörlund statistical convergence and the statistical deferred Nörlund summability for sequences of real numbers defined over a Banach space. We have also established a theorem presenting a connection between these two interesting notions. Moreover, based upon our proposed methods, we have proved a new Korovkin-type approximation theorem with algebraic test functions for a sequence of real numbers on a Banach space and demonstrated that our theorem effectively extends and improves most of the earlier existing results (in classical and statistical versions). Finally, we have presented an example involving the generalized Meyer–König and Zeller operators of a real sequence demonstrating that our theorem is a stronger approach than its classical and statistical versions.

Keywords: statistical convergence; statistical deferred Nörlund summability; positive linear operators; sequences of real variables; banach space; korovkin-type approximation theorems

1. Introduction and Motivation

Statistical convergence plays a vital role as an extension of the classical convergence in the study of convergence analysis of sequence spaces. The credit goes to Fast [1] and Steinhaus [2] for they have independently defined this notion; however, Zygmund [3] was the first to introduce this idea in the form of "almost convergence". This concept is also found in random graph theory (see [4,5]) in the sense that almost convergence, which is same as the statistical convergence, and it means convergence with a probability of 1, whereas in usual statistical convergence the probability is not necessarily 1. Subsequently, this theory has been brought to a high degree of development by many researchers because of its wide applications in various fields of mathematics, such as in Real analysis, Probability theory, Measure theory and Approximation theory and so on. For more details study in this direction, see [6–18].

Let $\mathfrak{K} \subseteq \mathbb{N}$ (set of natural numbers) and suppose that

$$\mathfrak{K}_n = \{k : k \in \mathbb{N} \quad \text{and} \quad k \in \mathfrak{K}\}.$$

The natural (or asymptotic) density of $\mathfrak{K}$ denoted by $d(\mathfrak{K})$, and is given by

$$d(\mathfrak{K}) = \lim_{n\to\infty} \frac{|\mathfrak{K}_n|}{n} = a,$$

where a finite real number, n is a natural number and $|\mathfrak{K}_n|$ is the cardinality of $\mathfrak{K}_n$.

A given real sequence (x_n) is said to be statistically convergent to ℓ if, for each $\epsilon > 0$, the set

$$\mathfrak{K}_\epsilon = \{k : k \in \mathbb{N} \quad \text{and} \quad |x_k - \ell| \geqq \epsilon\}$$

has zero natural density (see [1,2]). Thus, for each $\epsilon > 0$, we have

$$d(\mathfrak{K}_\epsilon) = \lim_{n\to\infty} \frac{|\mathfrak{K}_\epsilon|}{n} = \lim_{n\to\infty} \frac{1}{n} |\{k : k \leqq n \quad \text{and} \quad |x_k - \ell| \geqq \epsilon\}| = 0.$$

Here, we write

$$\text{stat} \lim_{n\to\infty} x_n = \ell.$$

In 2002, Móricz [19] introduced and studied some fundamental aspects of statistical Cesàro summability. Mohiuddine et al. [20] used this notion in a different way to establish some Korovkin-type approximation theorems. Subsequently, Karakaya and Chishti [21] introduced and studied the basic idea of the weighted statistical convergence and it was then modified by Mursaleen et al. [22]. Furthermore, Srivastava et al. [23,24], studied the notion of the deferred weighted as well as deferred Nörlund statistical convergence and used these notions to prove certain Korovkin-type approximation theorem with some new settings. Later on, some fundamental concept of the deferred Cesàro statistical convergence as well as the statistical deferred Cesàro summability, together with the associated approximation theorems was introduced by Jena et al. [25]. In 2019, Kandemir [26] studied the I-deferred statistical convergence in topological groups. Very recently, Paikray et al. [27] studied a new Korovkin-type theorem involving (p, q)-integers for statistically deferred Cesàro summability mean. On the other hand, Dutta et al. [28] studied another Korovkin type theorem over $C[0, \infty)$ by considering the exponential test functions 1, e^{-x} and e^{-2x} on the basis of the deferred Cesàro mean. For more recent works in this direction, see [23,29–38].

Essentially motivated by the aforementioned investigations and outcomes, in the present article we introduce the notion of the deferred Nörlund statistical convergence and the statistically deferred Nörlund summability of a real sequence. We then establish an inclusion relation between these two notions. Furthermore, we prove a new Korovkin-type approximation theorem with algebraic test functions for a real sequence over a Banach space via our proposed methods and also demonstrate that our outcome is a non-trivial generalization of ordinary and statistical versions of some well-studied earlier results.

2. Preliminaries and Definitions

Let (a_n) and (b_n) be sequences of non-negative integers such that, (i) $a_n < b_n$ and (ii) $\lim_{n\to\infty} b_n = \infty$. Suppose that (p_n) and (q_n) are the sequences of non-negative real numbers such that

$$\mathcal{P}_n = \sum_{m=a_n+1}^{b_n} p_m \quad \text{and} \quad \mathcal{Q}_n = \sum_{m=a_n+1}^{b_n} q_m.$$

The convolution of (p_n) and (q_n), the above-mentioned sequences is given by

$$\mathcal{R}_n = \sum_{v=a_n+1}^{b_n} p_v q_{b_n - v}.$$

We now recall the deferred Nörlund mean $D_a^b(N, p, q)$ as follows (see [24]):

$$t_n = \frac{1}{\mathcal{R}_n} \sum_{m=a_n+1}^{b_n} p_{b_n - m} q_m x_m.$$

We note that a sequence (x_n) is summable to ℓ via the method of deferred Nörlund summability involving the sequences (p_n) and (q_n) (or briefly, $D_a^b(N, p, q)$)-summable if,

$$\lim_{n\to\infty} t_n = \ell.$$

It is well known that the deferred Nörlund mean $D_a^b(N, p, q)$ is regular under the conditions (i) and (ii) (see, for details, Agnew [39]).

We further recall the following definition.

Definition 1. *(see [24]) Let (a_n) and (b_n) be sequences of non-negative integers and let (p_n) and (q_n) be the sequences of non-negative real numbers. A real sequence $\{x_n\}_{n\in\mathbb{N}}$ is deferred Nörlund statistically convergent to ℓ if, for every $\epsilon > 0$,*

$$\{m : m \leqq \mathcal{R}_n \quad \textit{and} \quad p_{b_n-m}q_m|x_m - \ell| \geqq \epsilon\}$$

has zero deferred Nörlund density, that is,

$$\lim_{n\to\infty} \frac{1}{\mathcal{R}_n}|\{m : m \leqq \mathcal{R}_n \quad \textit{and} \quad p_{b_n-m}q_m|x_m - \ell| \geqq \epsilon\}| = 0.$$

In this case, we write

$$stat_{DN} \lim x_n = \ell.$$

Let us now introduce the following definition in connection with our proposed work.

Definition 2. *Let (a_n) and (b_n) be sequences of non-negative integers and let (p_n) and (q_n) be the sequences of non-negative real numbers. A real sequence $\{x_n\}_{n\in\mathbb{N}}$ is statistically deferred Nörlund summable to ℓ if, for every $\epsilon > 0$,*

$$\{m : m \leqq n \quad \textit{and} \quad |t_m - \ell| \geqq \epsilon\}$$

has zero deferred Nörlund density, that is,

$$\lim_{n\to\infty} \frac{1}{n}|\{m : m \leqq n \quad \textit{and} \quad |t_m - \ell| \geqq \epsilon\}| = 0.$$

In this case, we write

$$stat \lim t_n = \ell.$$

Next, we wish to present a theorem in order to exhibit that every deferred Nörlund statistically convergent sequence is statistically deferred Nörlund summable. However, the converse is not generally true.

Theorem 1. *If a sequence (x_n) is deferred Nörlund statistically converges to a number ℓ, then it is statistically deferred Nörlund summable to ℓ (the same number); but in general the converse is not true.*

Proof. Suppose (x_n) is deferred Nörlund statistically convergent to ℓ. By the hypothesis, we have

$$\lim_{n\to\infty} \frac{1}{\mathcal{R}_n}|\{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_{b_n-m}q_m|x_m - \ell| \geqq \epsilon\}| = 0.$$

Consider two sets as follows:

$$\mathcal{K}_\epsilon = \lim_{m\to\infty} \{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_{b_n-m}q_m|x_m - \ell| \geqq \epsilon\}$$

and

$$\mathcal{K}_\epsilon^c = \lim_{m\to\infty} \{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_{b_n-m}q_m|x_m - \ell| < \epsilon\}.$$

Now,

$$
\begin{aligned}
|t_n - \ell| &= \left| \frac{1}{\mathcal{R}_n} \sum_{m=a_n+1}^{b_n} p_{b_n-m} q_m x_m - \ell \right| \\
&\leqq \left| \frac{1}{\mathcal{R}_n} \sum_{m=a_n+1}^{b_n} p_{b_n-m} q_m \left(x_m - \ell \right) \right| + |\ell| \left| \frac{1}{\mathcal{R}_n} \sum_{m=a_n+1}^{b_n} p_{b_n-m} q_m - 1 \right| \\
&\leqq \frac{1}{\mathcal{R}_n} \sum_{\substack{m=a_n+1 \\ (k \in \mathcal{K}_\epsilon)}}^{b_n} p_{b_n-m} q_m \left| x_m - \ell \right| + \frac{1}{\mathcal{R}_n} \sum_{\substack{m=a_n+1 \\ (k \in \mathcal{K}_\epsilon^c)}}^{b_n} p_{b_n-m} q_m \left| x_m - \ell \right| \\
&\quad + \left| \frac{1}{\mathcal{R}_n} \sum_{m=a_n+1}^{b_n} p_{b_n-m} q_m - 1 \right| \quad \left(\because \frac{1}{\mathcal{R}_n} \sum_{m=a_n+1}^{b_n} p_{b_n-m} q_m = 1 \right) \\
&\leqq \frac{1}{\mathcal{R}_n} |\mathcal{K}_\epsilon| + \frac{1}{\mathcal{R}_n} \mathcal{K}_\epsilon^c + 0 \to 0 \quad as \quad n \to \infty \quad (\because \lim_{n \to \infty} b_n = \infty),
\end{aligned}
$$

which implies that $t_n \to \ell$. Hence, (x_n) is statistically deferred Nörlund summable to ℓ. □

In view of the converse part of the theorem, we consider an example that shows that a sequence is statistically deferred Nörlund summable, even if it is not deferred Nörlund statistically convergent.

Example 1. *Suppose that*

$$a_n = 2n - 1 \;\; b_n = 4n - 1, \;\; and \;\; p_n = q_n = 1$$

and also consider a sequence (x_n) *by*

$$
x_n = \begin{cases} 0 & (n \;\; is \; even) \\ \\ 1 & (n \;\; is \; odd). \end{cases} \tag{1}
$$

One can easily see that, (x_n) *is neither ordinarily convergent nor convergent statistically. However, we have*

$$\frac{1}{\mathcal{R}_n} \sum_{m=a_n+1}^{b_n} p_{b_n-m} q_m x_m = \frac{1}{2n} \sum_{m=2n+1}^{4n} x_m = \frac{1}{2n} \frac{2n}{2} = \frac{1}{2}.$$

That is, (x_n) *is deferred Nörlund summable to* $\frac{1}{2}$ *and so also statistically deferred Nörlund summable to* $\frac{1}{2}$*; however, it is not deferred Nörlund statistically convergent.*

3. A New Korovkin-Type Approximation Theorem

In this section, we extend the result of Srivastava et al. [24] by using the notion of statistically deferred Nörlund summability of a real sequence over a Banach space.

Let $\mathcal{C}(X)$, be the space of all continuous functions (real valued) defined on a compact subset X $(X \subset \mathbb{R})$ under the norm $\|.\|_\infty$. Of course, $\mathcal{C}(X)$ is a Banach space. For $f \in \mathcal{C}(X)$, the norm $\|f\|$ of f is given by,

$$\|f\|_\infty = \sup_{x \in X} \{|f(x)|\}.$$

We say that the operator $\mathfrak{L}$ is a sequence of positive linear operator provided that

$$\mathfrak{L}(f; x) \geqq 0 \quad \text{whenever} \quad f \geqq 0.$$

Now we prove the following approximation theorem by using the statistical deferred Nörlund summability mean.

Theorem 2. *Let*

$$\mathfrak{L}_m : \mathcal{C}(X) \to \mathcal{C}(X)$$

be a sequence of positive linear operators. Then, $\forall \quad f \in \mathcal{C}(X)$,

$$stat_{DN} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f(x)\|_{\mathcal{C}(X)} = 0 \tag{2}$$

if and only if

$$stat_{\mathrm{DN}} \lim_{m\to\infty} \|\mathfrak{L}_m(1;x) - 1\|_{\mathcal{C}(X)} = 0, \tag{3}$$

$$stat_{\mathrm{DN}} \lim_{m\to\infty} \|\mathfrak{L}_m(x;x) - x\|_{\mathcal{C}(X)} = 0 \tag{4}$$

and

$$stat_{\mathrm{DN}} \lim_{m\to\infty} \|\mathfrak{L}_m(x^2;x) - x^2\|_{\mathcal{C}(X)} = 0. \tag{5}$$

Proof. Since each of the following functions

$$f_0(x) = 1, \quad f_1(x) = x \quad \text{and} \quad f_2(x) = x^2$$

belonging to $\mathcal{C}(X)$ and are continuous, the implication given by (2) implies (3) to (5) is obvious. Now in view of completing the proof of Theorem 2, we assume first that the conditions (3) to (5) hold true. If $f \in \mathcal{C}(X)$, then there exists a constant $\mathcal{M} > 0$ such that

$$|f(x)| \leqq \mathcal{M} \quad (\forall\, x \in X).$$

We thus find that

$$|f(s) - f(x)| \leqq 2\mathcal{M} \quad (s, x \in X). \tag{6}$$

Clearly, for given $\epsilon > 0$, there exists $\delta > 0$ for which

$$|f(s) - f(x)| < \epsilon \tag{7}$$

whenever

$$|s - x| < \delta, \quad \text{for all} \quad s, x \in X.$$

Let us choose

$$\varphi_1 = \varphi_1(s,x) = (s-x)^2.$$

If $|s - x| \geqq \delta$, we then obtain

$$|f(s) - f(x)| < \frac{2\mathcal{M}}{\delta^2}\varphi_1(s,x). \tag{8}$$

From the inequalities (7) and (8), we get

$$|f(s) - f(x)| < \epsilon + \frac{2\mathcal{M}}{\delta^2}\varphi_1(s,x),$$

which implies that

$$-\epsilon - \frac{2\mathcal{M}}{\delta^2}\varphi_1(s,x) \leqq f(s) - f(x) \leqq \epsilon + \frac{2\mathcal{M}}{\delta^2}\varphi_1(s,x). \tag{9}$$

Now, $\mathfrak{L}_m(1;x)$ being monotone and linear, so under the operator $\mathfrak{L}_m(1;x)$, we have

$$\begin{aligned}\mathfrak{L}_m(1;x)\left(-\epsilon - \frac{2\mathcal{M}}{\delta^2}\varphi_1(s,x)\right) &\leqq \mathfrak{L}_m(1;x)(f(s) - f(x)) \\ &\leqq \mathfrak{L}_m(1;x)\left(\epsilon + \frac{2\mathcal{M}}{\delta^2}\varphi_1(s,x)\right).\end{aligned}$$

Furthermore, $f(x)$ is a constant number in view that x is fixed. Consequently, we have

$$\begin{aligned}-\epsilon\mathfrak{L}_m(1;x) - \frac{2\mathcal{M}}{\delta^2}\mathfrak{L}_m(\varphi_1;x) &\leqq \mathfrak{L}_m(f;x) - f(x)\mathfrak{L}_m(1;x) \\ &\leqq \epsilon\mathfrak{L}_m(1;x) + \frac{2\mathcal{M}}{\delta^2}\mathfrak{L}_m(\varphi_1;x).\end{aligned} \tag{10}$$

Furthermore, we know that

$$\mathfrak{L}_m(f;x) - f(x) = [\mathfrak{L}_m(f;x) - f(x)\mathfrak{L}_m(1;x)] + f(x)[\mathfrak{L}_m(1;x) - 1]. \tag{11}$$

Using (10) and (11), we have

$$\mathfrak{L}_m(f;x) - f(x) < \epsilon\mathfrak{L}_m(1;x) + \frac{2\mathcal{M}}{\delta^2}\mathfrak{L}_m(\varphi_1;x) + f(x)[\mathfrak{L}_m(1;x) - 1]. \tag{12}$$

We now estimate $\mathfrak{L}_m(\varphi_1;x)$ as follows:

$$\begin{aligned}\mathfrak{L}_m(\varphi_1;x) &= \mathfrak{L}_m((s-x)^2;x) = \mathfrak{L}_m(s^2 - 2xs + x^2;x)\\ &= \mathfrak{L}_m(s^2;x) - 2x\mathfrak{L}_m(s;x) + x^2\mathfrak{L}_m(1;x)\\ &= [\mathfrak{L}_m(s^2;x) - x^2] - 2x[\mathfrak{L}_m(s;x) - x]\\ &\quad + x^2[\mathfrak{L}_m(1;x) - 1].\end{aligned}$$

Using (12), we obtain

$$\begin{aligned}\mathfrak{L}_m(f;x) - f(x) &< \epsilon\mathfrak{L}_m(1;x) + \frac{2\mathcal{M}}{\delta^2}\{[\mathfrak{L}_m(s^2;x) - x^2]\\ &\quad - 2x[\mathfrak{L}_m(s;x) - e^{-x}] + x^2[\mathfrak{L}_m(1;x) - 1]\}\\ &\quad + f(x)[\mathfrak{L}_m(1;x) - 1].\\ &= \epsilon[\mathfrak{L}_m(1;x) - 1] + \epsilon + \frac{2\mathcal{M}}{\delta^2}\{[\mathfrak{L}_m(s^2;x) - x^2]\\ &\quad - 2x[\mathfrak{L}_m(s;x) - x] + x^2[\mathfrak{L}_m(1;x) - 1]\}\\ &\quad + f(x)[L_m(1;x) - 1].\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, thus we have

$$\begin{aligned}|\mathfrak{L}_m(f;x) - f(x)| &\leqq \epsilon + \left(\epsilon + \frac{2\mathcal{M}}{\delta^2} + \mathcal{M}\right)|\mathfrak{L}_m(1;x) - 1|\\ &\quad + \frac{4\mathcal{M}}{\delta^2}|\mathfrak{L}_m(s;x) - x| + \frac{2\mathcal{M}}{\delta^2}|\mathfrak{L}_m(s^2;x) - x^2|\\ &\leqq \mathcal{K}(|\mathfrak{L}_m(1;x) - 1| + |\mathfrak{L}_m(s;x) - x|\\ &\quad + |\mathfrak{L}_m(s^2;x) - x^2|),\end{aligned} \tag{13}$$

where

$$\mathcal{K} = \max\left(\epsilon + \frac{2\mathcal{M}}{\delta^2} + \mathcal{M}, \frac{4\mathcal{M}}{\delta^2}, \frac{2\mathcal{M}}{\delta^2}\right).$$

Now, replacing $\mathfrak{L}_m(f;x)$ by

$$\frac{1}{\mathcal{R}_n}\sum_{m=a_n+1}^{b_n} p_{b_n-m}q_m\mathfrak{T}_m(f;x) = \mathcal{T}_m(f;x)$$

and noticing that, for a given $r > 0$, there exists $\epsilon > 0$ $(\epsilon < r)$, we get

$$\Omega_m(x;r) = \{m : m \leqq n \quad \text{and} \quad |\mathcal{T}_m(f;x) - f(x)| \geqq r\}.$$

Furthermore, for $i = 0, 1, 2$, we have

$$\Omega_{i,m}(x;r) = \left\{m : m \leqq n \quad \text{and} \quad |\mathcal{T}_m(f;x) - f_i(x)| \geqq \frac{r-\epsilon}{3\mathcal{K}}\right\},$$

so that,

$$\Omega_m(x;r) \leqq \sum_{i=0}^{2}\Omega_{i,m}(x;r).$$

Clearly, we obtain

$$\|\Omega_m(x;r)\|_{C(X)} \leqq \sum_{i=0}^{2} \|\Omega_{i,m}(x;r)\|_{C(X)}. \tag{14}$$

Now using the assumption as above for the implications (3) to (5) and in view of Definition 2, the right-hand side of (14) tends to zero as $n \to \infty$ leading to

$$\lim_{n\to\infty} \frac{\|\Omega_m(x;r)\|_{C(X)}}{\mathcal{R}_n} = 0 \ (\delta, r > 0).$$

Consequently, the implication (2) holds. This completes the proof of Theorem 2. □

Next, by using Definition 1, we present the following corollary as a consequence of Theorem 2.

Corollary 1. *Let $\mathfrak{L}_m : C(X) \to C(X)$ be a sequence of positive linear operators, and suppose that $f \in C(X)$. Then*

$$stat_{\mathrm{DN}} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f(x)\|_{C(X)} = 0$$

if and only if

$$stat_{\mathrm{DN}} \lim_{m\to\infty} \|\mathfrak{L}_m(1;x) - 1\|_{C(X)} = 0,$$
$$stat_{\mathrm{DN}} \lim_{m\to\infty} \|\mathfrak{L}_m(x;x) - x\|_{C(X)} = 0$$

and

$$stat_{\mathrm{DN}} \lim_{m\to\infty} \|\mathfrak{L}_m(x^2;x) - x^2\|_{C(X)} = 0.$$

We now present the following example for the sequence of positive linear operators that does not satisfy the associated conditions of the Korovkin approximation theorems proved previously in [24,33], but it satisfies the conditions of our Theorem 2. Consequently, our Theorem 2 is stronger than the earlier findings of both Srivastava et al. [24] and Paikray et al. [33].

We now recall the operator

$$x(1 + xD) \quad \left(D = \frac{d}{dx}\right),$$

which was applied by Al-Salam [40] and, in the recent past, by Viskov and Srivastava [41] (see also [42,43], and the monograph by Srivastava and Manocha [44] for various general families of operators and polynomials of this kind). Here, in our Example 2 below, we use this operator in conjunction with the Meyer–König and Zeller operators.

Example 2. *Let $X = [0,1]$ and we consider the Meyer–König and Zeller operators $\mathfrak{M}_n(f;x)$ on $C[0,1]$ given by (see [45]),*

$$\mathfrak{M}_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{k+n+1}\right)\binom{n+k}{k} x^k.(1+x)^{n+1}.$$

Furthermore, let $\mathfrak{L}_m : C[0,1] \to C[0,1]$ be a sequence of operators defined as follows:

$$\mathfrak{L}_m(f;x) = [1 + x_m]x(1 + xD)\mathfrak{M}_m(f) \quad (f \in C([0,1]), \tag{15}$$

where (x_m) is a real sequence defined in Example 1.

Now,

$$\mathfrak{L}_m(1;x) = [1 + x_m]x(1 + xD)1 = [1 + x_m]x,$$

$$\mathfrak{L}_m(s;x) = [1 + x_m]x(1 + xD)x = [1 + x_m]x(1 + x),$$

and

$$\mathfrak{L}_m(s^2;x) = [1+x_n]x(1+xD)\left\{x^2\left(\frac{n+2}{n+1}\right)+\frac{x}{n+1}\right\}$$
$$= [1+f_n(x)]\left\{x^2\left[\left(\frac{n+2}{n+1}\right)x+2\left(\frac{1}{n+1}\right)+2x\left(\frac{n+2}{n+1}\right)\right]\right\},$$

so that we have

$$stat_{\mathrm{DN}}\lim_{m\to\infty}\|\mathfrak{L}_m(1;x)-1\|_{C(X)}=0,$$
$$stat_{\mathrm{DN}}\lim_{m\to\infty}\|\mathfrak{L}_m(x;x)-x\|_{C(X)}=0$$

and

$$stat_{\mathrm{DN}}\lim_{m\to\infty}\|\mathfrak{L}_m(x^2;x)-x^2\|_{C(X)}=0,$$

that is, the sequence $\mathfrak{L}_m(f;x)$ satisfies the conditions (3) to (5). Therefore, by Theorem 2, we have

$$stat_{\mathrm{DN}}\lim_{m\to\infty}\|\mathfrak{L}_m(f;x)-f\|_{C(X)}=0.$$

Here, (x_n) is statistically deferred Nörlund summable, even if, it is neither Nörlund statistically convergent nor deferred Nörlund statistically convergent, so we certainly conclude that earlier works in [24,33] are not valid under the operators defined in (15), where as our Theorem 2 still serves for the operators defined by (15).

4. Concluding Remarks and Observations

In the last section of our investigation, we present various further remarks and observations correlating the different outcomes which we have proved here.

Remark 1. *Let $(x_m)_{m\in\mathbb{N}}$ be a real sequence given in Example 1. Then, since*

$$stat_{\mathrm{DN}}\lim_{m\to\infty}x_m=\frac{1}{2}\ on\ [0,1],$$

we have

$$stat_{\mathrm{DN}}\lim_{m\to\infty}\|\mathfrak{L}_m(f_i;x)-f_i(x)\|_{C(X)}=0\quad(i=0,1,2).\tag{16}$$

Thus, by Theorem 2, we can write

$$stat_{\mathrm{DN}}\lim_{m\to\infty}\|\mathfrak{L}_m(f;x)-f(x)\|_{C(X)}=0,\tag{17}$$

where

$$f_0(x)=1,\quad f_1(x)=x\quad and\quad f_2(x)=x^2.$$

As we know (x_m) is neither statistically convergent nor converges uniformly in the usual sense, thus the statistical and classical approximation of Korovkin-type theorems do not behave properly under the operators defined in (15). Hence, this application clearly indicates that our Theorem 2 is a non-trivial extension of the usual and statistical approximation of Korovkin-type theorems (see [1,46]).

Remark 2. *Let $(x_m)_{m\in\mathbb{N}}$ be a real sequence as given in Example 1. Then, since*

$$stat_{\mathrm{DN}}\lim_{m\to\infty}x_m=\frac{1}{2}\ on\ [0,1],$$

so (16) holds. Now, by applying (16) and Theorem 2, the condition (17) holds. Moreover, since the sequence (x_m) is not deferred Nörlund statistically convergent, the finding of Srivastava et al. [24] does not serve for our operator defined in (15). Thus, our Theorem 2 is certainly a non-trivial generalization of the findings of

Srivastava et al. [24] (see also [33,38]). Based upon the above outcomes, we conclude here that our chosen method has credibly worked under the operators defined in (15), and hence, it is stronger than the classical and statistical versions of the approximation of Korovkin-type theorems (see [24,33,38]) which were established earlier.

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